

Modular forms, modular symbols

(PARI-GP version 2.13.4)

Modular Forms

Dirichlet characters

Characters are encoded in three different ways:

- a `t_INT` $D \equiv 0, 1 \bmod 4$: the quadratic character $(D/\cdot)$;
- a `t_INTMOD` $\text{Mod}(m, q)$, $m \in (\mathbf{Z}/q)^*$ using a canonical bijection with the dual group (the Conrey character $\chi_q(m, \cdot)$);
- a pair $[G, \text{chi}]$, where $G = \text{znstar}(q, 1)$ encodes $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \leq k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$ and the vector $\text{chi} = [c_1, \dots, c_k]$ encodes the character such that $\chi(g_j) = e(c_j/d_j)$.

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$ `G = znstar(q, 1)`
convert datum D to $[G, \chi]$ `znchar(D)`
Galois orbits of Dirichlet characters `chargalois(G)`

Spaces of modular forms

Arguments of the form $[N, k, \chi]$ give the level weight and nebentypus χ ; χ can be omitted: $[N, k]$ means trivial χ .

initialize $S_k^{\text{new}}(\Gamma_0(N), \chi)$ `mfinit([N, k, \chi], 0)`
initialize $S_k(\Gamma_0(N), \chi)$ `mfinit([N, k, \chi], 1)`
initialize $S_k^{\text{old}}(\Gamma_0(N), \chi)$ `mfinit([N, k, \chi], 2)`
initialize $E_k(\Gamma_0(N), \chi)$ `mfinit([N, k, \chi], 3)`
initialize $M_k(\Gamma_0(N), \chi)$ `mfinit([N, k, \chi])`
find eigenforms `mfsplit(M)`
statistics on self-growing caches `getcache()`

We let $M = \text{mfinit}(\dots)$ denote a modular space.

describe the space M `mfdescribe(M)`
recover (N, k, χ) `mfparams(M)`
... the space identifier (0 to 4) `mfspace(M)`
... the dimension of M over $\mathbf{C}$ `mfdim(M)`
... a $\mathbf{C}$ -basis (f_i) of M `mfbasis(M)`
... a basis (F_j) of eigenforms `mfeigenbasis(M)`
... polynomials defining $\mathbf{Q}(\chi)(F_j)/\mathbf{Q}(\chi)$ `mffields(M)`

matrix of Hecke operator T_n on (f_i) `mfheckemat(M, n)`
eigenvalues of w_Q `mfatkineigenvalues(M, Q)`
basis of period polynomials for weight k `mfperiodpolbasis(k)`
basis of the Kohnen $+$ -space `mfkohnenbasis(M)`
... new space and eigenforms `mfkohneneigenbasis(M, b)`
isomorphism $S_k^+(4N, \chi) \rightarrow S_{2k-1}(N, \chi^2)$ `mfkohnenbijection(M)`

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\text{new}}(\Gamma_0(N), \chi)$ `mfdim([N, k, \chi])`
dimension of $S_k(\Gamma_0(N), \chi)$ `mfdim([N, k, \chi], 1)`
dimension of $S_k^{\text{old}}(\Gamma_0(N), \chi)$ `mfdim([N, k, \chi], 2)`
dimension of $M_k(\Gamma_0(N), \chi)$ `mfdim([N, k, \chi], 3)`
dimension of $E_k(\Gamma_0(N), \chi)$ `mfdim([N, k, \chi], 4)`
Sturm's bound for $M_k(\Gamma_0(N), \chi)$ `mfsturm(N, k)`

$\Gamma_0(N)$ **cosets**
list of right $\Gamma_0(N)$ cosets `mfcosets(N)`
identify coset a matrix belongs to `mflocoset`

Cusps

a cusp is given by a rational number or ∞ .

lists of cusps of $\Gamma_0(N)$ `mfcusps(N)`
number of cusps of $\Gamma_0(N)$ `mfnumcusps(N)`
width of cusp c of $\Gamma_0(N)$ `mfcuspswidth(N, c)`
is cusp c regular for $M_k(\Gamma_0(N), \chi)$? `mfcuspsisregular([N, k, \chi], c)`

Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients `mftobasis(mf, vec)`

There are also many predefined ones:
Eisenstein series E_k on $Sl_2(\mathbf{Z})$ `mfEk(k)`
Eisenstein-Hurwitz series on $\Gamma_0(4)$ `mfEH(k)`
unary θ function (for character ψ) `mfTheta({\psi})`
Ramanujan's Δ `mfDelta()`
 $E_k(\chi)$ `mfeisenstein(k, \chi)`
 $E_k(\chi_1, \chi_2)$ `mfeisenstein(k, \chi_1, \chi_2)`
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$ `mffrometaquo(a)`
newform attached to ell. curve $E/\mathbf{Q}$ `mffromell(E)`
identify an L -function as a eigenform `mffromlfun(L)`
 θ function attached to $Q > 0$ `mffromqt(Q)`
trace form in $S_k^{\text{new}}(\Gamma_0(N), \chi)$ `mftraceform([N, k, \chi])`
trace form in $S_k(\Gamma_0(N), \chi)$ `mftraceform([N, k, \chi], 1)`

Operations on modular forms

In this section, f, g and the $F[i]$ are modular forms

$f \times g$ `mfmul(f, g)`
 f/g `mfddiv(f, g)`
 f^n `mfpow(f, n)`
 $f(q)/q^v$ `mfshift(f, v)`
 $\sum_{i \leq k} \lambda_i F[i]$, $L = [\lambda_1, \dots, \lambda_k]$ `mflinear(F, L)`
 $f = g?$ `mfisequal(f, g)`
expanding operator $B_d(f)$ `mfbd(f, d)`
Hecke operator $T_n f$ `mfhecke(mf, f, n)`
initialize Atkin-Lehner operator w_Q `mfatkininit(mf, Q)`
... apply w_Q to f `mfatkin(w_Q, f)`
twist by the quadratic char $(D/\cdot)$ `mftwist(f, D)`
derivative wrt. $q \cdot d/dq$ `mfderiv(f)`
see f over an absolute field `mfreltoabs(f)`
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$ `mfderivE2(f)`
Rankin-Cohen bracket $[f, g]_n$ `mfbracket(f, g, n)`
Shimura lift of f for discriminant D `mfshimura(mf, f, D)`

Properties of modular forms

In this section, $f = \sum_n f_n q^n$ is a modular form in some space M with parameters N, k, χ .

describe the form f `mfdescribe(f)`
 (N, k, χ) for form f `mfparams(f)`
the space identifier (0 to 4) for f `mfspace(mf, f)`
 $[f_0, \dots, f_n]$ `mfcoefs(f, n)`
 f_n `mfcoef(f, n)`
is f a CM form? `mfisCM(f)`
is f an eta quotient? `mfisetaquo(f)`
Galois rep. attached to all $(1, \chi)$ eigenforms `mfgaloistype(M)`
... single eigenform `mfgaloistype(M, F)`
... as a polynomial fixed by $\text{Ker } \rho_F$ `mfgaloisprojrep(M, F)`
decompose f on `mfbasis(M)` `mftobasis(M, f)`
smallest level on which f is defined `mfconductor(M, f)`
decompose f on $\oplus S_k^{\text{new}}(\Gamma_0(d))$, $d \mid N$ `mftonew(M, f)`
valuation of f at cusp c `mfcuspsval(M, f, c)`
expansion at ∞ of $f \mid_k \gamma$ `mfslashexpansion(M, f, \gamma, n)`
 n -Taylor expansion of f at i `mftaylor(f, n)`
all rational eigenforms matching criteria `mfeigensearch`
... forms matching criteria `mfsearch`

Forms embedded into $\mathbf{C}$

Given a modular form f in $M_k(\Gamma_0(N), \chi)$ its field of definition $Q(f)$ has $n = [Q(f) : Q(\chi)]$ embeddings into the complex numbers. If $n = 1$, the following functions return a single answer, attached to the canonical embedding of f in $\mathbf{C}[[q]]$; else a vector of n results, corresponding to the n conjugates of f .

complex embeddings of $Q(f)$ `mfembed(f)`
... embed coefs of f `mfembed(f, v)`
evaluate f at $\tau \in \mathcal{H}$ `mfeval(f, \tau)`
 L -function attached to f `lfunmf(mf, f)`
... eigenforms of new space M `lfunmf(M)`

Periods and symbols

The functions in this section depend on $[Q(f) : Q(\chi)]$ as above.

initialize symbol fs attached to f `mfsymbol(M, f)`
evaluate symbol fs on path p `mfsymboleval(fs, p)`
Petersson product of f and g `mfpetersson(fs, gs)`
period polynomial of form f `mfperiodpol(M, fs)`
period polynomials for eigensymbol FS `mfmanin(FS)`

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X, Y]_{k-2}$, $L_k = \mathbf{Z}[X, Y]_{k-2}$. We let $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$; an element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b] - [a] \rightarrow$ the path from a to b . A path is coded by the pair $[a, b]$, where a, b are rationals or ∞ , denoting the point at infinity ($1 : 0$).

Let $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k -valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either $\mathbf{M}_k$ ($\varepsilon = 0$) or its $\pm$ -parts ($\varepsilon = \pm 1$) and fixes a minimal set of $\mathbf{Z}[G]$ -generators (g_i) of Δ .

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$
the level M
the weight k
the sign ε
Farey symbol attached to G
... attached to $H < G$
 $H \backslash G$ and right G -action
 $\mathbf{Z}[G]$ -generators (g_i) and relations for Δ
decompose $p = [a, b]$ on the (g_i)

`msinit( $N, k, \{\varepsilon = 0\}$ )`
`msgetlevel( $M$ )`
`msgetweight( $M$ )`
`msgetsign( $M$ )`
`mspolygon( $M$ )`
`msfarey( $F, inH$ )`
`mscosets( $genG, inH$ )`
`mspathgens( $M$ )`
`mspathlog( $M, p$ )`

Create a symbol

Eisenstein symbol attached to cusp c
cuspidal symbol attached to $E/\mathbf{Q}$
symbol having given Hecke eigenvalues
is s a symbol ?

`msfromcusp( $M, c$ )`
`msfromell( $E$ )`
`msfromhecke( $M, v, \{H\}$ )`
`msissymbol( $M, s$ )`

Operations on symbols

the list of all $s(g_i)$
evaluate symbol s on path $p = [a, b]$
Petersson product of s and t

`mseval( $M, s$ )`
`mseval( $M, s, p$ )`
`mspetersson( $M, s, t$ )`

Operators on subspaces

An operator is given by a matrix of a fixed $\mathbf{Q}$ -basis. H , if given, is a stable $\mathbf{Q}$ -subspace of $\mathbf{M}_k(G)$: operator is restricted to H .

matrix of Hecke operator T_p or U_p
matrix of Atkin-Lehner w_Q
matrix of the $*$ involution

`mshecke( $M, p, \{H\}$ )`
`msatkinlehner( $M, Q\{H\}$ )`
`msstar( $M, \{H\}$ )`

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a $\mathbf{Q}$ -basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

cuspidal subspace $S_k(G)^\varepsilon$
Eisenstein subspace $E_k(G)^\varepsilon$
new part of $S_k(G)^\varepsilon$
split H into simple subspaces (of $\dim \leq d$)
dimension of a subspace
($a_1, \dots, a_B$) for attached newform
 $\mathbf{Z}$ -structure from $H^1(G, L_k)$ on subspace A

`mscuspidal( $M$ )`
`mseisenstein( $M$ )`
`msnew( $M$ )`
`mssplit( $M, H, \{d\}$ )`
`msdim( $M$ )`
`msqexpansion( $M, H, \{B\}$ )`
`mslattice( $M, A$ )`

Overconvergent symbols and p -adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \leq 1$), which is an eigenvector for T_p with nonzero eigenvalue a_p , we can attach a p -adic L -function L_p . The function L_p is defined on continuous characters of $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle \chi \rangle^{s_1} \tau^{s_2}$, where (s_1, s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ , taking values in spaces of p -adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* = 0 (fastest), and that $v_p(a_p) \geq \textit{flag}$ otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to Φ allowing to compute L_p to high accuracy.

initialize Mp to lift symbols
lift symbol ϕ
eval overconvergent symbol Φ on path p
mu for p -adic L -functions
 $L_p^{(r)}(\chi^s)$, $s = [s_1, s_2]$
 $\hat{L}_p(\tau^i)(x)$

`mspadicinit( $M, p, n, \{\textit{flag}\}$ )`
`mstooms( $Mp, \phi$ )`
`msomseval( $Mp, \Phi, p$ )`
`mspadicmoments( $Mp, S, \{D = 1\}$ )`
`mspadicL( $mu, \{s = 0\}, \{r = 0\}$ )`
`mspadicseries( $mu, \{i = 0\}$ )`

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