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NOTES ON FERMAT'S LAST THEOREM I

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The story of 'Fermat's Last Theorem'
has been told so often it hardly bears retelling.
H. M. Edwards

Dramatis Personæ

Euclid of Alexandria	~ -300
Diophantus of Alexandria	$\sim$ 250
Pierre de Fermat	1601–1655
Leonhard Euler	1707–1783
Joseph Louis Lagrange	1736–1813
Sophie Germain	1776–1831
Carl Friedrich Gauss	1777–1855
Augustin Louis Cauchy	1789–1857
Gabriel Lamé	1795–1870
Peter Gustav Lejeune Dirichlet	1805–1859
Joseph Liouville	1809–1882
Ernst Eduard Kummer	1810–1893
Harry Schultz Vandiver	1882–1973
Gerhard Frey	
Kenneth A. Ribet	
Andrew Wiles	$\sim$1953–

Fermat's Last Theorem states that there are no positive integers x, y and z with

$$x^n + y^n = z^n$$

if n is an integer greater than two. For n equals two there are many solutions:

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad 8^2 + 15^2 = 17^2, \quad \dots ,$$

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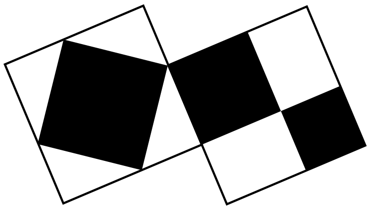
the Pythagorean triples. In the margin of his copy of Diophantus' *Arithmetica* the French jurist Fermat wrote c 1637 that for greater n no such triples can be found; he added that he had a marvellous proof for this, which, however, the margin was too small to contain:

> *Cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere; cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caparet.*

Every other result which Fermat had announced in like manner had long ago been dealt with; only this one, the *last*, remained.

Problem 8 in Book II of Claude Bachet's translation of Diophantus asks for a rule for writing a square as the sum of two squares. The resulting equation $z^2 = y^2 + x^2$ is that of the Theorem of Pythogoras, which says that in every right-angled triangle the square on the hypotenuse is the sum of the squares on the other two sides. The logo of Macquarie University's ceNTRe for Number Theory Research



provides a graphical proof, showing in particular that Pythogoras' Theorem has only the depth of the identity $(x+y)^2 = x^2 + 2xy + y^2$.

It is a little more difficult to find all solutions in integers, but not much more. If $x^2 + y^2 = z^2$, we can suppose that x, y and z pairwise have no common factor, for such a factor would be common to all three quantities and can be factored out, leaving an equation of the original shape. Thus at least two of x, y and z must be odd. But the square of an odd number (so, of the shape $2m+1$) leaves a remainder of 1 on division by 4, whilst the square of an even number (so, of the shape $2m$) leaves a remainder of 0 on division by 4. It follows that z must be odd and that one of x and y, say $x = 2x'$, must be even. Then we obtain

$$4x'^2 = z^2 - y^2 = (z+y)(z-y) \quad \text{so} \quad x'^2 = \tfrac{1}{2}(z+y)\tfrac{1}{2}(z-y).$$

But if the product of two numbers that have no factor in common is a square, then each of the two numbers is a square.

This is clear on splitting the two numbers into their prime factors and checking the contribution of each distinct prime. To apply the principle we need only note that both $\frac{1}{2}(z+y)$ and $\frac{1}{2}(z-y)$ are integers, because both z and y are odd; and that they have no common factor. The latter is clear, because if d were a common factor then d is a factor both of their sum z, and their difference y. Yet we began by determining that y and z are *relatively prime* — that they have no common factor. So both $\frac{1}{2}(z+y)$ and $\frac{1}{2}(z-y)$ are squares, say,

$$\tfrac{1}{2}(z+y) = u^2 \quad \text{and} \quad \tfrac{1}{2}(z-y) = v^2.$$

Thus ${x'}^2 = u^2 v^2$, and summarising, we have

$$x = 2uv, \quad y = u^2 - v^2, \quad \text{and} \quad z = u^2 + v^2 .$$

We obtain all Pythagorean triples without common factor by choosing integers u and v without common factor and of different *parity* — that is, one odd and the other even; and with u greater than v.

Of course, it is easy to verify that, indeed,

$$(2uv)^2 + (u^2 - v^2)^2 = (u^2 + v^2)^2 .$$

Fermat did show that the equation

$$x^4 + y^4 = z^4$$

has no solution in positive integers. In fact he shows a little more, that already

$$x^4 + y^4 = w^2$$

has no solution in positive integers. As above, we may suppose that x is even, and y and w are odd. Then, by the preceding argument, it follows that there are integers a and b so that

$$x^2 = 2ab, \quad y^2 = a^2 - b^2, \quad \text{and} \quad w = a^2 + b^2 .$$

From the expression for y^2 it follows that a is odd and b is even, and from that for x^2 we may deduce that there are integers c and d so that

$$a = c^2 \quad \text{and} \quad b = 2d^2 .$$

Hence

$$y^2 = c^4 - 4d^4 .$$

Again applying the results from the case $n = 2$ we see that there are integers e and f so that

$$y = e^2 - f^2, \quad d^2 = ef \quad \text{and} \quad c^2 = e^2 + f^2 .$$

Clearly e and f must be relatively prime, so there are integers u and v so that $e = u^2$, $f = v^2$ and

$$u^4 + v^4 = c^2 .$$

But now Fermat makes a truly marvellous observation. He notes that c is less than w. So what this argument shows is that given a solution (x, y, w) there is a *smaller* solution (u, v, c)! That is, eventually, absurd. By the *method of infinite descent*, here introduced, it follows that there is no solution in positive integers to $x^4 + y^4 = w^2$, and *a fortiori* — all the more so, none for $x^4 + y^4 = z^4$.

It was many years later, in 1753, that Euler dealt with the case $n = 3$. There was an alleged error in the argument, later dealt with by Gauss. Dirichlet and Legendre proved the case $n = 5$ in 1825 and Lamé settled the case $n = 7$ in 1839; Dirichlet had proved the case $n = 14$ in 1832.

On March 1, 1847 Lamé informed the Parisian *Académie des Sciences* that he had settled the general case. Lamé attributed the basic idea of his proof to Liouville. The idea consisted of working with numbers of the shape

$$a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{n-1}\zeta^{n-1},$$

where $a_0, a_1, \ldots, a_{n-1}$ are integers and ζ is a complex number with the property that $\zeta^n = 1$, but $\zeta \neq 1$. Here Lamé assumed n to be an odd prime number. It had been known for some time that this assumption does not, of course, impose any restriction in the proof of Fermat's Last Theorem.

With the aid of these numbers, $x^n + y^n$ may be split into n factors:

$$(x+y)(x+\zeta y)(x+\zeta^2 y)\cdots(x+\zeta^{n-1}y)$$

and Fermat's equation then assumes the shape

$$(x+y)(x+\zeta y)(x+\zeta^2 y)\cdots(x+\zeta^{n-1}y) = z^n.$$

To this Lamé applies a generalisation of the principle already described in the case $n = 2$, whereby if a product of numbers without common factor is an n-th power, then each is an n-th power. Lamé assumes that this principle holds for the *cyclotomic* integers he has just introduced and proceeds with an argument that shows that necessarily one of x or y is zero.

After Lamé, Liouville addressed the meeting. He pointed out that the idea of using complex numbers was nothing new; one could already meet such numbers in the work of Euler, Lagrange, Gauss and Jacobi. Moreover, it seemed to him, said Liouville, that Lamé implicitly assumed that unique factorisation into primes also held for cyclotomic integers.

A second difficulty is numbers that divide 1, or *units* as we now call them. There is a problem, in that for example $-4 \cdot -9 = 36$ with 36 a square, and -4 and -9 relatively prime, whilst neither is a square. In the cyclotomic case one can see readily that there are many more units than just ± 1. For example, $\zeta + \zeta^{n-1}$ is a unit whenever $n > 1$ is odd. Properties of divisibility by $\zeta + \zeta^{n-1}$ play an important role in Lame's argument. But $\zeta + \zeta^{n-1}$ divides 1, and therefore every number.

N'y a-t-il pas là une lacune à remplir?
J. Liouville

Some of this material has been liberally borrowed from the introduction to the thesis of Hendrik Lenstra Jr., 'Euclidean number fields', *Math. Intelligencer* **2** (1980), 6–15; 73–77; 99–103; the rest comes from things I just knew and from notes of mine of some 17 years ago which were probably much aided by thoughts taken from W. J. LeVeque, *Topics in Number Theory*, Addison-Wesley 1961, Vol 2.