Special Cases of Compactly Supported Radial Basis Functions

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Abstract: Two instances of positive definite radial basis functions with compact support are presented. The first is "Euclid's hat", obtained by convolution of the characteristic function of the Euclidean unit ball in $\mathbb{R}^d$. If d is odd, the result is a piecewise polynomial function of the Euclidean distance to the origin. In even-dimensional spaces things are somewhat more complicated, but still easy to calculate. The second instance is the radialization of tensor products of univariate B-splines of even order. For this we give a recursion via a positive univariate convolution integral that holds for arbitrary orders and dimensions, and we explicitly calculate the result for the two-dimensional case. Since the cubic bivariate tensor-product B-spline on uniform knots does not differ too much from its radialization, we finally consider the radial function obtained by direct evaluation of even-order univariate B-splines as functions of Euclidean distance in $\mathbb{R}^d$. Calculating Fourier transforms via power series, it turns out that at least for dimensions $d = 2, 3$ and orders 2 and 4 of univariate B-splines this approach does not yield positive definite functions.

1 Introduction

A scalar function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is called a *positive definite radial basis function on* $\mathbb{R}^d$, if the quadratic form

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k\phi(\|x_j - x_k\|_2),\ \alpha = (\alpha_1, \ldots, \alpha_N)^T \in \mathbb{R}^N \tag{1.1}$$

is positive definite for all sets $X = \{x_1, \ldots, x_N\}$ of pairwise distinct points in $\mathbb{R}^d$. Such a function allows unique interpolation on X by translates $\phi(\|\cdot - x_j\|_2), 1 \leq j \leq N$. Furthermore, the solution

$$s_f = \sum_{j=1}^{N}\alpha_j\phi(\|\cdot - x_j\|_2) \tag{1.2}$$

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of the interpolation problem can be calculated and evaluated very efficiently, if the function ϕ is of compact support. Unfortunately, there are no readily available compactly supported positive definite radial basis functions for dimensions $d \geq 2$ in the literature. This paper is intended as a first step to improve the situation.

When going over from a radial function $\phi(\|\cdot\|_2)$ on $I\!R^d$ to a more general function $\Phi : I\!R^d \rightarrow I\!R$ with the symmetry property $\Phi(-x) = \Phi(x)$ for all $x \in I\!R^d$, the notion of positive definiteness is the same as above, the quadratic form now being

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k\Phi(x_j - x_k),\ \alpha = (\alpha_1, \ldots, \alpha_N)^T \in I\!R^N \tag{1.3}$$

instead of (1.1). Due to results of Madych and Nelson [3] and Iske [2] the positive definiteness of continuous and absolutely integrable functions $\Phi : I\!R^d \rightarrow I\!R$ is equivalent to the Fourier transform $\hat{\Phi}$ being nonnegative on $I\!R^d$ and positive at least on an open subset of $I\!R^d$. We shall call $\hat{\Phi}$ *strictly nonnegative* if this condition is satisfied. We note in passing that the cited result is a consequence of the simple identity

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k\Phi(x_j - x_k) = (2\pi)^{-d}\int_{I\!R_d}\hat{\Phi}(\omega)\left|\sum_{j=1}^{N}\alpha_j e^{ix_j^T\omega}\right|^2 d\omega \tag{1.4}$$

for the quadratic form (1.3). At the same time, this identity serves to inform the reader about out choice of constants in Fourier transforms.

By the convolution theorem, all $2m \geq 2$-fold convolutions of compactly supported functions Φ on $I\!R^d$ with Fourier transforms that are nonzero on at least an open subset of $I\!R^d$ will have strictly nonnegative Fourier transforms and thus be positive definite. Such functions obviously exist in abundance, but so far they are either not radial or not explicitly known or numerically inaccessible. The Fourier transforms of the convolutions are plain even powers of the original Fourier transforms, but the convolutions themselves are hard to handle both as inverse Fourier transforms and as convolution integrals over L_2 balls. The simplest case, treated in section 2, will simply convolve two instances of the characteristic function of the L_2 unit ball in $I\!R^d$. This mimics the generation of the second-order univariate B-spline (the *hat* function) via convolution of two characteristic functions of intervals in $I\!R$, and generates a function that can be called "*Euclid's hat*". It turns out to be a piecewise polynomial function of Euclidean distance r in odd dimensions, while it recursively boils down to piecewise linear combinations of $\arccos(r/2)$ and odd polynomials times $\sqrt{1 - r^2/4}$ in even dimensions. Its support is $[0, 2]$, of course, and its approximation properties match those of the radial basis function $\phi(r) = r$ on Sobolew space $W^{(d+1)/2}(I\!R^d)$.

While *radial* compactly supported and positive definite basis functions are rare, there are plenty of non-radial positive definite basis functions with compact support, e.g. tensor products of $(2m)$-th order univariate B-splines on uniform knot sequences $[-m, m] \cap Z\!\!Z$. Their Fourier transforms are strictly nonnegative, because they are $(2m)$-th powers of tensor products of univariate sinc functions, We turn these functions into radial functions by taking spherical means, that is we apply the *radialization* operator

$$(Rad\ \ f)(r) := \frac{1}{\sigma_d}\int_{S_d} f(ry)dy, \quad f : I\!R^d \rightarrow I\!R, \tag{1.5}$$

to $(2m)$-th-order tensor-product B-splines. Here, we denote the unit sphere in $\mathbb{R}^d$ by S_d and its surface area by σ_d. One can easily see that the radialization commutes with the Fourier transform whenever both can be applied. Thus the Fourier transform of $Rad\,\Phi$ is the radialization of $\hat{\Phi}$, and strictly nonnegative if $\hat{\Phi}$ is strictly nonnegative. This approach is a second possibility to generate positive definite radial functions with compact support, and we shall apply it to even-order tensor-product B-splines in sections 3 and 4. For dimension $d = 2$ and any even order $2m$ we derive a simple algorithm to calculate the radialization explicitly, while for larger dimensions we only supply a recursion formula involving a non-oscillatory integration. The appoximation behavior of the resulting compactly supported positive definite radial basis functions is comparable to that of $\phi(r) = r^{2m-d}$ or $\phi(r) = r^{2m-d}\log r$ on odd or even-dimensional spaces $\mathbb{R}^d$, respectively, and the approximation order of interpolants to functions from Sobolew space $W^m(\mathbb{R}^d)$ for $m > d/2$ is $m - d/2$ in these cases.

Results of the previous approach for $d = 2$ and $m = 2$ show that the radialization of the cubic bivariate tensor-product B-spline does not deviate much from the univariate cubic B-spline in regions where the function values are not too small. The reason is that the cubic bivariate tensor-product B-spline looks quite radial except near the boundary of its support. Therefore it seems reasonable to use the $(2m)$-th order univariate B-spline directly as a function of Euclidean distance in order to get another compactly supported radial and possibly also positive definite function. Due to Paley-Wiener theory, the Fourier transform of this is an entire function of exponential type, and we calculate its Taylor series explicitly in section 5. The series can be stably evaluated near the origin, and this suffices to show that it attains negative values for $m = 1, 2$ and $d = 2, 3$. This rules out the possibility of generally using univariate even-order B-splines directly as functions of the Euclidean distance.

2 Euclid's hat function

The univariate hat function

$$\beta_2(t) = \left\{ \begin{array}{ll} 1 - |t| & |t| \leq 1 \\ 0 & \text{else} \end{array} \right\}$$

is a second-order symmetric univariate B-spline with knots $-1, 0, 1$. It is the convolution

$$\beta_2 = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}$$

of two instances of the characteristic function $\chi_{[-1/2,1/2]}$ on $[-1/2, 1/2]$. If this approach is generalized to the multivariate setting, the result is a radial function only if convolutions of characteristic functions of *Euclidean* balls are taken. Using the unit balls of the Chebyshev or L_∞ norm yields the tensor product of instances of β_2, i.e.: a 2^d-sided pyramid that could be called "*Chebyshev's hat*" and that has knot lines joining the center with the corners of the domain.

We now proceed to calculate "*Euclid's hat*" by convolution

$$\beta_{2,d} := \chi_{B_d} * \chi_{B_d}$$

where B_d is the Euclidean unit ball in $\mathbb{R}^d$. This, being part of the Diplom thesis [7] of the second author, involves to calculate the volume of the cap of the unit ball in $\mathbb{R}^d$ which is cut off by a plane at distance $t \leq 1$ to the origin. Thus, for $t \in [0,1]$, we have

$$\begin{aligned} \beta_{2,d}(2t) &= 2\int_t^1 vol(\sqrt{1-s^2}B_{d-1})ds \\ &= 2\omega_{d-1}\int_t^1 (1-s^2)^{(d-1)/2}ds, \end{aligned}$$

where ω_{d-1} is the volume of the unit ball B_{d-1} in $\mathbb{R}^{d-1}$. We substitute $t = \cos\psi,\ s = \cos\varphi$ and get

$$\beta_{2,d}(2\cos\psi) = 2\omega_{d-1}\int_0^\psi \sin^d\varphi d\varphi,\ 0 \leq \psi \leq \pi/2.$$

An easy integration by parts yields the recursion

$$dF_d(\psi) = (d-1)F_{d-2}(\psi) - \cos\psi\sin^{d-1}\psi,\ d \geq 2$$

for

$$F_d(\psi) = \int_0^\psi \sin^d\varphi d\varphi.$$

Clearly,

$$\begin{aligned} F_0(\psi) &= \psi, \\ F_1(\psi) &= 1-\cos\psi, \\ F_2(\psi) &= \tfrac{1}{2}(\psi - \sin\psi\cos\psi), \end{aligned}$$

and all F_{2k+1} are odd polynomials in $\cos\psi$ due to the recursion. This implies that in spaces of odd dimension $d = 2k+1$ the Euclidean hat function is an odd polynomial

$$\begin{aligned} \beta_{2,2k+1}(2t) &= 2\omega_{2k}F_{2k+1}(\arccos t) \\ &= 2\omega_{2k}\tfrac{2k}{2k+1}F_{2k-1}(\arccos t) - \tfrac{1}{2k+1}t(1-t^2)^k \\ &= \tfrac{1}{2k+1}(2\pi\beta_{2,2k-1}(2t) - t(1-t^2)^k) \end{aligned}$$

for $t \in [0,1]$ with $\beta_{2,1}(2t) = 2(1-t)$. In spaces with even dimension $d = 2k$ we similarly get

$$\beta_{2,2k}(2t) = \frac{1}{2k}(2\pi\beta_{2,2k-2}(2t) - t\sqrt{1-t^2}(1-t^2)^{k-1})$$

with

$$\beta_{2,2}(2t) = 2(\arccos t - t\sqrt{1-t^2}).$$

This will always generate a function of t that can be written as a linear combination of $\arccos t/2$ and an odd polynomial in t multiplied by $\sqrt{1-t^2/4}$. In these formulae we made use of $\omega_n/\omega_{n-2} = 2\pi/n$.

Theorem 2.1 *The Euclidean hat function $\phi(\|\cdot\|_2) = \Phi(\cdot)$, calculated recursively as above, is a continuous positive definite radial function with compact support. For functions f with a finite value of*

$$\|f\|^2 := (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{|\hat{f}(\omega)|^2 d\omega}{\hat{\Phi}(\omega)}, \tag{2.1}$$

the interpolants s_f as in (1.2) to data at $X = \{x_1, \ldots, x_N\}$ satisfy an error bound

$$|f(x) - s_f(x)| \le \|f\| \cdot P(x), \ x \in I\!R^d. \tag{2.2}$$

The function $P(x)$ can be viewed as the norm of the error functional at x in the dual of the "native" space of functions induced by the norm (2.1), and it can be evaluated (see [4]) as

$$P^2(x) = \Phi(0) - \sum_{j=1}^{N} u_j(x)\Phi(x - x_j) + \sum_{j,k=1}^{N} u_j(x)u_k(x)\Phi(x_j - x_k) \tag{2.3}$$

using the Lagrange interpolants u_j with $u_j(x_k) = \delta_{jk}$. For all positive constants ρ there are positive constants h_0 and c such that for all data sets X and all points $x \in I\!R^d$ with

$$h_\rho(x) := \sup_{\|y-x\|_2 \le \rho} \min_{1 \le j \le N} \|y - x_j\|_2 \le h_0$$

we have

$$P(x) \le c \cdot h_\rho^{1/2}(x). \tag{2.4}$$

Remark: The same kind of error bound holds for $\phi(r) = r$ (when constants are added to the space of interpolating functions) and for the Sobolew space radial function $\phi(r) = r \cdot K_1(r)$ on $W^{(d+1)/2}(I\!R^d)$.

Proof: The second part of the assertion follows from arguments of Wu and Schaback [8] when applied to this special setting. Everything hinges on the decay properties at infinity of the Fourier transform of $\Phi = \phi(\|\cdot\|_2)$, which is

$$\hat{\Phi}(\omega) = \left(\|\omega\|_2^{-d/2} J_{d/2}(\|\omega\|_2) \right)^2,$$

up to a multiplicative constant, as follows from standard calculations as in Stein and Weiss [6], p. 155 and p. 171. The decay of $J_{d/2}(r)$ at infinity is $\mathcal{O}(r^{-1/2})$ due to Abramowitz and Stegun [1], equation 9.2.1. Altogether, the behavior of $\hat{\Phi}$ thus is

$$\hat{\Phi}(\omega) = \left\{ \begin{array}{ll} \mathcal{O}(1) & \text{for } \omega \text{ near } 0 \\ \mathcal{O}(\|\omega\|_2^{-d-1}) & \text{for } \omega \text{ near } \infty \end{array} \right\}. \tag{2.5}$$

This is precisely the behavior of $(1 + \|\omega\|_2^2)^{-k}$ for $k = (d+1)/2$, and we see that Sobolew space $W^k(I\!R^d)$ with this k is containing all functions f with finite values of (2.1), which is the norm of the "native" space for Φ in the sense of [4] and [5]. Then (2.2) and (2.3) follow immediately, while (2.4) can be obtained from [8] by the same techniques as used there for $\phi(r) = r$ with Fourier transform behaving like (2.5) at infinity. □

Remark. Functions f with finite values of (2.1) have Fourier transforms that necessarily share the zeros ζ_k of $J_{d/2}^2(\|\cdot\|_2)$. This defines a subspace of $W^{(d+1)/2}(I\!R^d)$ of functions that are orthogonal on $I\!R^d$ to functions $f_\omega(x) := e^{ix^t\omega}$ for all $\omega \in I\!R^d$ with $\|\omega\|_2 = \zeta_k$. This is a peculiar situation that arises in all similar cases where $\hat{\Phi}$ is strictly nonnegative, but has zeros on sets with measure zero. Its implications with respect to approximation behavior are unclear, but highly interesting.

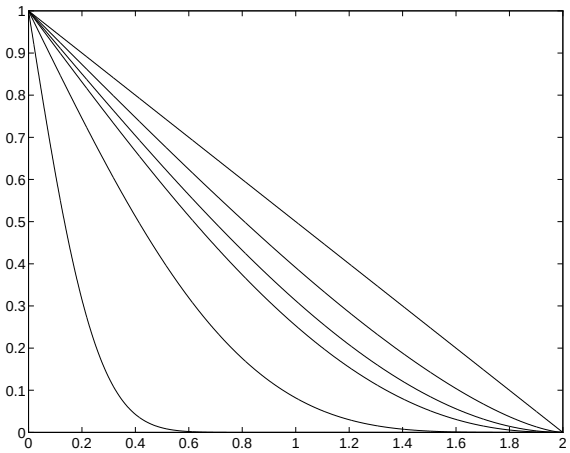

Figure 1. Profiles of Euclids hats for dimensions $d = 1, 2, 3, 4, 10, 100,$ from right to left.

3 Radialized B-splines in two dimensions

Let β_{2m} be the univariate B-spline of order $2m$ with uniform knot set $[-m, m] \cap \mathbb{Z}$. We do not need to specify the normalization here. The radialization of the d-fold tensor product is the function

$$\rho_{2m,d}(r) := \frac{1}{\sigma_d} \int_{S_d} \prod_{j=1}^{d} \beta_{2m}(r y_j) dy, \ r \geq 0. \tag{3.1}$$

For $d = 2$ we parametrize S_d in the usual way and get, by use of symmetry,

$$\rho_{2m,2}(r) = \frac{8}{2\pi} \int_0^{\pi/4} \beta_{2m}(r \sin\varphi) \beta_{2m}(r \cos\varphi) d\varphi.$$

For positive arguments we represent β_{2m} as a piecewise polynomial via

$$\beta_{2m}(r) = \sum_{j=1}^{m} \chi_{[j-1,j]}(r) \sum_{k=0}^{2m-1} c_{jk} r^k$$

and evaluate $\rho_{2m,2}$ explicitly by application of the formulae

$$\rho_{2m,2}(r) = \frac{4}{\pi} \sum_{j,\ell=1}^{m} \sum_{k,n=0}^{2m-1} c_{jk} c_{ln} r^{k+n} F(r; j, \ell, k, n), \tag{3.2}$$

$$\begin{aligned} F(r; j, \ell, k, n) &:= \int_0^{\pi/4} \chi_{[j-1,j]}(r \cos\varphi) \chi_{[\ell-1,\ell]}(r \sin\varphi) \cos^k\varphi \sin^n\varphi d\varphi \\ &= \int_{\varphi_1(r;j,\ell)}^{\varphi_2(r;j,\ell)} \cos^k\varphi \sin^n\varphi d\varphi, \end{aligned}$$

where the limits of integration can be explicitly calculated and programmed. Integrals of the form

$$\int_{\varphi_1}^{\varphi_2} \cos^k\varphi \sin^n\varphi d\varphi =: H_{k,n}(\varphi_1, \varphi_2)$$

are explicitly known and numerically available without integration. We use the recursion

$$H_{k,n} = H_{k,n-2} - H_{k+2,n-2}$$

to reduce evrything to $H_{k,1}$ and $H_{k,0}$, and then

$$\begin{aligned} H_{k,1}(\varphi_1, \varphi_2) &= \tfrac{1}{k+1}(\cos^{k+1}\varphi_1 - \cos^{k+1}\varphi_2) \\ kH_{k,0}(\varphi_1, \varphi_2) &= \sin\varphi_2 \cos^{k-1}\varphi_2 - \sin\varphi_1 \cos^{k-1}\varphi_1 + (k-1)H_{k-2,0}(\varphi_1, \varphi_2), \end{aligned}$$

while $H_{1,0}$ and $H_{0,0}$ are elementary. Though (3.2) contains $\mathcal{O}(m^4)$ terms, there can be only $\mathcal{O}(m)$ pairs (j, ℓ) that yield nonzero contributions for a specific value of r, because the arc $(r\cos\varphi, r\sin\varphi)$ meets only $\mathcal{O}(m)$ cubes $[j-1, j] \times [\ell - 1, \ell]$. For each such pair (j, ℓ) we calculate $\varphi_1(r; j, \ell)$ and $\varphi_2(r; j, \ell)$ such that the interval $[\varphi_1, \varphi_2]$ precisely describes the set of values for φ that satisfy

$$\begin{aligned} j-1 &\leq r\cos\varphi &&\leq j \\ \ell - 1 &\leq r\sin\varphi &&\leq \ell \\ 0 &\leq \varphi &&\leq \pi/4. \end{aligned}$$

This can be done in a straightforward way, using arcsin and arccos functions appropriately. Once φ_1 and φ_2 are found, we calculate the whole set of values $F(r; j, \ell, k, n)$ for $0 \leq k, n \leq 2m-1$ using the recursions.

This still is not a peculiarly efficient method to get $\rho_{2m,2}(r)$ for a given value of r. For large-scale applications one would prepare a sufficiently good piecewise polynomial or rational approximation to $\rho_{2m,2}$ beforehand, using the above method to generate input values.

We include a few figures for illustration. In Figure 2 the two curves are hard to distinguish. Thus we add a semilogarithmic plot as Figure 3. There the univariate cubic B-spline dashes down at abscissa 2.0, while the radialized bivariate cubic tensor-product B-spline extends up to $2\sqrt{2} = 2.82$. It is quite remarkable that the difference between the plain cubic B-spline and the radialization of its tensor product is so small in regions where both take on reasonably large values. This suggests to use the cubic univariate B-spline directly as a function of Euclidean distance. However, the resulting compactly supported radial function fails to be positive definite, as we shall prove in section 5.

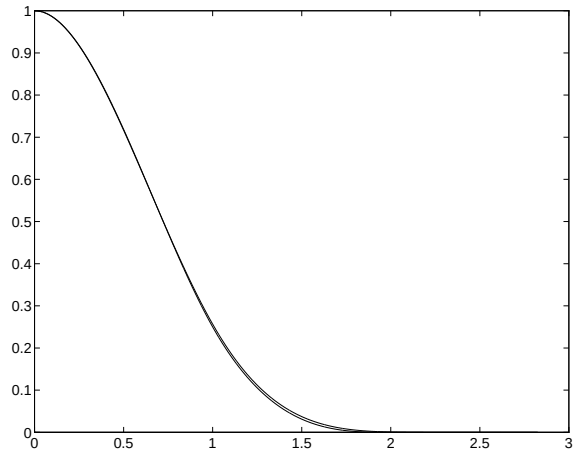

Figure 2. Univariate cubic B-spline and the radialization of the bivariate cubic tensor-product B-spline.

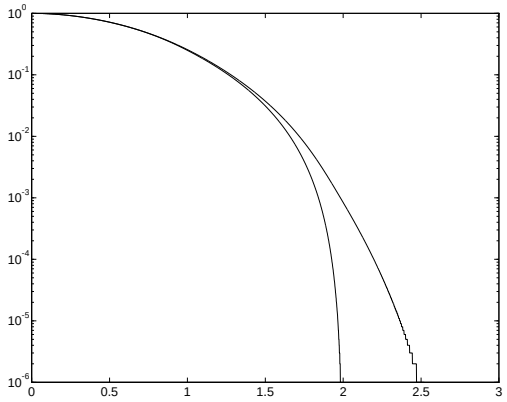

Figure 3. Same as Fig. 2, but semilogarithmic.

4 Radialized B-splines in 3 or more dimensions

For dimensions $d \geq 3$ we now want to calculate (3.1) recursively and split the integral via

$$\begin{aligned}
\rho_{2m,d} &= \tfrac{2}{\sigma_d} \int_0^1 \beta_{2m}(ry_d) \int_{\sqrt{1-y_d^2} S_{d-1}} \prod_{j=1}^{d-1} \beta_{2m}(rz_j) dz dy_d \\
&= \tfrac{2}{\sigma_d} \int_0^1 \beta_{2m}(ry_d) \left(1-y_d^2\right)^{\frac{d-1}{2}} \int_{S_{d-1}} \prod_{j=1}^{d-1} \beta_{2m}(r\sqrt{1-y_d^2}u_j) du dy_d \\
&= \tfrac{2\sigma_{d-1}}{\sigma_d} \int_0^1 \beta_{2m}(ry_d) \left(1-y_d^2\right)^{\frac{d-1}{2}} \rho_{2m,d-1}(r\sqrt{1-y_2^2}) dy_d \\
&= \tfrac{2\sigma_{d-1}}{\sigma_d} \int_0^\pi \beta_{2m}(r\cos\varphi) \sin^{d-2}\varphi \cdot \rho_{2m,d-1}(r\sin\varphi) d\varphi.
\end{aligned}$$

The integrand is nonnegative, leading to stability when applying numerical integration techniques. For $d = 3$ one can use this recursion twice, and the result boils down to a trigonometric integration that is similar to the one in the preceding section. However, we do not include any numerical results for this case, but rather state a result on the approximation order of interpolants with these compactly supported radial basis functions.

Theorem 4.1 *The radialization $\rho_{2m,d}$ of the d-variate tensor product of the $(2m)$-th order univariate B-spline β_{2m} with knot set $[-m,m] \cap \mathbb{Z}$ is a positive definite function. It yields interpolants that satisfy error bounds like (2.2) on Sobolew space $W^m(\mathbb{R}^d)$ with (2.4) replaced by*

$$P(x) \leq c \cdot h_\rho(x)^{m-d/2}. \tag{4.1}$$

Remark. This error behavior is similar to that of thin-plate splines $\phi(r) = r^{2m-d}$ for odd dimensions d or $r^{2m-d}\log r$ for even dimensions. It is *not* comparable to the behavior of $\phi(r) = r^{2m-1}$ as one would expect.

Proof: Again we have to check the decay of the Fourier transform at infinity. Since the Fourier transform commutes with radialization, and since the asymptotic decay is not affected by taking spherical means, the decay of the Fourier transform is $\mathcal{O}(\|\omega\|_2^{-2m})$ for $\|\omega\|_2$ tending

to infinity. Note that the radialized Fourier transform cannot vanish. Thus the "native" space in the sense of [4] and [5] is equivalent to Sobolew space $W^m(\mathbb{R}^d)$. The corresponding order of approximation then is $m - d/2$, as follows from the technique in [8]. □

5 B-splines as radial functions

The numerical results of section 3 clearly show that the radialized bivariate tensor-product cubic B-spline does not deviate too much from the cubic B-spline evaluated directly as a function of the Euclidean distance in $\mathbb{R}^2$. The latter would be a very convenient tool for multivariate interpolation, if it were positive definite. However, this section serves to prove that it isn't. In view of the results of Iske [2] it suffices to prove that the Fourier transform of such a B-spline as a radial function in $\mathbb{R}^d$ changes sign somewhere. Classical Paley-Wiener theory implies that there must be a nicely convergent Taylor series for this Fourier transform. We shall calculate this series and simply evaluate it.

Again, let β_{2m} be the univariate symmetric B-spline of order $2m$ with knot set $[-m, m] \cap \mathbb{Z}$. We use the radial form of the Fourier transform (see e.g. Stein and Weiss [6]) together with the usual power series for Bessel functions. This yields the radial Fourier transform

$$\begin{aligned}
\widehat{\beta_{2m}(\|\cdot\|_2)}(s) &\doteq \int_0^m \beta_{2m}(r) r^{d-1} (rs)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(rs)\, dr \\
&\doteq \int_0^m \beta_{2m}(r) r^{d-1} \sum_{k=0}^{\infty} \frac{(-r^2 s^2/4)^k}{k!\Gamma(k+1+d/2)}\, dr \qquad (5.1) \\
&\doteq \sum_{k=0}^{\infty} \frac{(-s^2/4)^k}{k!\Gamma(k+1+d/2)} \int_0^m \beta_{2m}(r) r^{2k+d-1}\, dr,
\end{aligned}$$

where $\doteq$ means equality up to a positive function of s on $\mathbb{R}_{>0}$. If we normalize β_{2m} to be the exact Peano kernel of the $2m$-th order central divided difference $[-m, \ldots, m]$, then for any $f \in C^{2m}[-m, m]$ we have

$$[-m, \ldots, m] f = \int_{-m}^{m} \beta_{2m}(t) f^{(2m)}(t)\, dt.$$

With the usual truncated power function this yields

$$\int_0^m \beta_{2m}(r) r^{2k+d-1}\, dr = \frac{(2k+d-1)!}{(2m+2k+d-1)!} [-m, \ldots, m](r)_+^{2m+2k+d-1},$$

and these divided differences are easy to calculate for low values of m, say for $m = 1,\ 2$. The results can be inserted into (5.1) and the power series will converge nicely for moderate values of s. We evaluated the series for $m = 1, 2$ and $d = 2, 3$ to find regions where it changed sign to become slightly but surely negative.

We include two plots of Fourier transforms of B-splines as functions of Euclidean distance. Figure 4 contains the case $m = 1$ for dimensions $d = 1, 2, 3$, while Figure 5 presents $m = 2$ for the same dimensions. The three curves can be distinguished at the left margin: the height of the cut-off peak at 0 decreases with increasing dimension. Thus the reader can identify the

univariate B-spline with its $\mathrm{sinc}^{2m}(t/2)$ Fourier transform that has a zero at 2π without sign change. The two other curves belong to dimensions 2 and 3, and they clearly go below zero.

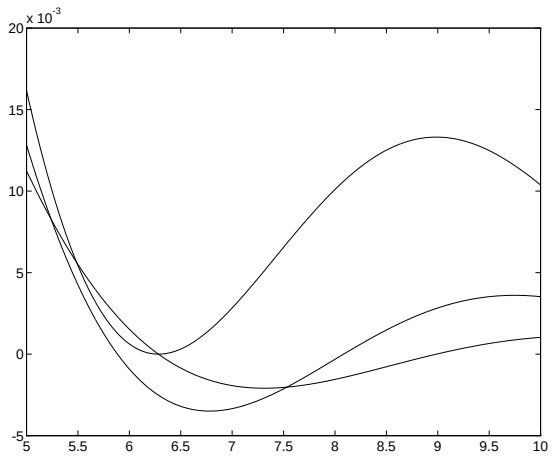

Figure 4. Fourier transforms of B-splines β_2
as functions of Euclidean distance in $I\!R^d$ for $d = 1, 2, 3$.

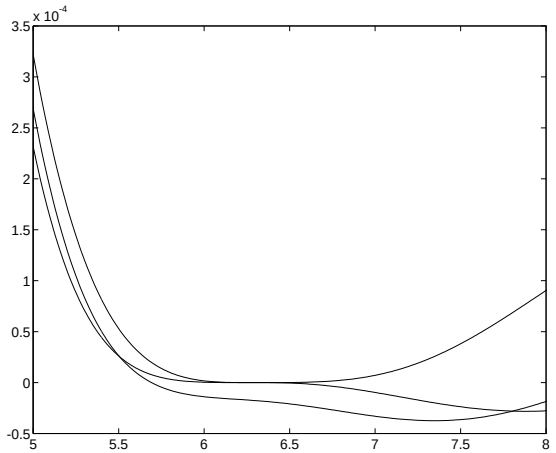

Figure 5. Fourier transforms of B-splines β_4
as functions of Euclidean distance in $I\!R^d$ for $d = 1, 2, 3$.

This rules out the possibility to use $\beta_{2m}(\|\cdot\|_2)$ as a a compactly supported radial basis function in general. However, we checked only special cases, and there may be B-splines of larger support or with non-uniform knots that still work. Note that due to section 2 there are nonnegative univariate piecewise polynomials with compact support that generate positive definite radial functions in certain odd-dimensional spaces, when evaluated as functions of Euclidean distance. In addition, we only checked for positive definiteness. The quadratic form as in (1.1) may be still be nondegenerate or the matrix $(\phi(\|x_j - x_k\|_2))_{1 \le j,k \le N}$ may still be nonsingular for every choice of sets $X = \{x_1, \ldots, x_N\}$ of pairwise distinct points in $I\!R^d$.

References

[1] Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York 1970

[2] Iske, A., Ph.D. dissertation, Göttingen 1994

[3] Madych, W.R., and Nelson, S. A., Multivariate Interpolation: a variational theory. Manuscript, 1983

[4] Schaback, R., Comparison of radial basis function interpolants, Preprint 1992

[5] Schaback, R., Reproduction of Polynomials by radial basis functions, Preprint 1993

[6] Stein, E.M., and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, 1971

[7] Wendland, H., Diplom thesis, Göttingen 1994

[8] Wu, Zong–min, and R. Schaback, Local Error Estimates for Radial Basis Function Interpolation *IMA J. of Numer. Anal.* 13, 13-27, 1993

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