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Cohomology of Modules in the Principal Block of a Finite Group

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Abstract. In this paper, we prove the conjectures made in a joint paper of the author with Carlson and Robinson, on the vanishing of cohomology of a finite group G. In particular, we prove that if k is a field of characteristic p, then every non-projective kG-module M in the principal block has nontrivial cohomology in the sense that $H^*(G, M) \neq 0$, if and only if the centralizer in G of every element of order p is p-nilpotent (this was proved for p odd in the above mentioned paper, but the proof here is independent of p). We prove the stronger statement that whether or not these conditions hold, the union of the varieties of the modules in the principal block having no cohomology coincides with the union of the varieties of the elementary abelian p-subgroups whose centralizers are not p-nilpotent (i.e., the nucleus). The proofs involve the new idempotent functor machinery of Rickard.

Contents

1. Introduction

Recent developments in modular representation theory of finite groups have involved a re-evaluation of the role of infinitely generated modules. In particular, Rickard [5] has introduced some infinitely generated modules which are idempotent in the stable category, in the sense that the tensor square is isomorphic to the original module plus a projective. This work, together with a version of Dade's lemma for infinitely generated modules, has allowed Benson, Carlson and Rickard [1, 2] to

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formulate and prove a generalization of the usual theory of complexity and varieties for modules to the infinitely generated situation.

In this paper, we shall demonstrate how the recent work described above can be used to address some older questions about finitely generated modules. In particular, we shall prove the conjectures formulated in the paper of Benson, Carlson and Robinson [3]. Before stating our main theorem, we state a more easily understood consequence, which provides an affirmative solution to Conjecture 1.2 of that paper. The proof may be found in Section 5.

Theorem 1.1. *Let k be a field of characteristic p, and let G be a finite group. Then the centralizer of every element of order p in G is p-nilpotent, if and only if for every non-projective module M in the principal block $B_0(kG)$, $H^n(G,M) \neq 0$ for some $n \neq 0$.*

We remark that in general, for a kG-module M, $H^n(G,M) \neq 0$ for some $n \neq 0$ if and only if $\hat{H}^n(G,M) \neq 0$ for infinitely many values of n both positive and negative, cf. Theorem 1.1 of [3]. We also remark that in the statement of the above theorem, it does not matter whether we restrict our attention to finitely generated kG-modules.

In the language of [3], our main theorem is the following. This almost provides an affirmative answer to Conjecture 10.10 of that paper, which does not mention passing down to summands. Terminology used in this introduction is explained in Section 2, and the proof may be found in Section 5.

Theorem 1.2. *Let k be an algebraically closed field of characteristic p, and let G be a finite group. Then every finitely generated kG-module in the principal block is a direct summand of a nuclear homology module.*

It follows from this theorem that every kG-module in the principal block is a filtered colimit of nuclear homology modules. In order to prove that every finitely generated module in the principal block actually is a nuclear homology module, it would suffice to show that the characters of nuclear homology modules span the principal block, and then the argument would run as in the proof of Proposition 4.4 of [3]. It does not seem immediately clear that this character theoretic statement is true.

As pointed out in Corollary 10.12 of [3] (passage to direct summands does not affect this), it follows from this theorem that the nucleus Y_G (which is the union of the images in the cohomology variety V_G, of the elementary abelian p-subgroups whose centralizers are not p-nilpotent) coincides with the representation theoretic nucleus Θ_G (which is the union of the varieties of the finitely generated modules in the principal block having no cohomology).

Corollary 1.3. *For any finite group G, we have $Y_G = \Theta_G$.*

In the case where $Y_G = \{0\}$, this implies Theorem 1.1. More precisely, we prove the following strengthened form of Theorem 1.4 of [3]:

Theorem 1.4. *Suppose that G is a finite group and k is a field of characteristic p. Then the following are equivalent:*

(A) *Every finitely generated module in the principal block $B_0(kG)$ is a trivial homology module.*

(A′) *Every simple module in $B_0(kG)$ is a direct summand of a trivial homology module.*
(A″) *Every (finitely generated) trivial source module in $B_0(kG)$ is a direct summand of a trivial homology module.*
(A‴) *Every module (finitely or infinitely generated) in $B_0(kG)$ is a filtered colimit of trivial homology modules.*
(B) *For every finitely generated non-projective module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some $n > 0$.*
(B′) *For every finitely generated non-projective periodic module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some $n > 0$.*
(B″) *For every (finitely or infinitely generated) non-projective module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some $n > 0$.*
(B‴) *For every (finitely or infinitely generated) module M of complexity one in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some $n > 0$.*
(C) *For every non-projective (finitely generated) trivial source module M in $B_0(kG)$, we have $H^n(G, M) \neq 0$ for some $n > 0$.*
(D) *The centralizer of every element of order p in G is p-nilpotent.*
(D′) *The centralizer of every nontrivial p-subgroup of G is p-nilpotent.*
(E) *For every non-projective finitely generated indecomposable module M in $B_0(kG)$, with vertex R and Green correspondent $f(M)$, we have $H^n(N_G(R), f(M)) \neq 0$ for some $n > 0$.*

On the way to proving these theorems, we prove a remarkable property of modules of complexity one. In general, such a module decomposes as a direct sum of modules whose variety consists of a single line through the origin in the cohomology variety $V_G(k)$. The following theorem is proved at the end of Section 3.

Theorem 1.5. *Suppose that M is a kG-module whose variety $\mathcal{V}_G(M)$ consists of a single line L through the origin in $V_G(k)$. Let E be an elementary abelian p-subgroup of G, minimal with respect to the property that L is contained in the image of the map $\mathrm{res}^*_{G,E} : V_E(k) \to V_G(k)$ induced by restriction from G to E in cohomology. Let $L = \mathrm{res}^*_{G,E}(\ell_0)$ with ℓ_0 a line through the origin in $V_E(k)$, and let D be the subgroup of $N_G(E)$ consisting of the elements which stabilize ℓ_0 setwise. Then the direct sum of M with some projective kG-module is induced from D.*

We remark that this paper makes Sections 8 and 9 of [3] obsolete, and there is no longer anything special about odd primes in our proofs. We also remark that we have not been able to tackle the original question which motivated [3], namely whether every simple module in the principal block necessarily has nonvanishing cohomology. One possible approach to this might be to try to prove that the variety of a simple module in the principal block cannot be contained in the nucleus. Since the property of being simple is not easy to work with, it may be better to consider modules whose endomorphism rings, modulo traces from suitable subgroups, are isomorphic to the field.

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2. Terminology and Background Material

Let G be a finite group and k an algebraically closed field of characteristic p. Let $\mathsf{stmod}(kG)$ be the stable category of finitely generated kG-modules, considered as a triangulated category. The homomorphisms $\underline{\mathrm{Hom}}_{kG}(M,N)$ in this category are homomorphisms in the usual module category, modulo those that factor through some projective module. The triangles in $\mathsf{stmod}(kG)$ come from the short exact sequences in $\mathsf{mod}(kG)$ in the normal way. Similarly, $\mathsf{StMod}(kG)$ is the stable category of all (not necessarily finitely generated) kG-modules, which is again a triangulated category.

We write $V_G(k)$ for the maximal ideal spectrum of $H^*(G,k)$. Note that for p odd, elements of odd degree square to zero, so that $H^*(G,k)$ modulo its nil radical is commutative. Thus $V_G(k)$ is a homogeneous affine variety. Associated to any (not necessarily finitely generated) kG-module M, there is a collection $\mathcal{V}_G(M)$ of closed homogeneous irreducible subvarieties of $V_G(k)$ (see [2] for details). These varieties have good properties with respect to tensor products, and M is projective if and only if $\mathcal{V}_G(M) = \emptyset$.

We next remark that there is a mistake in the definition of Y_G given in Section 10 of [3]. The **nucleus** Y_G should be defined as the subvariety of $V_G(k)$ given as the union of the images of the maps $\mathrm{res}^*_{G,H} : V_H(k) \to V_G(k)$ induced by $\mathrm{res}_{G,H} : H^*(G,k) \to H^*(H,k)$, as H runs over the set of subgroups of G for which $C_G(H)$ is not p-nilpotent (and not the union of the images of $\mathrm{res}^*_{G,C_G(H)} : V_{C_G(H)}(k) \to V_G(k)$ as stated there; also in the proof of Theorem 10.2 of that paper, $V_{C_G(H)}$ should be replaced by V_H, and no other changes are necessary).

The **representation theoretic nucleus** Θ_G is the subset of $V_G(k)$ given as the union of the varieties $V_G(M)$ as M runs over the finitely generated modules in the principal block $B_0(kG)$ with $H^n(G,M) = 0$ for all n. By Theorem 6.4 of [3], it suffices to consider periodic modules in this definition.

We say that a finitely generated kG-module M is a **trivial homology module** or a **TH module** if there exists a finite complex $(C_i, \delta_i : C_i \to C_{i-1})$ of finitely generated kG-modules and homomorphisms such that the following conditions hold:

(i) Each C_i is a projective kG-module, and $C_i = 0$ for $i < 0$ and for i sufficiently large.

(ii) For $i > 0$, $H_i(C_*)$ is a direct sum of copies of the trivial kG-module k.

(iii) $H_0(C_*) \cong M$.

We say that a finitely generated kG-module M is a **nuclear homology module** or an **NH module** if it satisfies the same conditions, but with (ii) replaced by:

(ii′) For $i > 0$, $H_i(C_*)$ is a direct sum of copies of the trivial kG-module k and finitely generated modules M' in $B_0(kG)$ with $V_G(M') \subseteq Y_G$.

We write $\mathcal{TH}$ and $\mathcal{NH}$ for the thick subcategories of $\mathsf{stmod}(kG)$ consisting of the direct summands of trivial homology modules and of nuclear homology modules respectively.

Next, we recall from Section 5 of Rickard [5] that given any thick subcategory $\mathcal{C}$ of $\mathsf{stmod}(kG)$, there are functors $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}$ on $\mathsf{StMod}(kG)$ satisfying the following properties:

(a) For any X in $\mathsf{StMod}(kG)$, $\mathcal{E}_{\mathcal{C}}(X)$ is a filtered colimit of objects in $\mathcal{C}$.

(b) For any X in $\mathsf{StMod}(kG)$, $\mathcal{F}_{\mathcal{C}}(X)$ is $\mathcal{C}$-local, in the sense that for any object M in $\mathcal{C}$, $\underline{\mathrm{Hom}}_{kG}(M, \mathcal{F}_{\mathcal{C}}(X)) = 0$.

(c) There is a triangle in $\mathsf{StMod}(kG)$

$$\mathcal{E}_{\mathcal{C}}(X) \to X \to \mathcal{F}_{\mathcal{C}}(X) \to \Omega^{-1}\mathcal{E}_{\mathcal{C}}(X).$$

In fact (see the remark after Proposition 5.7 of [5]) the functors $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}$ are characterized by these properties.

Our goal will be to show that the functor $\mathcal{F}_{\mathcal{NH}}$ is the zero functor on the principal block, which will enable us to prove that every finitely generated module in the principal block is a direct summand of a nuclear homology module.

If V is a closed homogeneous subvariety of $V_G(k)$, we write $\mathcal{C}_V$ for the subcategory of $\mathsf{stmod}(kG)$ consisting of the finitely generated modules M with $V_G(M) \subseteq V$. Then the corresponding functors $\mathcal{E}_V = \mathcal{E}_{\mathcal{C}_V}$ and $\mathcal{F}_V = \mathcal{F}_{\mathcal{C}_V}$ are given by tensoring with certain (usually infinitely generated) modules e_V and f_V. These are orthogonal idempotents in $\mathsf{StMod}(kG)$, in the sense that $e_V \otimes e_V \cong e_V \oplus (\text{projective})$, $f_V \otimes f_V \cong f_V \oplus (\text{projective})$, and $e_V \otimes f_V$ is projective. The triangle for a module X in this situation is given by tensoring the triangle

$$e_V \xrightarrow{\lambda_V} k \xrightarrow{\mu_V} f_V \to \Omega^{-1} e_V$$

with X.

3. Inducing Idempotents

Let L be a line through the origin in $V_G(k)$. Then by the Quillen stratification theorem, there is an elementary abelian p-subgroup E, uniquely determined up to conjugacy, with the property that L is in the image of $\mathrm{res}^*_{G,E} : V_E(k) \to V_G(k)$, but L is not in the image of $\mathrm{res}^*_{G,E'} : V_{E'}(k) \to V_G(k)$ for any proper subgroup E' of E. In this situation, we say that L **originates** in E. We write $C = C_G(E)$ for the centralizer, and $N = N_G(E)$ for the normalizer in G of E. Let ℓ_0 be a line through the origin in $V_E(k)$ with $L = \mathrm{res}^*_{G,E}(\ell_0)$, and let D be the subgroup of $N_G(E)$ consisting of the elements which stabilize ℓ_0 setwise. Then ℓ_0 and D are uniquely determined up to conjugacy in N. Since ℓ_0 originates in E, the centralizer C is equal to the pointwise stabilizer of ℓ_0. Any finite group of automorphisms of the line ℓ_0 is cyclic of order prime to p, so we have $C \trianglelefteq D \leq N$ with D/C a cyclic group of order prime to p. Finally, we set $\ell = \mathrm{res}^*_{D,E}(\ell_0) \subseteq V_D(k)$, so that $L = \mathrm{res}^*_{G,D}(\ell)$.

Theorem 3.1. *With the above notation, let e_ℓ be the idempotent kD-module corresponding to ℓ and e_L be the idempotent kG-module corresponding to L. Then*

$$e_\ell{\uparrow^G} \cong e_L \oplus (\text{projective}).$$

Proof. Consider the composite map

$$e_\ell{\uparrow^G} \xrightarrow{\lambda_\ell \uparrow^G} k_D{\uparrow^G} \xrightarrow{\nu} k,$$

where $\nu : k_D{\uparrow^G} \to k$ is the augmentation map. On restriction to E, this becomes (modulo projectives) the composite map

$$\bigoplus_{g \in N/D} g \otimes e_\ell{\downarrow_E} \to \left(\bigoplus_{g \in N/D} g \otimes k \right) \oplus (\text{induced modules}) \xrightarrow{\nu\downarrow_E} k.$$

Here, $g \in N/D$ means that g runs over a set of left coset representatives of D in N. The first map is the sum of all the maps

$$\begin{array}{ccc} g \otimes e_\ell{\downarrow_E} & \cong & e_{g(\ell)}{\downarrow_E} \\ \downarrow & & \downarrow \\ g \otimes k & \cong & k \end{array}$$

and the second map $\nu\downarrow_E$ sends $\sum_i g_i \otimes \lambda_i$ to $\sum_i \lambda_i$. On restriction to a cyclic shifted subgroup corresponding to a point in ℓ_0, the summands $g \otimes e_\ell{\downarrow_E}$ for $g \notin D$ give projective modules, while $1 \otimes e_\ell$ restricts to give $k \oplus$ (projective), because ℓ_0 isn't fixed by any $g \in N \setminus D$. Moreover, this copy of k maps isomorphically to $1 \otimes k$ in the second module, and then isomorphically to k in the third module. So if we complete to a triangle

$$e_\ell{\uparrow^G} \to k \to f \to \Omega^{-1} e_\ell{\uparrow^G},$$

then f restricted to this cyclic shifted subgroup is projective.

The module $e_\ell{\uparrow^G}$ is a filtered colimit of modules in $\mathcal{C}_L$, since e_ℓ is a filtered colimit of modules in $\mathcal{C}_\ell$. For M in $\mathcal{C}_L$, $\mathrm{Hom}_k(M, f)$ is projective, by a combination of Dade's lemma (the infinite dimensional version given in Section 3 of [2]) and Chouinard's theorem [4], so f is $\mathcal{C}_L$-local.

By Rickard's characterization (see the remark after Proposition 5.7 of [5]), the triangle

$$e_\ell{\uparrow^G} \to k \to f \to \Omega^{-1} e_\ell{\uparrow^G}$$

is isomorphic to

$$e_L \to k \to f_L \to \Omega^{-1} e_L. \qquad \square$$

Corollary 3.2. *If M is a module whose variety $\mathcal{V}_G(M) = \{L\}$ with L as above, then $M \oplus$ (projective) is induced from D.*

Proof. If $\mathcal{V}_G(M) \subseteq \{L\}$ then using the theorem, we have

$$M \oplus (\text{projective}) \cong M \otimes e_L \cong M \otimes e_\ell{\uparrow^G} \cong (M{\downarrow_D} \otimes e_\ell){\uparrow^G},$$

and so $M \oplus$ (projective) is induced from D. $\square$

This completes the proof of Theorem 1.5.

4. An Equivalence of Categories

We can combine the results of the last section with the Mackey decomposition theorem to obtain an equivalence of categories as follows. Let $\mathcal{C}$ be the full subcategory of $\mathsf{StMod}(kG)$ consisting of kG-modules M with $\mathcal{V}_G(M) \subseteq \{L\}$ (or equivalently $M \cong e_L \otimes M$), and let $\mathcal{C}'$ be the full subcategory of $\mathsf{StMod}(kD)$ consisting of modules M' with $\mathcal{V}_D(M') \subseteq \{\ell\}$ (or equivalently $M' \cong e_\ell \otimes M'$). Using the Mackey decomposition theorem, we see that if M' is in $\mathcal{C}'$ then $M'{\uparrow^G}{\downarrow_D}$ is isomorphic to a direct sum of M' with a module M'' satisfying $\mathcal{V}_D(M'') \cap \{\ell\} = \emptyset$. So we have

$$e_\ell \otimes (M'{\uparrow^G}{\downarrow_D}) \cong M'.$$

Since every object in $\mathcal{C}$ is induced from an object in $\mathcal{C}'$ by Corollary 3.2, it follows that the functors $(e_\ell \otimes -) \circ \operatorname{res}_{G,D} : \mathcal{C} \to \mathcal{C}'$ and $\operatorname{ind}_{D,G} : \mathcal{C}' \to \mathcal{C}$ are mutually inverse equivalences of categories.

Lemma 4.1. *If M is a kG-module in $\mathcal{C}$, which lies in the principal block $B_0(kG)$, then $e_\ell \otimes M{\downarrow_D}$ is a direct sum of a projective module and a module in the principal block $B_0(kD)$.*

Conversely, if M is a kG-module in $\mathcal{C}$ with no summand in the principal block $B_0(kG)$, then $e_\ell \otimes M{\downarrow_D}$ is a direct sum of a projective module and a module with no summand in the principal block $B_0(kD)$.

Proof. Let e be a block idempotent of kG, and let $\operatorname{Br}_E : Z(kG) \to Z(kD)$ be the Brauer map with respect to E. If b is any block of kD, say with defect group R, then $E \le R \le C$, and so $RC_G(R) \le C$. So the Brauer correspondent b^G is defined, and by Brauer's third main theorem, b^G is equal to $B_0(kG)$ if and only if $b = B_0(kD)$. It follows that if e_0 is the principal block idempotent of kG and e_1 is the principal block idempotent of kD, then $\operatorname{Br}_E(e_0) = e_1$, and $\operatorname{Br}_E(e)e_1 \neq 0$ if and only if $e = e_0$.

If M is a finitely generated kG-module with $e.M = M$, then Nagao's lemma says that

$$M{\downarrow_D} \cong \operatorname{Br}_E(e).M{\downarrow_D} \oplus M_1$$

where M_1 is a direct sum of modules which are projective relative to subgroups $Q \le C$ with $E \not\le Q$. Since the variety of $e_\ell \otimes M{\downarrow_D}$ has trivial intersection with the image of $V_{E'} \to V_D$ for any proper subgroup E' of E, it follows that

$$e_\ell \otimes M{\downarrow_D} \cong \operatorname{Br}_E(e).(e_\ell \otimes M{\downarrow_D}) \oplus M_2$$

where M_2 is projective.

If $M = e.M$ is not finitely generated, express it as a filtered colimit of finitely generated modules M_α in $\mathcal{C}$. Each $e_\ell \otimes M_\alpha{\downarrow_D}$ may be written as a direct sum of $\operatorname{Br}_E(e).(e_\ell \otimes M_\alpha{\downarrow_D})$ and a projective module killed by $\operatorname{Br}_E(e)$. There are no maps between these two types of summands, so when we pass to the colimit, we obtain a decomposition of $e_\ell \otimes M{\downarrow_D}$ of the desired form. $\square$

Theorem 4.2. *The functors $(e_\ell \otimes -) \circ \operatorname{res}_{G,D} : \mathcal{C} \to \mathcal{C}'$ and $\operatorname{ind}_{D,G} : \mathcal{C}' \to \mathcal{C}$ are mutually inverse equivalences of categories, and induce mutually inverse equivalences between the full subcategories $B_0(kG) \cap \mathcal{C}$ and $B_0(kD) \cap \mathcal{C}'$.*

Proof. This follows immediately from the lemma and the discussion preceding it. $\square$

5. The Main Theorems

We continue with the same notation. Namely, L is a line through the origin in $V_G(k)$ originating in an elementary abelian p-subgroup E of G. We set $C = C_G(E)$, $N = N_G(E)$ and $L = \operatorname{res}^*_{G,E}(\ell_0)$, with ℓ_0 a line through the origin in $V_E(k)$. We set D equal to the stabilizer in N of the line ℓ_0. We set $\ell_1 = \operatorname{res}^*_{C,E}(\ell_0) \subseteq V_C(k)$ and $\ell = \operatorname{res}^*_{D,E}(\ell_0) \subseteq V_D(k)$.

Lemma 5.1. *Suppose that C is p-nilpotent. Then for any module M' in $B_0(kD)$ satisfying $\mathcal{V}_D(M') = \{\ell\}$, we have $\hat{H}^n(D, M') \neq 0$ for some n.*

Proof. The argument for this is given in the proof of Proposition 6.8 of [3]; we repeat it here for convenience. Let $\bar{C} = C/O_{p'}(C)$ and $\bar{D} = D/O_{p'}(C)$. Then $\bar{C}$ is a p-group, and $\bar{D}/\bar{C}$ is a cyclic p'-group. By Lemma 6.7 of [3], we may choose a homogeneous element $\zeta \in H^m(\bar{C},k) = H^m(C,k)$ for some m, so that $\ell_1 \cap V_C(\langle\zeta\rangle) = \{0\}$, and so that the one dimensional subspace $\langle\zeta\rangle \subseteq H^*(C,k)$ is $\bar{D}/\bar{C}$-invariant and affords a faithful one dimensional representation of $\bar{D}/\bar{C}$. For a suitable one dimensional representation ε of D with kernel C, ζ may be regarded as an element of $\mathrm{Ext}^m_{kD}(k,\varepsilon)$. Thus ζ is represented by a homomorphism $\hat{\zeta} : \Omega^m(k) \to \varepsilon$, and we write L_ζ for the kernel of $\hat{\zeta}$. So there is a short exact sequence of kD-modules

$$0 \to L_\zeta \to \Omega^m(k) \to \varepsilon \to 0.$$

Tensoring with M', we obtain a short exact sequence

$$0 \to L_\zeta \otimes M' \to \Omega^m(k) \otimes M' \to \varepsilon \otimes M' \to 0.$$

The tensor product theorem for varieties (Theorem 10.8 of [2]) implies that $L_\zeta \otimes M'$ is projective, and so we obtain a stable isomorphism $\Omega^m(M') \cong \varepsilon \otimes M'$.

Since M' is non-projective, for some value of r we have

$$\widehat{\mathrm{Ext}}^0_{kD}(\varepsilon^r, M') = \underline{\mathrm{Hom}}_{kG}(\varepsilon^r, M') \neq 0,$$

where ε^r denotes the rth tensor power of ε. This is because every simple module in $B_0(kD)$ is isomorphic to some such ε^r. Thus

$$\hat{H}^{mr}(D,M') \cong \hat{H}^0(D,\Omega^{-mr}(M')) \cong \hat{H}^0(D,\varepsilon^{-r} \otimes M') \cong \widehat{\mathrm{Ext}}^0_{kD}(\varepsilon^r, M') \neq 0.$$

Here, ε^{-r} denotes the rth tensor power of the dual module ε^*. □

Theorem 5.2. *Suppose that M is a module in $B_0(kG)$ with $\mathcal{V}_G(M) = \{L\}$, and that C is p-nilpotent. Then $\hat{H}^n(G,M) \neq 0$ for some n.*

Proof. By Theorem 4.2, there is a module M' in $B_0(kD)$ with $M'\uparrow^G \cong M \oplus$ (projective). By Shapiro's lemma we have $\hat{H}^n(G,M) \cong \hat{H}^n(D,M')$. By Lemma 5.1, this is nonzero for some n. □

Corollary 5.3. *Suppose that M is a non-projective kG-module in $B_0(kG)$ with the property that $\mathcal{V}_G(M)$ contains no closed homogeneous subset of the nucleus Y_G. Then $\hat{H}^n(G,M) \neq 0$ for some n.*

Proof. We use the argument given in Theorem 6.4 of [3] to reduce to the complexity one case. Let K be an algebraically closed extension of k of large transcendence degree. Since M is non-projective, $\mathcal{V}_G(M)$ contains a closed homogeneous irreducible subset V which is not contained in Y_G. So $\mathcal{V}_G(K \otimes_k M)$ contains a generic line L for V. Choose elements $\zeta_1, \dots, \zeta_s \in H^*(G,K)$ so that

$$\mathcal{V}_G(K \otimes_k M) \cap \mathcal{V}_G\langle\zeta_1\rangle \cap \dots \cap \mathcal{V}_G\langle\zeta_s\rangle = \{L\}.$$

Here, $\mathcal{V}_G\langle\zeta_i\rangle$ is the collection of closed homogeneous subsets of the hypersurface $V_G\langle\zeta_i\rangle$ defined by ζ_i. Then we have

$$\mathcal{V}_G((K \otimes_k M) \otimes_K L_{\zeta_1} \otimes_K \dots \otimes_K L_{\zeta_s}) = \{L\}.$$

Next, we note that in Lemma 6.3 of [3], although M_2 needs to be finitely generated, M_1 does not. So every non-projective summand of $M_1 \otimes L_\zeta$ is in the same block as M_1. So every non-projective summand of $(K \otimes_k M) \otimes_K L_{\zeta_1} \otimes_K \cdots \otimes_K L_{\zeta_s}$ is in $B_0(KG)$, and by the theorem, we have

$$\hat{H}^n(G, (K \otimes_k M) \otimes_K L_{\zeta_1} \otimes_K \cdots \otimes_K L_{\zeta_s}) \neq 0$$

for infinitely many values of n, positive and negative.

Similarly, in Lemma 6.2 of [3], although M_1 must be finitely generated, M_2 need not be. So if ζ is a homogeneous element in cohomology, then $\hat{H}^n(G, M_2) = 0$ for all n implies $\hat{H}^n(G, L_\zeta \otimes M_2) = 0$ for all n. So we may deduce that $\hat{H}^n(G, K \otimes_k M) \neq 0$ for infinitely many values of n, positive and negative. Finally, this implies that the same is true of $\hat{H}^n(G, M)$. □

Proposition 5.4. *If M is an $\mathcal{NH}$-local kG-module, then $\mathcal{V}_G(M)$ contains no closed homogeneous subset of the nucleus Y_G.*

Proof. If $\mathcal{V}_G(M)$ contains a closed homogeneous subset V of Y_G, then

$$\underline{\mathrm{Hom}}_{kG}(\mathcal{E}_V(M), M) \neq 0,$$

while if M is $\mathcal{NH}$-local, $\mathcal{E}_{\mathcal{NH}}(M) = 0$. However, any map from $\mathcal{E}_V(M)$ to M factors through $\mathcal{E}_{\mathcal{NH}}(M)$, because the subcategory of $\mathsf{stmod}(kG)$ consisting of modules with variety in V is contained in $\mathcal{NH}$. □

Theorem 5.5. *If M is a module in $B_0(kG)$, then $\mathcal{E}_{\mathcal{NH}}(M) \cong M$ and $\mathcal{F}_{\mathcal{NH}}(M) = 0$.*

Proof. Consider the variety of $\mathcal{F}_{\mathcal{NH}}(M)$. By Proposition 5.4, it contains no closed homogeneous subset of the nucleus Y_G. So if $\mathcal{F}_{\mathcal{NH}}(M)$ is nonzero in $\mathsf{StMod}(kG)$ (i.e., non-projective), its variety must contain some closed homogeneous subset which is not in the nucleus. Then by Corollary 5.3, we have $\hat{H}^n(G, \mathcal{F}_{\mathcal{NH}}(M)) \neq 0$ for infinitely many values of n. So for some n, we have $\underline{\mathrm{Hom}}_{kG}(\Omega^n(k), \mathcal{F}_{\mathcal{NH}}(M)) \neq 0$. Since $\Omega^n(k)$ is an NH module, this contradicts the fact that $\mathcal{F}_{\mathcal{NH}}(M)$ is $\mathcal{NH}$-local. It follows that $\mathcal{F}_{\mathcal{NH}}(M) = 0$, and therefore that $\mathcal{E}_{\mathcal{NH}}(M) \cong M$. □

Proof of Theorem 1.2. By Theorem 5.5, if M is in $B_0(kG)$, then $\mathcal{E}_{\mathcal{NH}}(M) \cong M$. So M is a filtered colimit of NH modules, and since it is finitely generated, it follows that it is a direct summand of an NH module. □

Proof of Corollary 1.3. It is shown in Corollary 10.12 of [3] that this follows from Theorem 1.2. □

Theorem 5.6. *Suppose that the centralizer of every element of order p in G is p-nilpotent. Then every finitely generated module in the principal block is a trivial homology module.*

Proof. The condition on G is equivalent to the condition that $Y_G = \{0\}$. So under these conditions, nuclear homology modules are the same as trivial homology modules. So the theorem follows from Theorem 1.2, using Theorem 3.5 and Propositions 4.4 and 4.5 of [3]. □

Proof of Theorem 1.4. It is proved in [3] that (A) $\Leftrightarrow$ (A′) $\Leftrightarrow$ (A″) $\Rightarrow$ (B) $\Leftrightarrow$ (B′) $\Rightarrow$ (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (D′) $\Leftrightarrow$ (E). It is clear that (A‴) $\Rightarrow$ (A′), (B″) $\Rightarrow$ (B) and (B″) $\Rightarrow$ (B‴) $\Rightarrow$ (B′). Theorem 5.6 shows that (D) $\Rightarrow$ (A‴). Finally, Corollary 5.3 shows that (D) $\Rightarrow$ (B″). □

Proof of Theorem 1.1. This is just the statement that (B$''$) $\Leftrightarrow$ (D) in Theorem 1.4, so this is now proved. □

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