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Topological and Algebraic Copies of $\mathbb{N}^*$ in $\mathbb{N}^*$

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Abstract. We show that the only topological and algebraic copies of $\mathbb{N}^*$ to be found in $\mathbb{N}^*$ are the trivial ones, namely $k \cdot \mathbb{N}^*$ for $k \in \mathbb{N}$. A similar statement holds for copies of $\mathbb{Z}^*$ in $\mathbb{Z}^*$. As a consequence, we obtain the corollary that $\mathbb{N}^*$ contains no copies of $\mathbb{Z}^*$ whatever.

Contents

1. Introduction

Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the discrete space of positive integers and let $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. Similarly write $\mathbb{Z}^* = \beta\mathbb{Z}\backslash\mathbb{Z}$.

It is well known (see [6]) that any infinite closed subspace of $\mathbb{N}^*$ contains a topological copy of all of $\beta\mathbb{N}$. It was then a natural question raised by van Douwen [4] as to whether there are topological and algebraic copies of $(\beta\mathbb{N}, +)$ in $\mathbb{N}^*$. (While [4] was published in 1991, the actual question dates back to 1978.) Here the operation $+$ on $\beta\mathbb{N}$ is the right continuous extension of ordinary addition on $\mathbb{N}$ which has $\mathbb{N}$ contained in (in fact equal to) the topological center of $\beta\mathbb{N}$. By "right continuous" we mean that the function $\rho_p : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$, defined by $\rho_p(q) = q + p$, is continuous for each $p \in \beta\mathbb{N}$. The "topological center" consists of those points for which λ_x is also continuous, where $\lambda_x(p) = x + p$.

The semigroup $(\beta\mathbb{N}, +)$ has had numerous significant applications to Ramsey Theory (a branch of combinatorial number theory) dating back to the Galvin-Glazer proof of the Finite Sum Theorem. (See [3].) See also the surveys [7,8] for several other applications. Many of these applications depend on a reasonably detailed knowledge of the structure of $(\beta\mathbb{N}, +)$.

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In [14] it was shown that there is no continuous one-to-one homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$, answering van Douwen's question in the negative. Observe that as an immediate consequence of this theorem, we have that the only continuous one-to-one homomorphisms from $\beta\mathbb{N}$ to $\beta\mathbb{N}$ are those given by $\phi(p) = k \cdot p$ for some $k \in \mathbb{N}$. To see this, one simply notes that $\phi(1) \notin \mathbb{N}^*$ (for if it were, the whole image of $\beta\mathbb{N}$ would be contained in $\mathbb{N}^*$), so that $\phi(1) = k$ for some $k \in \mathbb{N}$. Then by induction $\phi(n) = k \cdot n$ for all $n \in \mathbb{N}$. Consequently, one has $\phi(p) = k \cdot p$ for all $p \in \beta\mathbb{N}$. (One should be cautioned that by $k \cdot p$ we mean the value of the continuous function extending the function $n \mapsto k \cdot n$, and not the sum of p with itself k times.)

The question then naturally arises as to what sort of restrictions there are on continuous one-to-one homomorphisms from $\mathbb{N}^*$ to $\mathbb{N}^*$. What one ought to expect is not obvious. Consider the set $H = \bigcap_{n=1}^{\infty} cl(\mathbb{N}2^n)$, where the closure is taken in $\beta\mathbb{Z}$ (or equivalently in $\beta\mathbb{N}$). Then H is a subsemigroup of $\beta\mathbb{N}$ and contains a good deal of the known algebraic structure of $\beta\mathbb{N}$. For example, free groups on $2^{\mathfrak{c}}$ generators (where $\mathfrak{c}$ is the cardinality of the continuum) are known to exist in $\beta\mathbb{N}$, and these are contained in H [9]. And copies of H are plentiful. In fact [11], H contains $\mathfrak{c}$ pairwise disjoint copies of itself.

By contrast we show here that the only continuous one-to-one homomorphisms from $\mathbb{N}^*$ to itself are those given by $\phi(p) = k \cdot p$. We also show that a similar statement applies to $\mathbb{Z}^*$, and obtain as a corollary the fact that $\mathbb{N}^*$ contains no copies of $\mathbb{Z}^*$ at all. These results can be found in Section 3, Section 2 being devoted to preliminary results.

We take the points of $\beta\mathbb{N}$ and $\beta\mathbb{Z}$ to be the ultrafilters on $\mathbb{N}$ and $\mathbb{Z}$ respectively. We identify the points of $\mathbb{N}$ with the principal ultrafilters on $\mathbb{N}$ and we identify an ultrafilter p on $\mathbb{N}$ with the ultrafilter $\{A \subseteq \mathbb{Z} : A \cap \mathbb{N} \in p\}$ on $\mathbb{Z}$. By means of these identifications we pretend that $\mathbb{N} \subseteq \beta\mathbb{N} \subseteq \beta\mathbb{Z}$ and $\mathbb{Z} \subseteq \beta\mathbb{Z}$. Given any discrete X and any $A \subseteq X$, $\overline{A} = \{p \in \beta X : A \in p\}$ and given $p \in \beta X$, $\{\overline{A} : A \in p\}$ is a neighborhood basis for p consisting of clopen sets. Given $A \subseteq X$ we write $A^* = \overline{A} \backslash X$.

Given p and q in $\beta\mathbb{Z}$ and $A \subseteq \mathbb{Z}$ one has $A \in p + q$ if and only if $\{x \in \mathbb{Z} : A - x \in q\} \in p$. Alternatively if $\langle x_i \rangle_{i \in I}$ and $\langle y_j \rangle_{j \in J}$ are nets in $\mathbb{Z}$ converging to p and q respectively one has $p + q = \lim_{i \in I} \lim_{j \in J} (x_i + y_j)$. Alternatively, $p + q = p\text{-}\lim_{m \in \mathbb{Z}} q\text{-}\lim_{n \in \mathbb{Z}} m + n$, where given any sequence $\langle x_n \rangle_{n \in \mathbb{Z}}$ in any topological space X, one has $p\text{-}\lim_{n \in \mathbb{Z}} x_n = y$ if and only if for every neighborhood U of y, $\{n \in \mathbb{Z} : x_n \in U\} \in p$. We also utilize, as we remarked earlier, the product $k \cdot p$ for $k \in \mathbb{Z}$ and $p \in \beta\mathbb{Z}$. Here, given $A \subseteq \mathbb{Z}$ and $k \neq 0$, one has $A \in k \cdot p$ if and only if $A/k \in p$, where $A/k = \{y \in \mathbb{Z} : y \cdot k \in A\}$ (and $0 \cdot p = 0$). By $-p$ we mean $-1 \cdot p = \{-A : A \in p\}$. (Note that $p + -p \neq 0$ unless $p \in \mathbb{Z}$.) It is then a routine matter to check that the following limited version of the distributive law holds: given $k \in \mathbb{Z}$ and $p, q \in \beta\mathbb{Z}$, $k \cdot (p + q) = k \cdot p + k \cdot q$. (See [4] for the fact that wider versions of the distributive law fail.) Observe that as a consequence of this limited version of the distributive law, the functions ϕ from $\mathbb{N}^*$ and $\mathbb{Z}^*$ defined by $\phi(p) = k \cdot p$ are (continuous and one-to-one) homomorphisms.

The reader is referred to [8] for an elementary introduction to the semigroups $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$, with the caution that the order of operation is reversed there from the way we use it here. We will use without specific mention the fact that any compact right topological semigroup has idempotents [5, Corollary 2.10]. We will also use frequently the fact that elements of $\mathbb{Z}$ commute with every element of $\beta\mathbb{Z}$

and the fact that if $p, q \in \beta\mathbb{Z}$ and $n \in \mathbb{Z}$ and $n + p = q$, then $p = -n + q$.

An important fact for our constructions is that $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{Z}, +)$ have smallest two sided ideals which we denote by $K(\beta\mathbb{N})$ and $K(\beta\mathbb{Z})$ respectively. The smallest ideal of any compact right topological semigroup has a rich structure, for which the reader is referred to [2, Theorem 1.3.11]. (The reference [2] is also the appropriate place to look for unfamiliar algebraic terminology.) We will not utilize much of the internal structure of the smallest ideals in this paper, but we will use the fact that $K(\beta\mathbb{Z}) = K(\beta\mathbb{N}) \cup -K(\beta\mathbb{N})$. (To see this note first that $K(\beta\mathbb{N}) \cup -K(\beta\mathbb{N})$ is an ideal of $\beta\mathbb{Z}$ so $K(\beta\mathbb{Z}) \subseteq K(\beta\mathbb{N}) \cup -K(\beta\mathbb{N})$. For the reverse inclusion, let say $p \in -K(\beta\mathbb{N})$ and pick a minimal left ideal L of $-K(\beta\mathbb{N})$ with $p \in L$. Then $L \subseteq -\mathbb{N}^*$ and $-\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$ so by [1, II.1.8(ii) and (iii)] and the fact, from Zorn's Lemma, that any left ideal contains a minimal left ideal, L is a minimal left ideal of $\beta\mathbb{Z}$.)

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The following useful result is apparently due originally to Frolík.

1.1 Lemma. *Let A and B be σ-compact subsets of $\beta\mathbb{Z}$ such that $(cl\ A) \cap (cl\ B) \neq \emptyset$. Then $A \cap (cl\ B) \neq \emptyset$ or $(cl\ A) \cap B \neq \emptyset$.*

Proof. See [10, Lemma 1.1] or [6, Chapter 14]. □

2. Preliminaries

Our first preliminaries are modifications of results in [13]. Our plan of attack is as follows. We will assume we have some continuous one-to-one homomorphism ϕ from $\mathbb{N}^*$ to $\mathbb{N}^*$ or from $\mathbb{Z}^*$ to $\mathbb{Z}^*$. We then find a point $q \in \mathbb{Z}^*$ and $k \in \mathbb{Z}$ such that $\phi(p) + q = k \cdot p + q$ for all p and such that $p \mapsto p + q$ is one-to-one.

2.1 Definition. $T = \bigcap_{n=1}^{\infty} cl(\mathbb{N}n)$.

It is easy to verify that T is a compact subsemigroup of $\mathbb{N}^*$. Further, the idempotents of $\mathbb{N}^*$ are all in T as can be seen by considering the natural homomorphisms from $\beta\mathbb{N}$ to $\mathbb{Z}_n$.

2.2 Lemma. *Let $p \in \beta\mathbb{N}\backslash K(\beta\mathbb{Z})$. There is a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that $x_t | x_{t+1}$ and $t! | x_{t+1}$ for all $t \in \mathbb{N}$ and such that for all $q \in (cl\{x_t : t \in \mathbb{N}\}) \cap \mathbb{N}^*$, $p \notin \beta\mathbb{Z} + -q + p$ and $p \notin \beta\mathbb{Z} + q + p$.*

Proof. Pick some $q \in T \cap K(\beta\mathbb{N})$. This is possible since T contains every idempotent. Since $\beta\mathbb{Z} + q + p \subseteq K(\beta\mathbb{N})$ and $\beta\mathbb{Z} + -q + p \subseteq K(\beta\mathbb{Z}) \cap \mathbb{N}^* = K(\beta\mathbb{N})$, p does not belong to either of these compact sets. Thus for some $Y \in p$, $\overline{Y} \cap ((\beta\mathbb{Z} + q + p) \cup (\beta\mathbb{Z} + -q + p)) = \emptyset$. For each $n \in \mathbb{Z}$ let $K_n = \{m \in \mathbb{N} : Y \in n + m + p\}$ and let $L_n = \{m \in \mathbb{N} : Y \in n - m + p\}$. Because for all $n \in \mathbb{Z}$, one has $Y \notin n + q + p$, none of the sets $K_n \in q$. Similarly $q \notin cl\ L_n$. Thus $T \backslash (\bigcup_{n \in \mathbb{Z}} cl(K_n \cup L_n)) \neq \emptyset$. In particular for each $m, l \in \mathbb{N}$, $\mathbb{N}m \backslash \bigcup_{n=-l}^{l} (K_n \cup L_n) \neq \emptyset$. We may thus choose inductively a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ as follows. Let $x_1 \in \mathbb{N} \backslash (K_0 \cup L_0)$. Given x_t, let $x_{t+1} \in (\mathbb{N}x_t t!) \backslash \bigcup_{n=-t}^{t} (K_n \cup L_n)$.

To complete the proof, let $q \in cl\{x_t : t \in \mathbb{N}\} \cap \mathbb{N}^*$. Suppose that there is some $r \in \beta\mathbb{Z}$ with $p = r + -q + p$ or $p = r + q + p$. Assume $p = r + -q + p$. Then

$Y \in r + -q + p$ so pick $n \in \mathbb{Z}$ with $Y \in n + -q + p$. Also $\{x_t : t > |n|\} \in q$ so pick $t > |n|$ such that $Y \in n - x_t + p$. But then $x_t \in L_n$, a contradiction. Similarly if $p = r + q + p$ we get $n \in \mathbb{Z}$ and $t > |n|$ with $Y \in n + x_t + p$ and conclude that $x_t \in K_n$. $\square$

2.3 Lemma. *Let $p \in \beta\mathbb{Z}\backslash K(\beta\mathbb{Z})$ and let $k \in \mathbb{Z}\backslash\{0\}$. There exists $q \in \beta\mathbb{N}$ such that $k \cdot q + p$ is right cancellable in $\beta\mathbb{Z}$.*

Proof. Note that if $p \in \mathbb{Z}$, then $k + p$ is right cancellable (an easy exercise) so we may assume $p \in \mathbb{Z}^*$. Further, it suffices to establish the result for $p \in \mathbb{N}^*$. (For if $p \in -\mathbb{N}^*$, get $q \in \beta\mathbb{N}$ such that $-k \cdot q + -p$ is right cancellable. Then $k \cdot q + p$ is right cancellable.) We thus assume $p \in \mathbb{N}^*$.

Pick a sequence $\langle x_t\rangle_{t=1}^{\infty}$ as guaranteed by Lemma 2.2 for p. For $t > |k|$, one has $(x_t/|k|) \in \mathbb{N}$. Pick $q \in cl\{(x_t/|k|) : t > |k|\} \cap \mathbb{N}^*$. We show that $k \cdot q + p \notin \mathbb{Z}^* + k \cdot q + p$ so that, by [10, Corollary 3.3], $k \cdot q + p$ is right cancellable. (The statement in [10] refers to left cancellability because we were using the reverse order of operation there.)

So suppose we have some $r \in \mathbb{Z}^*$ such that $k \cdot q + p = r + k \cdot q + p$. We will proceed now on the assumption that $k < 0$, the other case being very similar. We have that $k \cdot q \in cl\{-x_t : t \in \mathbb{N}\}$ so $k \cdot q + p \in cl(\{-x_t : t \in \mathbb{N}\} + p)$ while $r + k \cdot q + p \in cl((\mathbb{Z}\backslash\{0\}) + k \cdot q + p)$ so by Lemma 1.1, one of the following conclusions must hold:

(1) There exist some $t \in \mathbb{N}$ and some $v \in \beta\mathbb{Z}$ such that $-x_t + p = v + k \cdot q + p$.
(2) There exist $w \in cl\{-x_t : t \in \mathbb{N}\}$ and $s \in Z\backslash\{0\}$ such that $w + p = s + k \cdot q + p$.

Conclusion (1) implies that $p = x_t + v + k \cdot q + p$, which contradicts Lemma 2.2 since $-k \cdot q \in cl\{x_t : t \in \mathbb{N}\} \cap \mathbb{N}^*$. Thus conclusion (2) must hold. If one had $w \in \mathbb{Z}$, one would again get a contradiction to Lemma 2.2, so $w \in \mathbb{Z}^*$. Pick $m \in \mathbb{N}$ such that $m > |s|$. Let $g : \mathbb{Z} \longrightarrow \{0, 1, \ldots, m-1\}$ be the natural congruence map (i.e. $g(n) \equiv n \mod m$) and denote also by g its extension to $\beta\mathbb{Z}$. Since $m|x_t$ for all $t > m$ we have $g(w) = 0$ and $g(k \cdot q) = 0$. But then $g(p) = g(w + p) = g(s + k \cdot q + p) = g(s) + g(p)$ so $g(s) = 0$. Since $|s| < m$ and $s \neq 0$, this is a contradiction. $\square$

We now present the last of our preliminaries that are closely based on results from [13].

2.4 Lemma. (a) *Let $e + e = e \in \mathbb{N}^*\backslash K(\beta\mathbb{N})$, let $r \in e + \beta\mathbb{N} + e$, and let $n \in \mathbb{N}$. If r commutes with every member of $e + n \cdot \beta\mathbb{N} + e$, then $r \in \mathbb{Z} + e$.*
(b) *Let $e + e = e \in \mathbb{Z}^*\backslash K(\beta\mathbb{Z})$, let $r \in e + \beta\mathbb{Z} + e$, and let $n \in \mathbb{Z}\backslash\{0\}$. If r commutes with every member of $e + n \cdot \beta\mathbb{Z} + e$, then $r \in \mathbb{Z} + e$.*

Proof. We show first that (a) implies (b). It is an easy exercise to show that $\mathbb{N}^*$ and $-\mathbb{N}^*$ are left ideals of $\beta\mathbb{Z}$ so that, since $e + r = r + e$ one has $e, r \in \mathbb{N}^*$ or $e, r \in -\mathbb{N}^*$. If $e, r \in -\mathbb{N}^*$, then $-r$ commutes with every member of $-e + n \cdot \beta\mathbb{Z} + -e$, so we may assume $e, r \in \mathbb{N}^*$. Finally, if r commutes with every member of $e + n \cdot \beta\mathbb{Z} + e$, then it certainly commutes with every member of $e + |n| \cdot \beta\mathbb{N} + e$.

We thus establish (a). We claim that in fact r commutes with every member of $e + \beta\mathbb{N} + e$ so let $q \in e + \beta\mathbb{N} + e$ be given. Pick $i \in \{0, 1, \ldots, n-1\}$ such that $\mathbb{N}n - i \in q$. Then $q + i \in n \cdot \beta\mathbb{N}$ and $q = e + q + e$ so $q + i = e + q + i + e$ so $q + i \in e + n \cdot \beta\mathbb{N} + e$.

Thus r commutes with $q+i$ and so $r+q = r+q+i+-i = q+i+r+-i = q+r$. Thus by [13, Theorem 11] $r \in \mathbb{Z}+e$ as claimed. □

One can define an ordering on the idempotents of $\beta\mathbb{Z}$ by agreeing that $e \leq f$ if and only if $e = f+e = e+f$. The following lemma is due to Ruppert and most of the proof can be found in [12]. We include the entire proof because it is so short.

2.5 Lemma. *Let $e \in \beta\mathbb{Z}$ be an idempotent. Then $e \in K(\beta\mathbb{Z})$ if and only if e is minimal with respect to the order $\leq$.*

Proof. Necessity. Since $e \in K(\beta\mathbb{Z})$, $\beta\mathbb{Z}+e$ is a minimal left ideal. Assume $f \leq e$. Then $f = f+e$ so $f \in \beta\mathbb{Z}+e$ so $\beta\mathbb{Z}+f \subseteq \beta\mathbb{Z}+e$ so $\beta\mathbb{Z}+f = \beta\mathbb{Z}+e$. Thus $e \in \beta\mathbb{Z}+f$ so $e = e+f = f$.

Sufficiency. Pick (by Zorn's Lemma) a minimal left ideal $L \subseteq \beta\mathbb{Z}+e$. Since L is a compact right topological semigroup, pick an idempotent $t \in L$ and note that $t = t+e$. Let $f = e+t$. Then $f \in L \subseteq K(\beta\mathbb{Z})$. Also $f+f = e+t+e+t = e+t+t = e+t = f$ so f is an idempotent. Then $e+f = e+e+t = e+t = f$ and $f+e = e+t+e = e+t = f$ so $f \leq e$ so $f = e$ so $e \in K(\beta\mathbb{Z})$. □

2.6 Lemma. *Let $\phi : \mathbb{N}^* \longrightarrow \beta\mathbb{Z}$ be a one-to-one homomorphism. If $e+e = e \in \mathbb{N}^*\backslash K(\beta\mathbb{Z})$, then $\phi(e) \notin K(\beta\mathbb{Z})$.*

Proof. By Lemma 2.5 pick an idempotent $f \leq e$ with $f \neq e$. Since ϕ is a homomorphism, $\phi(f) \leq \phi(e)$ and, since ϕ is one-to-one, $\phi(f) \neq \phi(e)$. Thus by Lemma 2.5 $\phi(e) \notin K(\beta\mathbb{Z})$. □

We will utilize the factorial representation of integers. As is well known and easy to check, any $x \in \mathbb{N}$ can be uniquely represented in the form $\sum_{n\in F} t_n \cdot n!$ where each $t_n \in \{1, 2, \ldots, n\}$ and F is a finite nonempty subset of $\mathbb{N}$. Further, given $m \in \mathbb{N}$ and $x = \sum_{n\in F} t_n \cdot n!$ as above, $m!|x$ if and only if $\min F \geq m$.

2.7 Definition. (a) For $x \in \mathbb{N}$, define $F(x) \subseteq \mathbb{N}$ and define

$$\alpha(x) : F(x) \longrightarrow \prod_{n\in F(x)} \{1, 2, \ldots, n\} \quad \text{by} \quad x = \sum_{n\in F(x)} \alpha(x)(n) \cdot n!.$$

(b) Define $g : \mathbb{N} \longrightarrow \mathbb{N}$ by $g(x) = \alpha(x)(\max\ F(x))$. Denote also by g the continuous extension from $\beta\mathbb{N}$ to $\beta\mathbb{N}$.
(c) For each $k \in \mathbb{N}$, $C_k = \{x \in \mathbb{N} : g(x) = k\}$.

Thus $g(x)$ is the leftmost nonzero digit of x when x is written in its factorial representation like an ordinary decimal number, i.e. with the least significant digits to the right.

2.8 Lemma. *Let $p \in \beta\mathbb{N}$ and let $q \in T$. Then $g(p+q) = g(q)$.*

Proof. Let $x \in \mathbb{N}$. If $n! > x$, then $g(x+y) = g(y)$ for all $y \in \mathbb{N}n!$ so that also $g(x+q) = g(q)$. By continuity, $g(p+q) = g(q)$ for all $p \in \beta\mathbb{N}$. □

2.9 Lemma. *Let $A = \{x \in \mathbb{N} : F(x) \subseteq \mathbb{N}2\}$. Then $T\backslash\overline{A}$ is an ideal of T.*

Proof. Let $p, q \in T$ with $\{p,q\}\backslash\overline{A} \neq \emptyset$. Suppose $p+q \in \overline{A}$ and let $B = \{x \in \mathbb{N} : A - x \in q\}$. Pick $x \in B$ (with $x \notin A$ if $p \notin \overline{A}$). Pick $n \in \mathbb{N}$ with $n! > x$ and pick $y \in \mathbb{N}n! \cap (A - x)$ with $y \notin A$ if $q \notin \overline{A}$. Then $F(x+y) = F(x) \cup F(y)$ and either $x \notin A$ or $y \notin A$ so $x + y \notin A$, a contradiction. $\square$

2.10 Lemma. *Let $A = \{x \in \mathbb{N} : F(x) \subseteq \mathbb{N}2\}$ and let*

$$S = \left(T \cap \overline{A} \cap cl\left(\bigcup_{k=1}^{\infty} C_k{}^*\right)\right) \Big\backslash \bigcup_{k=1}^{\infty} \overline{C_k}\ .$$

Then S is a compact subsemigroup of $\mathbb{N}^*$ and $S \cap K(\beta\mathbb{N}) = \emptyset$.

Proof. By Lemma 2.9 $T\backslash\overline{A}$ is an ideal of T so $K(T) \cap \overline{A} = \emptyset$. By [9, Lemma 1.5], $K(T) = T \cap K(\beta\mathbb{N})$ so $T \cap \overline{A} \cap K(\beta\mathbb{N}) = \emptyset$ and hence $S \cap K(\beta\mathbb{N}) = \emptyset$. To see that $S \neq \emptyset$, for each $k \in \mathbb{N}$ pick $p_k \in cl\{k \cdot (2n)! : n \in \mathbb{N}$ and $n \geq k\} \cap \mathbb{N}^*$. Then $p_k \in T \cap \overline{A} \cap C_k{}^*$. Let p be a cluster point of the sequence $\langle p_k \rangle_{k=1}^{\infty}$. Then $p \in T \cap \overline{A} \cap cl(\bigcup_{k=1}^{\infty} C_k{}^*)$. Suppose that $p \in \overline{C_n}$ for some $n \in \mathbb{N}$. Then $\overline{C_n}$ is a neighborhood of p missing all but one term of the sequence $\langle p_k \rangle_{k=1}^{\infty}$, a contradiction.

Finally, we let $p, q \in S$ and show $p + q \in S$. To see that $A \in p + q$ we show that $A \subseteq \{x \in \mathbb{N} : A - x \in q\}$. Let $x \in A$ and pick $n \in \mathbb{N}$ such that $x < n!$. Then $A \cap \mathbb{N}n! \subseteq A - x$.

To see that $p+q \in cl(\bigcup_{k=1}^{\infty} C_k{}^*)$, let $B \in p+q$. Pick any $x \in \mathbb{N}$ such that $B - x \in q$. Then $cl(B-x) \cap (\bigcup_{k=1}^{\infty} C_k{}^*) \neq \emptyset$ so pick $k \in \mathbb{N}$ and pick $r \in cl(B-x) \cap C_k{}^*$. Then $r + x \in B^*$. Pick $n \in \mathbb{N}$ such that $x < n!$ and let $D = \{y \in C_k : y > (n+2)!\}$. Then $D \in r$ and for any $y \in D$, $g(y+x) = g(y) = k$ or $g(y+x) = g(y) + 1 = k+1$ and hence $r + x \in \overline{C_k} \cup \overline{C_{k+1}}$. Thus $\overline{B} \cap \bigcup_{l=1}^{\infty}(C_l{}^*) \neq \emptyset$.

Suppose $p + q \in \overline{C_k}$ for some $k \in \mathbb{N}$. Then by Lemma 2.8, $g(q) = g(p+q) = k$ so $q \in \overline{C_k}$, a contradiction. $\square$

2.11 Lemma. *Let ϕ be a continuous homomorphism from $\mathbb{N}^*$ to $\mathbb{N}^*$ or from $\mathbb{Z}^*$ to $\mathbb{Z}^*$, let $e + e = e \in \mathbb{N}^*$, and assume that for some $m, n \in \mathbb{Z}\backslash\{0\}$, one has $\phi(m+e) = n + \phi(e)$. Then for all $p \in \beta\mathbb{Z}$, $\phi(m \cdot p + e) = n \cdot p + \phi(e)$.*

Proof. By induction we have for all $t \in \mathbb{N}$, $\phi(m \cdot t + e) = n \cdot t + \phi(e)$ and we note that this is true for $t = 0$ as well. Now given $t \in \mathbb{N}$, $\phi(-m \cdot t + e) + \phi(m \cdot t + e) = \phi(e)$ so $\phi(-m \cdot t + e) + n \cdot t + \phi(e) = \phi(e)$ so that $\phi(-m \cdot t + e) + \phi(e) = -n \cdot t + \phi(e)$ and hence $\phi(-m \cdot t + e) = -n \cdot t + \phi(e)$. Denote by $\lambda_x^\bullet$ the function which multiplies on the left by x. We then have that the continuous functions $\phi \circ \rho_e \circ \lambda_m^\bullet$ and $\rho_{\phi(e)} \circ \lambda_n^\bullet$ agree on $\mathbb{Z}$ and consequently agree on $\beta\mathbb{Z}$. $\square$

3. Algebraic and topological copies of $\mathbb{N}^*$ and $\mathbb{Z}^*$

We show in this section that the only copies of $\mathbb{N}^*$ in $\mathbb{N}^*$ and of $\mathbb{Z}^*$ in $\mathbb{Z}^*$ are the trivial ones. The proof for $\mathbb{Z}^*$ is somewhat less complicated, so we present it first.

3.1 Theorem. *Assume that ϕ is a continuous one-to-one homomorphism from $\mathbb{Z}^*$ into $\mathbb{Z}^*$. There is some $k \in \mathbb{Z}\backslash\{0\}$ such that for all $p \in \mathbb{Z}^*$, $\phi(p) = k \cdot p$.*

Proof. Let S be as in Lemma 2.10 and pick an idempotent $e \in S$. Then $e \notin K(\beta\mathbb{Z})$ so by Lemma 2.6 $\phi(e) \notin K(\beta\mathbb{Z})$. Now $\phi(e) = \phi(e) + \phi(e) \in cl((\mathbb{Z}\backslash\{0\}) + \phi(e))$ and $\phi(e) \in \phi[cl(\bigcup_{k=1}^{\infty}({C_k}^*))] = cl(\bigcup_{k=1}^{\infty}\phi[{C_k}^*])$. Thus by Lemma 1.1, one of the following must hold:

(1) There exist some $k \in \mathbb{N}$, some $r \in {C_k}^*$ and some $p \in \beta\mathbb{Z}$ such that $\phi(r) = p + \phi(e)$.

(2) There is some $n \in \mathbb{Z}\backslash\{0\}$ such that $n + \phi(e) \in cl(\bigcup_{k=1}^{\infty}\phi[{C_k}^*])$.

Assume (1) holds. Then $\phi(r+e) = p + \phi(e) + \phi(e) = p + \phi(e) = \phi(r)$ so, since ϕ is one-to-one, $r = r + e$. But by Lemma 2.8, $g(r+e) = g(e)$ so we have $g(e) = g(r+e) = g(r) = k$ so $e \in \overline{C_k}$ contradicting the fact that $e \in S$.

Thus (2) holds. Then $n + \phi(e) \in cl(\bigcup_{k=1}^{\infty}\phi[\overline{C_k} \cap \mathbb{N}^*]) \subseteq \phi[\mathbb{N}^*]$. Pick $r \in \mathbb{N}^*$ such that $\phi(r) = n + \phi(e)$. Then $\phi(e+r+e) = \phi(e) + n + \phi(e) + \phi(e) = n + \phi(e) = \phi(r)$ so $e + r + e = r$. We claim now that r is in the center of $e + \beta\mathbb{N} + e$. Indeed, given $q \in e + \beta\mathbb{N} + e$ we have $\phi(r+q) = n + \phi(e) + \phi(q) + \phi(e) = \phi(e) + \phi(q) + n + \phi(e) = \phi(e+q) + \phi(r) = \phi(q+r)$ so $r + q = q + r$. Thus by Lemma 2.4 we have some $m \in Z$ such that $r = m + e$. That is, $n + \phi(e) = \phi(m+e)$. Note that since $n \neq 0$, $m \neq 0$.

Now by Lemma 2.11 we have for all $p \in \beta\mathbb{Z}$, $\phi(m \cdot p + e) = n \cdot p + \phi(e)$. Let $r = \phi(1+e)$. Observe that $r \in \phi(e) + \beta\mathbb{Z} + \phi(e)$. We claim that r commutes with any member of $\phi(e) + n \cdot \beta\mathbb{Z} + \phi(e)$. Indeed, let $p \in \beta\mathbb{Z}$ be given. Then $r + \phi(e) + n \cdot p + \phi(e) = \phi(1+e+e) + \phi(m \cdot p + e) = \phi(1 + e + m \cdot p + e) = \phi(e + m \cdot p + e + 1 + e) = \phi(e) + n \cdot p + \phi(e) + r$. Thus by Lemma 2.4 there is some $k \in \mathbb{Z}$ such that $\phi(1+e) = k + \phi(e)$. Note that $k \neq 0$ since ϕ is one-to-one. Thus by Lemma 2.11, for all $p \in \beta\mathbb{Z}$, $\phi(p+e) = k \cdot p + \phi(e)$.

By Lemma 2.3, pick $q \in \beta\mathbb{N}$ such that $k \cdot q + \phi(e)$ is right cancellable in $\beta\mathbb{Z}$. Then given any $p \in \beta\mathbb{Z}$ we have $\phi(p+q+e) = k \cdot p + k \cdot q + \phi(e)$ and $\phi(p+q+e) = \phi(p) + k \cdot q + \phi(e)$ so by right cancellation we have $k \cdot p = \phi(p)$ as required. □

We need an additional preliminary result before obtaining the corresponding statement about $\mathbb{N}^*$.

3.2 Lemma. *Let* ϕ *be a continuous one-to-one homomorphism from* $\mathbb{N}^*$ *to* $\mathbb{N}^*$ *and let* $e + e = e \in \mathbb{N}^*\backslash K(\beta\mathbb{N})$. *There do not exist* $m, n \in \mathbb{N}$ *such that* $\phi(m+e) = -n + \phi(e)$ *or* $\phi(-m+e) = n + \phi(e)$.

Proof. Observe that by Lemma 2.6 $\phi(e) \notin K(\beta\mathbb{N})$. Observe also that the second conclusion implies the first. Indeed if $\phi(-m+e) = n + \phi(e)$, then $\phi(-m+e) + \phi(m+e) = \phi(e)$ so $n + \phi(e) + \phi(m+e) = \phi(e)$ so $\phi(m+e) = -n + \phi(e)$.

So assume we have $\phi(m+e) = -n + \phi(e)$. Then by Lemma 2.11 $\phi(m \cdot p + e) = -n \cdot p + \phi(e)$ for all $p \in \beta\mathbb{Z}$. Pick by Lemma 2.3 some $q \in \beta\mathbb{N}$ such that $-n \cdot q + \phi(e)$ is right cancellable in $\beta\mathbb{Z}$. Pick any $p \in \mathbb{N}^*$. Then $\phi(m \cdot p) + -n \cdot q + \phi(e) = \phi(m \cdot p + m \cdot q + e) = -n \cdot p + -n \cdot q + \phi(e)$ so by right cancellation $\phi(m \cdot p) = -n \cdot p$, a contradiction since $\phi(m \cdot p) \in \mathbb{N}^*$. □

3.3 Theorem. *Assume that* ϕ *is a continuous one-to-one homomorphism from* $\mathbb{N}^*$ *into* $\mathbb{N}^*$. *There is some* $k \in \mathbb{N}$ *such that for all* $p \in \mathbb{N}^*$, $\phi(p) = k \cdot p$.

Proof. Let S be as in Lemma 2.10 and pick an idempotent $e \in S$. Then $e \notin K(\beta\mathbb{Z})$ so by Lemma 2.6, $\phi(e) \notin K(\beta\mathbb{Z})$. Now $\phi(e) = \phi(e) + \phi(e) \in cl(\mathbb{N} + \phi(e))$ and $\phi(e) \in \phi[cl(\bigcup_{k=1}^{\infty}(C_k{}^*))] = cl(\bigcup_{k=1}^{\infty} \phi[C_k{}^*])$. Thus by Lemma 1.1, one of the following must hold:

(1) There exist some $k \in \mathbb{N}$, some $r \in C_k{}^*$ and some $p \in \beta\mathbb{N}$ such that $\phi(r) = p + \phi(e)$.
(2) There is some $n \in \mathbb{N}$ such that $n + \phi(e) \in cl(\bigcup_{k=1}^{\infty} \phi[C_k{}^*])$.

Possibility (1) cannot hold exactly as in the proof of Theorem 3.1. Thus we have some $n \in \mathbb{N}$ such that $n + \phi(e) \in \phi[\mathbb{N}^*]$. Pick $r \in \mathbb{N}^*$ such that $n + \phi(e) = \phi(r)$. Then $\phi(e + r + e) = \phi(r)$ so $r = e + r + e$. We conclude as before that r is in the center of $e + \beta\mathbb{N} + e$ and hence, by Lemma 2.4, $r \in \mathbb{Z} + e$. Thus we have some $m \in \mathbb{Z}$ such that $\phi(m+e) = n+\phi(e)$. Since $n \neq 0$, $m \neq 0$. By Lemma 3.2 we can't have $m < 0$ so $m \in \mathbb{N}$. By Lemma 2.11, $\phi(m \cdot p + e) = n \cdot p + \phi(e)$ for all $p \in \beta\mathbb{N}$.

Let $r = \phi(1 + e)$. Then as in the proof of Theorem 3.1, $r \in \phi(e) + \beta\mathbb{N} + \phi(e)$ and r commutes with every member of $\phi(e) + n \cdot \beta\mathbb{N} + \phi(e)$. Thus by Lemma 2.4, one has some $k \in \mathbb{Z}$ such that $r = k + \phi(e)$. Then $1 + e \neq e$ so $k \neq 0$. Since $\phi(1 + e) = k + \phi(e)$ we have by Lemma 3.2 that $k > 0$.

Picking by Lemma 2.3 some $q \in \beta\mathbb{N}$ such that $k \cdot q + \phi(e)$ is right cancellative in $\beta\mathbb{Z}$ we have for any $p \in \beta\mathbb{N}$ that $\phi(p)+k \cdot q+\phi(e) = \phi(p+q+e) = k \cdot p+k \cdot q+\phi(e)$ so $\phi(p) = k \cdot p$. □

We now present a simple corollary of Theorem 3.1.

3.4 Corollary. *There does not exist a continuous one-to-one homomorphism taking $\mathbb{Z}^*$ to $\mathbb{N}^*$.*

Proof. Any such homomorphism would have to be given by $\phi(p) = k \cdot p$ and any such a function takes on values in $-\mathbb{N}^*$. □

Note that algebraic copies of $\mathbb{Z}$ are plentiful in $\mathbb{N}^*$, namely as $\mathbb{Z} + e$ for any idempotent e. None of these are topological copies however as $\mathbb{Z} + e$ cannot be discrete. (Given a net $\langle x_i \rangle_{i \in I}$ in $\mathbb{N}$ converging to e one has $\lim_{i\in I}(x_i+e) = e+e = e$, so e is a limit point of $\mathbb{Z} + e$.) It is trivial to get discrete copies of $\mathbb{N}$. We see now that in fact one can get discrete copies of $\omega = \mathbb{N} \cup \{0\}$.

3.5 Theorem. *There is a discrete copy of ω in $\mathbb{N}^*$.*

Proof. For $x \in \mathbb{N}$ let $G(x)$ be the binary support of x. That is, $x = \sum_{t \in G(x)} 2^t$. Let $A = \{x \in \mathbb{N} : G(x) \subseteq \mathbb{N}2\}$. As in the introduction, let $H = \bigcap_{n=1}^{\infty} cl\mathbb{N}2^n$. Then $\overline{A} \cap H$ is a compact semigroup so pick an idempotent $e \in \overline{A} \cap H$. Pick $p \in \mathbb{N}^*$ such that $\{2^{2n+1} : n \in \mathbb{N}\} \in p$. Let $q = e + p + e$. Define $\phi : \omega \longrightarrow \mathbb{N}^*$ inductively by $\phi(0) = e$ and $\phi(n+1) = \phi(n)+q$. Observe that for $n > 0$, $\phi(n)$ is the sum of q with itself n times. Since $q = e+q = q+e$ one has immediately that ϕ is a homomorphism. To see that $\phi[\omega]$ is discrete, for each $n \in \omega$, let $A_n = \{x \in \mathbb{N} : |G(x)\backslash\mathbb{N}2| = n\}$. To complete the proof we show by induction that for each $n \in \mathbb{N}$, $A_n \in \phi(n)$. The case $n = 0$ is immediate since $A = A_0$. To see that $A_1 \in q$, we show that $A \subseteq \{x \in \mathbb{N} : A_1 - x \in p + e\}$. So let $x \in A$ be given and let $k = \max G(x)$. Let $B = \{2^{2n+1} : n \in \mathbb{N}$ and $n > k\}$. To see that $B \subseteq \{y \in \mathbb{N} : (A_1 - x) - y \in e\}$, let $n \in N$ with $n > k$. Then $\mathbb{N}2^{2n+2} \subseteq (A_1 - x) - 2^{2n+1}$. Now given $n \in \mathbb{N}$, we show that $A_n \subseteq \{x \in \mathbb{N} : A_{n+1} - x \in q\}$, so let $x \in A_n$ and let $k = \max G(x)$. Then $\{y \in A_1 : \min G(y) > k\} \subseteq A_{n+1} - x$. $\square$

3.6 QUESTION. Is there a discrete copy of $\mathbb{Z}$ in $\mathbb{N}^*$?

REFERENCES

1. J. Berglund and K. Hofmann, *Compact Semitopological Semigroups and Weakly Almost Periodic Functions*, Lecture Notes in Math., no. 42, Springer-Verlag, Berlin, 1967.
2. J. Berglund, H. Junghenn, and P. Milnes, *Analysis on Semigroups*, Wiley, New York, 1989.
3. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), 417–455.
4. E. van Douwen, *The Čech-Stone compactification of a discrete groupoid*, Topology and its Applications **39** (1991), 43–60.
5. R. Ellis, *Lectures in Topological Dynamics*, Benjamin, New York, 1969.
6. L. Gillman and M. Jerison, *Rings of Continuous Functions*, van Nostrand, Princeton, 1960.
7. N. Hindman, *The semigroup* $\beta\mathbb{N}$ *and its applications to number theory*, The Analytical and Topological Theory of Semigroups (K. Hofmann, J. Lawson, and J. Pym, eds.), Walter de-Gruyter & Co., Berlin, 1990, pp. 347–360.
8. ______, *Ultrafilters and combinatorial number theory*, Number Theory Carbondale 1979 (M. Nathanson, ed.), Lecture Notes in Math., no. 751, Springer-Verlag, Berlin, 1979, pp. 119–184.
9. N. Hindman and J. Pym, *Free groups and semigroups in* $\beta\mathbb{N}$, Semigroup Forum **30** (1984), 177–193.
10. N. Hindman and D. Strauss, *Cancellation in the Stone-Čech compactification of a discrete semigroup*, Proc. Edinburgh Math. Soc. **37** (1994), 379–397.
11. A. Lisan, *Free groups in* $\beta\mathbb{N}$ *which miss the minimal ideal*, Semigroup Forum **37** (1988), 233–239.
12. W. Ruppert, *Compact Semitopological Semigroups: an Intrinsic Theory*, Lecture Notes in Math., no. 1079, Springer-Verlag, Berlin, 1984.
13. D. Strauss, *Ideals and commutativity in* $\beta\mathbb{N}$, Topology and its Applications **60** (1994), 281–293.
14. ______, $\mathbb{N}^*$ *does not contain an algebraic and topological copy of* $\beta\mathbb{N}$, J. London Math. Soc. **46** (1992), 463-470.

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