FETI DOMAIN DECOMPOSITION METHODS FOR SCALAR ADVECTION–DIFFUSION PROBLEMS

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Abstract. In this paper, we show that iterative substructuring methods of Finite Element Tearing and Interconnecting type can be successfully employed for the solution of linear systems arising from the finite element approximation of scalar advection–diffusion problems. Using similar ideas as those of a recently developed Neumann–Neumann method, we propose a one–level algorithm and a class of two–level algorithms, obtained by suitably modifying the local problems on the subdomains. We present some numerical results for some significant test cases. Our methods appear to be optimal for flows without closed streamlines and possibly very small values of the viscosity. They also show very good performances for rotating flows and moderate Reynolds numbers. Therefore, the algorithms proposed appear to be well–suited for many convection–dominated problems of practical interest.

Key words. Advection–diffusion, stabilization, domain decomposition, FETI, preconditioners.

AMS subject classifications. 65F10, 65N22, 65N30, 65N55

1. Introduction.

In this paper, we consider the boundary value problem

$$
\begin{aligned}
&Lu := -\nu \Delta u + \mathbf{a} \cdot \nabla u + cu = f, \quad \text{in } \Omega,\\
&u = 0, \quad \text{on } \partial\Omega_D, \qquad\qquad (1)\\
&\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega_N.
\end{aligned}
$$

Here, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded, connected polyhedral domain with a Lipschitz continuous boundary $\partial\Omega$ and outward normal denoted by $\mathbf{n}$. We consider a partition $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$, where $\partial\Omega_N$ can possibly be empty. For simplicity, we only deal with homogeneous Dirichlet and Neumann conditions, but more general non homogeneous boundary data can also be used; see section 4 and, e.g., [31, Ch. 6].

The viscosity ν is positive, but can be arbitrarily small for advection–dominated problems. For simplicity, we assume that ν is constant. The velocity field $\mathbf{a}$ is given and we suppose that $\mathbf{a} \in L^\infty(\Omega)^n$ and $\nabla \cdot \mathbf{a} \in L^\infty(\Omega)$. The scalar function $c \in L^\infty(\Omega)$ is a reaction coefficient that may arise from a finite difference discretization of a time derivative, and $f \in L^2(\Omega)$ is a source term.

The aim of this paper is to build a family of iterative methods of Finite Element Tearing and Interconnecting (FETI) type for a conforming finite element (FE) approximation of problem (1). We show that by borrowing some ideas from a Neumann–Neumann method for the same problem, see [2], and by using some recent developments in the analysis of FETI methods, see [25, 39], FETI algorithms can be employed successfully for advection–diffusion problems as well. We are primarily interested in convection–dominated problems.

FETI methods were first introduced for the solution of conforming approximations of elasticity problems in [15]. In this approach, the original domain Ω is decomposed into non–overlapping subdomains Ω_i, $i = 1, \ldots, N$. On each subdomain Ω_i a local stiffness matrix is obtained from the finite element discretization of local Neumann

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problems. Analogously, a set of right hand sides is built. The continuity of the solution, corresponding to the set of primal variables, is then enforced across the interface defined by the inner subdomain boundaries by using Lagrange multipliers. In the original FETI algorithm, the primal variables are then eliminated by solving local Neumann problems, and an equation for the Lagrange multipliers is obtained. Several preconditioners have been used extensively for the resulting dual problem. They ensure that the condition number of the preconditioned system is independent of the number of substructures and increases only slowly with the number of unknowns associated to the substructures. In addition, if suitable scaling matrices are introduced, the condition number can also be made independent of possibly large jumps of the coefficients; see section 3.1. We refer to [14, 16, 26, 10, 38, 34, 25], for Poisson and elasticity problems, to [12], for acoustic scattering problems, to [13, 11, 27, 38], for shell and plates problems, and to [39, 33], for edge element approximations of Maxwell's equations.

A number of domain decomposition methods have also been proposed for advection–diffusion equations. Overlapping methods have been considered, e.g., in [6, 7, 5, 30]. For iterative substructuring methods, we refer to [32, Ch. 6] and to the references therein. In particular, we mention [8, 18, 40, 9, 4], where the matching conditions on the subdomain interfaces are chosen according to the direction of the flow, and [1, 3, 2], where the same kinds of boundary conditions are employed for all the subproblems and are chosen in order to ensure their stability. The methods considered in [29, 28, 30, 22, 23] also belong to the latter family; in these methods, optimized conditions of higher order are imposed on the boundaries of the subdomains in order to improve the convergence. Many multigrid methods have also been proposed for advection–diffusion problems, and we refer, e.g., to [19, 20] and to the references therein.

In our work, we will employ some ideas from [1, 2], and use the same Robin boundary conditions for all the subproblems. We stress the fact that we, by no means, claim that this is the only way of constructing FETI methods for advection–diffusion problems and other methods are possible. We only know of few previous papers; see [23], where a FETI coarse space is employed for an iterative substructuring method with optimized interface conditions. When discussing our numerical results, we will also refer to [36], where some preconditioners based on relaxation and incomplete factorizations are carefully tested for some stabilized methods.

As is customary, see [31, Ch. 6], we suppose

$$c - \frac{1}{2}\nabla \cdot \mathbf{a} \geq c_0 > 0, \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{a} \cdot \mathbf{n} \geq 0, \quad \text{on } \partial\Omega_N. \tag{3}$$

For any $\mathcal{D} \subset \Omega$, the bilinear form associated to the operator L and the boundary conditions in (1) is

$$a_{\mathcal{D}}(u, v) := \int_{\mathcal{D}} (\nu \nabla u \cdot \nabla v + \mathbf{a} \cdot \nabla u\, v + c\, uv)\, dx, \quad u,\, v \in H^1(\Omega; \partial\Omega_D), \tag{4}$$

where $H^1(\Omega; \partial\Omega_D)$ is the subspace of $H^1(\Omega)$ of functions that vanish on $\partial\Omega_D$. In case $\mathcal{D} = \Omega$, we drop the subscript. The variational formulation of problem (1) is: Find

$u \in H^1(\Omega; \partial\Omega_D)$ such that

$$a(u,v) = \int_\Omega f v \, dx, \quad v \in H^1(\Omega; \partial\Omega_D). \tag{5}$$

The bilinear form $a_{\mathcal{D}}(\cdot,\cdot)$ can be decomposed into a symmetric and a skew–symmetric part

$$a_{\mathcal{D}}(u,v) = b_{\mathcal{D}}(u,v) + s_{\mathcal{D}}(u,v),$$

with

$$b_{\mathcal{D}}(u,v) := \int_{\mathcal{D}} \left(\nu \nabla u \cdot \nabla v + \left(c - \frac{1}{2} \nabla \cdot \mathbf{a} \right) uv \right) dx + \frac{1}{2} \int_{\partial\mathcal{D} \setminus \partial\Omega_D} \mathbf{a} \cdot \mathbf{n} \, uv \, ds, \tag{6}$$

$$s_{\mathcal{D}}(u,v) := \frac{1}{2} \int_{\mathcal{D}} (\mathbf{a} \cdot \nabla u \, v - \mathbf{a} \cdot \nabla v \, u) \; dx, \tag{7}$$

for all u and v in $H^1(\Omega; \partial\Omega_D)$.

We note that conditions (2) and (3) ensure that the bilinear form $a(\cdot,\cdot)$ is coercive in $H^1(\Omega; \partial\Omega_D)$ and problem (5) is then well–posed.

The outline of the remainder of this paper is as follows. In section 2, we introduce a partition of the domain Ω, together with some local finite element spaces and local bilinear forms. In section 3, we define our one– and two–level FETI methods. The numerical results for three significant test cases are presented in section 4.

2. Finite element spaces and stabilized formulations. We consider a shape–regular triangulation $\mathcal{T}_h$ of the domain Ω. For each triangle $t \in \mathcal{T}_h$, let h_t be its diameter, and let h be the maximum of the diameters of the elements. We next consider a non overlapping partition of the domain Ω into subdomains,

$$\mathcal{F}_H = \left\{ \Omega_i \mid \quad 1 \le i \le N, \quad \bigcup_{i=1}^{N} \overline{\Omega_i} = \overline{\Omega} \right\},$$

such that each Ω_i is open and connected, and is the union of some elements in $\mathcal{T}_h$. We denote the diameter of Ω_i by H_i and define H as the maximum of the diameters of the subdomains. The elements of $\mathcal{F}_H$ are also called *substructures*. Let Γ_i be the part of $\partial\Omega_i$ that is common to other substructures

$$\Gamma_i := \overline{\partial\Omega_i \setminus \partial\Omega},$$

and let the interface Γ be the union of the Γ_i. We note that Γ and the Γ_i are closed sets.

For the approximation of problem (5), we consider the standard FE space of continuous, piecewise linear functions on Ω:

$$\widetilde{X}_h(\Omega) = \widetilde{X} := \{ u \in H^1(\Omega; \partial\Omega_D) |\; u_{|_t} \in \mathbb{P}_1(t), \; t \in \mathcal{T}_h, \}.$$

In order to define our FETI method, we need to introduce some local spaces. We define

$$H^1(\Omega_i; \partial\Omega_D) := \{ u_i \in H^1(\Omega_i) |\; u_i = 0 \text{ on } \partial\Omega_D \cap \partial\Omega_i \}.$$

We next introduce the FE spaces of continuous, piecewise linear functions on each subdomain Ω_i

$$X_h(\Omega_i) = X_i := \{u_i \in H^1(\Omega_i; \partial\Omega_D) |\ u_{i|_t} \in \mathbb{P}_1(t),\ t \in \mathcal{T}_h,\ t \subset \Omega_i\},$$

and the product space

$$X_h(\Omega) = X := \prod_{i=1}^{N} X_i \subset \prod_{i=1}^{N} H^1(\Omega_i; \partial\Omega_D)$$

We also need some local trace spaces

$$W_h(\Gamma_i) = W_i := \{u_i \text{ restricted to } \Gamma_i \ \ | \ u_i \in X_i\},$$

and the product space

$$W_h(\Gamma) = W := \prod_{i=1}^{N} W_i.$$

The local trace spaces W_i consist of continuous, piecewise linear functions on the Γ_i.

It is well known that for advection–dominated problems the original bilinear form $a(\cdot,\cdot)$ has to be modified in order to remove spurious oscillations of the Galerkin approximation on standard continuous polynomial spaces, if the mesh does not resolve boundary or internal layers. A large number of strategies have been proposed in the past twenty years and many of them consist of adding mesh dependent terms to the FE approximation; see, e.g., [24, 31] and the references therein . Here, we consider the Galerkin/least–squares method (GALS) originally developed in [21]. We introduce the modified bilinear form

$$a_\Omega^h(u,v) = a^h(u,v) := a(u,v) + \sum_{t \subset \Omega} \delta(h_t) \int_t Lu\, Lv\, dx, \quad u, v \in \widetilde{X}_h(\Omega), \tag{8}$$

and the corresponding local bilinear forms

$$\widehat{a}_{\Omega_i}^h(u,v) := a_{\Omega_i}(u,v) + \sum_{t \subset \Omega_i} \delta(h_t) \int_t Lu\, Lv\, dx, \quad u, v \in X_i, \quad i = 1, \ldots, N, \tag{9}$$

where $\delta(h_t)$ is a positive function that vanishes with h_t and may depend on the values of ν and $\mathbf{a}$ on the element t; see section 4 for a particular choice.

Our discrete problem becomes: Find $u \in \widetilde{X}$ such that

$$a^h(u,v) = \int_\Omega f v\ dx + \sum_{t \subset \Omega} \delta(h_t) \int_t f\, Lv\, dx, \quad v \in \widetilde{X}. \tag{10}$$

We will use modified local bilinear forms from now on. We note that we cannot in general ensure that the bilinear form $\widehat{a}_{\Omega_i}^h(\cdot,\cdot)$ is positive definite on the space X_i. This can be seen by considering the symmetric part of $a_{\Omega_i}(\cdot,\cdot)$, defined in (7), and noting that the boundary integral on $\partial\Omega_i \setminus \partial\Omega_D$ does not vanish in general and that, along the internal part of the boundary Γ_i, the sign of the coefficient $\mathbf{a} \cdot \mathbf{n}$ depends on the orientation of the normal $\mathbf{n}$ with respect to the flow $\mathbf{a}$. Following [2], we consider a modified bilinear form in Ω_i, obtained from the original one by subtracting this boundary integral along the internal part of the boundary:

$$a^h_{\Omega_i}(u,v) := \widehat{a}^h_{\Omega_i}(u,v) - \frac{1}{2}\int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,uv\,ds, \quad i = 1,\ldots,N.$$

The conditions (2) and (3) now ensure that these new local bilinear forms are positive definite on the spaces X_i. In addition, they are related to Robin problems on the substructures; if a function $u \in H^1(\Omega_i;\partial\Omega_D)$ satisfies

$$a_{\Omega_i}(u,v) - \frac{1}{2}\int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,uv\,ds = \int\limits_{\Omega_i} fv\,dx + \int\limits_{\Gamma_i} gv\,ds, \quad v \in H^1(\Omega_i;\partial\Omega_D),$$

and is sufficiently regular, it also satisfies

$$\begin{aligned}
&Lu = f, \quad \text{in } \Omega_i,\\
&u = 0, \quad \text{on } \partial\Omega_D \cap \partial\Omega_i,\\
&\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega_N \cap \partial\Omega_i,\\
&\frac{\partial u}{\partial n} - \frac{1}{2}\mathbf{a}\cdot\mathbf{n}\,u = g, \quad \text{on } \Gamma_i.
\end{aligned}$$

3. A FETI algorithm. In this section, we introduce a FETI method for the solution of problem (10).

We first assemble the local stiffness matrices K_i, relative to the local bilinear forms $a^h_{\Omega_i}(\cdot,\cdot)$, and the local load vectors f_i. The local matrices K_i can be represented as

$$K_i = \begin{bmatrix} K_i^{II} & K_i^{IB} \\ K_i^{BI} & K_i^{BB} \end{bmatrix},$$

where we divide a local vectors u_i into two subvectors, u_i^B and u_i^I, of degrees of freedom corresponding to nodes on Γ_i and on the rest of $\overline{\Omega_i}$, respectively. The load vectors f_i are divided in the same way.

We then consider the following problem

$$\begin{array}{rcccl} Kw & + & \widetilde{B}^t\lambda & = & f, \\ \widetilde{B}w & & & = & 0, \end{array} \tag{11}$$

where

$$w := \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \in X, \quad K := \mathrm{diag}\{\mathrm{K_1},\ldots,\mathrm{K_N}\}, \quad \mathrm{f} := \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}.$$

The matrix $\widetilde{B}$ has entries in $\{1,-1,0\}$ and is chosen such that the values of the solution w are equal at the nodes on the subdomain boundaries that are common to two or more substructures when $\widetilde{B}w = 0$. The vector λ is a Lagrange multiplier relative to the continuity constraint $\widetilde{B}w = 0$. In the following, we suppose that the constraints are linearly independent and therefore there are no redundant Lagrange multipliers. In this case, $\widetilde{B}$ has full rank. An analogous algorithm can be defined in the case of redundant multipliers; see, e.g., [34, 25].

We remark that we have $Ker(\widetilde{B}) = \widetilde{X}$ and that, when the matrix K is restricted to $\widetilde{X}$, the boundary integrals in the definition of the modified local bilinear forms $a^h_{\Omega_i}(\cdot,\cdot)$ cancel out. Problems (11) and (10) then have the same solution $u = w$.

The degrees of freedom that are not on the interface Γ belong to only one substructure and can be eliminated in parallel by block Gaussian elimination. Let g_i be the resulting right hand sides and let S_i be the Schur complement matrices

$$S_i : \; W_i \longrightarrow W_i,$$

relative to the degrees of freedom on Γ_i.

After eliminating the interior variables, we obtain the following problem for $u \in W$:

$$\begin{array}{lcccl} Su & + & B^t\lambda & = & g, \\ Bu & & & = & 0, \end{array} \tag{12}$$

where

$$u_i := w_{i|_{\Gamma_i}}, \quad S := \mathrm{diag}\{\mathrm{S_1}, \ldots, \mathrm{S_N}\}.$$

The matrix $B = [B_1, \ldots, B_N]$ is obtained from $\widetilde{B}$ by deleting the columns relative to the interior variables. We have, for $i = 1, \ldots, N$,

$$\begin{array}{rcl} S_i & = & K_i^{BB} - K_i^{BI}\left(K_i^{II}\right)^{-1}K_i^{IB}, \\ g_i & = & f_i^B - K_i^{BI}\left(K_i^{II}\right)^{-1}f_i^I. \end{array}$$

We can easily check that, since our local bilinear forms are positive–definite, the local Schur complements S_i are always invertible and, consequently, there is no natural coarse space associated to the substructures; we are in a similar case as that considered in [10].

Following [10], we first find u from the first equation in (12), and substitute its value in the second equation. We obtain the system

$$F\lambda = d, \tag{13}$$

where

$$F := B\,S^{-1}\,B^t, \quad d := B\,S^{-1}\,f.$$

Following [25, 39], we now define a preconditioner for (13).

We introduce the matrices,

$$R := \left[R_1, \; R_2, \; \ldots, \; R_N\right], \quad G := Q\,B\,R, \tag{14}$$

where R_i are vectors in W, related to the substructures $\{\Omega_i\}$ and Q is a suitable invertible matrix. More precisely, we suppose that R_i is obtained from a local vector $r_i \in W_i$ on Γ_i, by extending it by zero to the boundaries of the other substructures. We will make some particular choices for R and Q in sections 3.2 and 3.3.

Following [10, 39], we next define the projection

$$P := I - G(G^t F G)^{-1} G^t F,$$

onto the complement of $Range(G)$. We remark that this is not an orthogonal projection since the matrix F is not symmetric. Following [25, 39], we next define the preconditioner

$$\widehat{M}^{-1} := (BD^{-1}B^t)^{-1}\, BD^{-1}SD^{-1}B^t\, (BD^{-1}B^t)^{-1},$$

where D is a symmetric, non singular matrix that we will specify in the next section. It can be easily seen that $BD^{-1}B^t$ is invertible and is block diagonal.

As in [10, 39], we consider a projected preconditioned algorithm. Since our linear system is not symmetric, we consider the generalized minimal residual (GMRES) method; see, e.g., [35]. Other choices are clearly possible.

1. Initialize

$$\begin{aligned} \lambda^0 &= G(G^tFG)^{-1}G^t d \\ \text{Project: } w^0 &= P^t(d - F\lambda^0) \\ \text{Precondition: } z^0 &= \widehat{M}^{-1} w^0 \\ \text{Project: } y^0 &= P z^0 \\ v^1 &= y^0/\|y^0\| \end{aligned}$$

2. Iterate $k = 1, 2, \ldots, j, \ldots,$ until convergence

$$\begin{aligned} q^k &= F v^k \\ \text{Project: } w^k &= P^t q^k \\ \text{Precondition: } z^k &= \widehat{M}^{-1} w^k \\ \text{Project: } y^k &= P z^k \\ h_{i,k} &= {y^k}^t v^i, \quad i = 1, \ldots, k \\ \hat{v}^k &= y^k - \sum_{i=1}^{k} h_{i,k} v^i \\ h_{k+1,k} &= \|\hat{v}^k\| \\ v^{k+1} &= \hat{v}^k / h_{k+1,k} \end{aligned}$$

3. Form the approximate solution

$$\lambda^j = \lambda^0 + V_j u^j, \quad \text{where } u^j \text{ minimizes } \|\beta e_1 - H_j u\|,\ u \in \mathbb{R}^j.$$

Here, $\beta = \|y^0\|$, e_1 is the first column of the $(j+1) \times (j+1)$ identity matrix, V_j is the matrix, the columns of which are the vectors v_k, $k = 1, \ldots, j$, and H_j is a $(j+1) \times j$ matrix, the non–zero entries of which are the elements $h_{i,k}$; see [35].

Because of the choice of the initial vector λ^0, we can easily prove the following lemma.

LEMMA 3.1. *The following properties hold*

$$w^0 = d - F\lambda^0, \tag{15}$$

$$w^k = q^k, \quad \textit{for } k \geq 1, \tag{16}$$
$$\lambda^j - \lambda^0 \in Range(P). \tag{17}$$

We note that, thanks to Lemma 3.1, the action of P^t needs not be calculated in practice.

The method presented here is equivalent to using GMRES for solving the following preconditioned system

$$P\widehat{M}^{-1}P^tF\lambda = P\widehat{M}^{-1}P^td, \quad \lambda \in \lambda^0 + V, \tag{18}$$

with

$$V := Range(P).$$

In section 4, we will also consider some numerical results for the following preconditioned problem

$$\widehat{M}^{-1}F\lambda = \widehat{M}^{-1}d, \quad \lambda \in U, \tag{19}$$

with

$$U := Range(B),$$

which corresponds to the choice $P = I$ and $\lambda^0 = 0$ in our algorithm. In the following, we refer to GMRES applied to Equation (18) as two–level FETI (FETI–2), and as one–level FETI (FETI–1) when applied to Equation (19).

Before introducing our choices for the matrices D, R, and Q, we remark on the computational cost of our algorithms. The matrices S and S^{-1} need not be calculated in practice. The action of the local Schur complement S_i on a local vector requires the solution of a Dirichlet problem on the substructure Ω_i, while the action of S_i^{-1} requires the solution of a problem on Ω_i with Robin boundary conditions; see [37, Ch. 4]. At each step of FETI–2, the following problems have to be solved:

- one Dirichlet problem on each substructure, for the application of the preconditioner $\widehat{M}^{-1}$;
- two Robin problems on each substructure, for the two applications of F (cf. the definition of projection P);
- one coarse problem of dimension N, involving the matrix G^tFG (cf. the definition of P).

For FETI–1, we only have to solve one Dirichlet problem and one Robin problem on each substructure.

We recall that, for the two–level Robin–Robin method in [2], at each step, we have to solve one coarse problem, and two Dirichlet problems and one Robin problem on each substructure. For the corresponding one–level method, we only have to solve a Dirichlet problem and a Robin problem on each substructure.

3.1. The scaling matrix D. Following [2, 25], for the substructure Ω_i, we first consider a continuous piecewise linear function $\mu_i^\dagger \in W_i$ defined in the following way: For every node $x_h \subset \Gamma_i$, we set

$$\mu_i^\dagger(x_h) := \frac{\rho_i}{\sum_k \rho_k},$$

where, for every Ω_k that share the node x_h, ρ_k is the diagonal entry of the stiffness matrix K_k corresponding to the node x_h, and the sum is extended to all substructures that share x_h. Let $D_i : W_i \longrightarrow W_i$, be the diagonal matrix that represents the multiplication by $\mu_i^\dagger$.

We define the global scaling matrix $D : W \longrightarrow W$ as

$$D := \mathrm{diag}\{\mathrm{D_1}, \ldots, \mathrm{D_N}\}.$$

3.2. The matrix R. In this section, we consider a particular choice of the matrix R in the definition of our two–level FETI algorithm. It is given in terms of local vectors.

Following [2], we define a set of local functions that depend on the particular problem solved. We consider a substructure Ω_i and the solution ϕ_i to the following adjoint local problem: Find $\phi_i \in X_i$ such that

$$a^h_{\Omega_i}(v_i, \phi_i) = \int_{\Omega_i} v_i \, dx, \quad v \in X_i.$$

We then consider the following definition:

DEFINITION 3.1. *The local functions $\{r_i \in W_i,\ i = 1, \ldots, N\}$ are defined as*

$$r_i := \phi_i, \quad \text{restricted to } \Gamma_i.$$

The global functions R_i are obtained by extending the local vectors r_i by zero outside $\partial\Omega_i$.

We note that other choices are possible; see, e.g., [10] and [2, Sect. 4]. We could, e.g., choose r_i as a constant function on Γ_i. We have chosen to use the functions of Definition 3.1 since, for convection–dominated problems, they perform better than the constants in our numerical tests (the results are not presented here).

3.3. Some choices for the scaling matrix Q. Particular scaling matrices are often employed for FETI methods in order to improve their convergence properties. In [16, 25], some suitable matrices are employed for FETI methods for problems where the matrix S of local Schur complements has a non–vanishing kernel. The central idea of many of these choices is to make the scaling matrix as similar as possible or equal to the preconditioner $\widehat{M}^{-1}$. In [33], a different scaling matrix is considered for a problem where the matrix S is invertible; see [10]. There, such a matrix ensures that a scalable FETI method can be found for an edge element approximation on non–matching grids.

Our first choice for Q is simply the identity matrix. Our second choice is similar to that in [33]. We define the block–diagonal matrix

$$Q_1 := (BD^{-1}B^t)^{-1}. \tag{20}$$

Our third choice for Q is more in the spirit of [16, 25] and is defined by

$$Q_2 := \sum_{i=1}^{N} (BD^{-1}B^t)^{-1}\, B_i D_i^{-1} K_i^{BB} D_i^{-1} B_i^t \,(BD^{-1}B^t)^{-1}. \tag{21}$$

We note that Q_2 is obtained from the expression of the preconditioner $\widehat{M}^{-1}$, by replacing the local Schur complements S_i with the diagonal blocks K_i^{BB} of the local stiffness matrices.

4. Numerical experiments. We now present some numerical results for three test problems in two dimensions. We are primarily interested in the performance of our algorithm when varying the mesh size, the number of subdomains, the ratio between the advective and diffusive coefficients, and the type of vector field $\mathbf{a}$.

Our numerical results have been obtained with *Matlab 5.3*, using the mesh generator in the package *PDETool*, which, once a domain and a decomposition into subdomains are specified, creates a conforming, unstructured mesh, such that the subdomain boundaries do not cut through the elements. For this reason, for different partitions, the meshes that we have obtained are not exactly equal, but they can be made similar and with a comparable number of elements.

For the stabilization function $\delta(h_t)$, we have followed [21]. For every triangle $t \in \mathcal{T}_h$, let the local Peclet number be

$$Pe_t := \frac{h_t \, \|\mathbf{a}\|_{t;\infty}}{2\nu}, \qquad \text{with } \ \|\mathbf{a}\|_{t;\infty} := \sup_{x \in t} |\mathbf{a}(x)|,$$

and

$$\delta(h_t) := \begin{cases} \dfrac{\tau \, h_t}{2 \, \|\mathbf{a}\|_{t;\infty}}, & \text{if } Pe_t \geq 1, \\ \dfrac{\tau \, h_t^2}{4 \, \nu}, & \text{if } Pe_t < 1. \end{cases}$$

In our experiments, we have considered the value $\tau = 0.7$.

We have employed GMRES without restart. As a stopping criterion, we require the l^2–norm of the residual of the system $F\lambda = d$ to be reduced by a factor 10^{-6}. Other stopping criteria have been proposed for FETI methods for symmetric, positive–definite problems; see, e.g., [16, 10].

For the test problems considered, we also present a comparison with the Robin–Robin method developed in [2]; see also [1]. As we will show, our method and that of [2] give comparable numbers of iterations. However, some caution must be used in this comparison, since two different linear systems are solved. The Robin–Robin algorithm is a preconditioned iterative method in the conforming space $\widetilde{X}$ for problem (10), which involves the primal variable u. The FETI algorithm, on the other hand, is a preconditioned iterative method for problem (13) for the dual variable λ. Once an approximation for λ is found, an approximation for the primal variable u can be calculated; the approximate solution u^k that we obtain satisfies the equilibrium equation $Su^k + B^t\lambda^k = g$ exactly, and is in general discontinuous, since the residual considered in the FETI algorithm is equal to the jumps of our approximate solution:

$$Bu^k = d - F\lambda^k.$$

As the two methods employs different stopping criteria, some care should then be used in the comparison. In a future work, we intend to address the issue of the stopping criterion for FETI methods for convection–dominated problems in more detail.

4.1. Thermal boundary layer. We first consider a thermal boundary layer problem: see, e.g., [17, 40]. We choose $\Omega = (-1, 1)^2$ and the parallel velocity field

$$\mathbf{a} = \left(\frac{1+y}{2}, 0 \right);$$

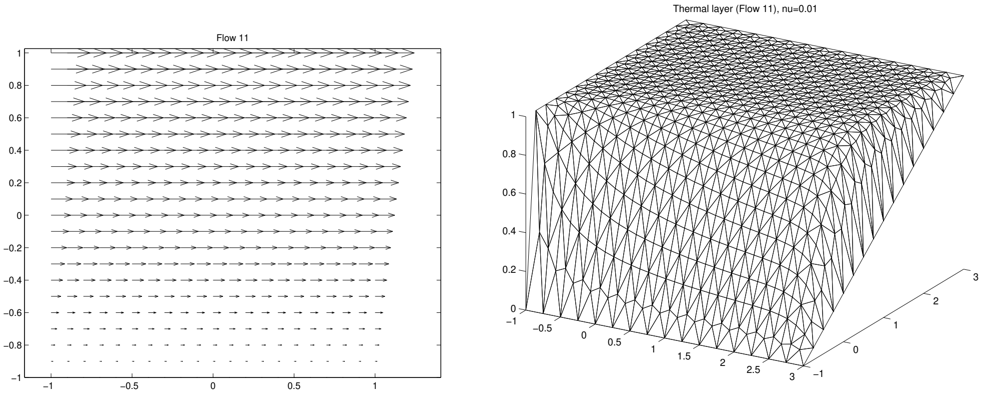

FIG. 1. **Thermal boundary layer problem**. *The flow field (left) and the solution for* $\nu = 0.01$ *(right).*

see Figure 1. We consider the following Dirichlet conditions on $\partial\Omega$:

$$
\begin{aligned}
& u = 1 \quad \begin{cases} x = -1, \ -1 < y \le 1, \\ y = 1, \ -1 \le x \le 1, \end{cases} \\
& u = 0, \quad y = -1, \ -1 \le x \le 1, \\
& u = \frac{1+y}{2}, \quad x = 1, \ -1 \le y \le 1,
\end{aligned}
$$

and the values $f = 0$ and $c = 10^{-4}$. The solution for $\nu = 0.01$ is shown in Figure 1. We note that there are two boundary layers along $\partial\Omega$.

In Table 1, we show the number of iterations for FETI–1 and FETI–2 for different choices of Q, versus the viscosity ν. We consider partitions into 4×4, 8×8, and 16×16 equal square substructures.

For large values of the viscosity, our algorithms exhibit convergence properties similar to most of domain decomposition methods for symmetric, coercive problems; see, e.g., [37]. The number of iterations increases with the number of subdomains for the one–level algorithm, but, with a suitable coarse problem, the number of iterations is independent of the number of subdomains. For smaller values of the viscosity ($\nu \le 0.01$), the difference between FETI–1 and FETI–2s becomes smaller. In particular, for the 16 and 64 subdomains cases, the iteration counts are the same for the two classes of algorithms. For the 256 subdomains case, the FETI–2 algorithms show a slight improvement with respect to FETI–1.

In addition, we note that for a fixed number of substructures, the number of iterations appears to converge to a constant value as ν tends to zero, for the four algorithms considered. For the 16 and 64 subdomains cases, these values are approximately the same for the four algorithms. For the 256 subdomains case, the number of iterations of FETI–1 is a little higher than those of the FETI–2 algorithms. We note that, for convection–dominated problems and parallel flows, this is the same behavior as exhibited by some other iterative substructuring methods, see, e.g., [9, 40, 3], and some preconditioned iterative methods based on incomplete factorizations, see, e.g., [36]. We conclude that the FETI–2 algorithms do not seem to present any considerable advantage over FETI–1 for the choices of Q considered in section 3.3.

In Table 2, we present some results for different meshes. An initial mesh consists of

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	13	11	10	11
0.1	13	11	11	11
0.01	9	9	9	9
0.001	10	10	10	10
1e-04	11	11	11	10
1e-05	11	12	11	10
1e-06	11	12	10	10

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	44	9	9	9
0.1	34	11	10	10
0.01	16	13	13	13
0.001	16	15	14	15
1e-04	18	18	17	18
1e-05	19	19	18	19
1e-06	19	19	18	19

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	159	8	8	7
0.1	98	10	9	9
0.01	38	18	18	18
0.001	33	27	27	27
1e-04	46	38	38	40
1e-05	51	42	41	45
1e-06	51	42	41	45

TABLE 1

Thermal boundary layer. *FETI method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for FETI–1 (first columns) and FETI–2 with different choices of the scaling matrix* Q*. Cases of* 16 *(first table),* 64 *(second table), and* 256 *(third table) substructures, with meshes consisting of* 1792*,* 1920*, and* 1984 *elements, respectively.*

480 triangles and we make three successive uniform refinements. We show the number of iterations versus the viscosity ν, for the case of 64 substructures and FETI–1 and FETI–2 with $Q = I$, for four different meshes. We note that our iteration counts appear to be bounded as the mesh–size becomes smaller. In particular, for viscosities $\nu \leq 0.001$, both FETI–1 and FETI–2 converge faster once the mesh is refined enough. Note also the relative large number of iterations for a very coarse mesh (480 elements). This is a similar behavior as that of some other iterative substructuring methods see, e.g., [9, 40, 3, 2].

We finally report some results obtained with the one– and two–level Robin–Robin algorithms described in [2]. For the same test problem, Table 3 shows the number of iterations versus the viscosity ν. We consider the same partitions and meshes as before. By comparing the values in Tables 1 and 3, we observe that the FETI and the Robin–Robin methods give very similar iteration counts. We refer to [25], for some theoretical and algorithmic connections between the two families of methods for symmetric, positive–definite problems.

elem.	$\nu=1$	$\nu=0.1$	$\nu=0.01$	$\nu=0.001$	$\nu=1e-04$	$\nu=1e-05$	$\nu=1e-06$
480	43	32	16	24	32	36	37
1920	44	34	16	16	18	19	19
7680	46	36	17	14	16	18	17
30720	47	38	19	14	15	17	17

elem.	$\nu=1$	$\nu=0.1$	$\nu=0.01$	$\nu=0.001$	$\nu=1e-04$	$\nu=1e-05$	$\nu=1e-06$
480	8	10	14	21	27	29	29
1920	9	11	13	15	18	19	19
7680	11	12	15	13	16	17	17
30720	13	14	16	13	14	16	16

TABLE 2

Thermal boundary layer. *FETI method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for different meshes, for FETI–1 (first table) and FETI–2 with* $Q = I$ *(second table). Case of* 64 *substructures.*

	16 substr.		64 substr.		256 substr.	
ν	1 lev.	2 lev.	1 lev.	2 lev.	1 lev.	2 lev.
1	13	10	44	9	163	7
0.1	13	12	34	12	97	9
0.01	10	9	16	13	37	17
0.001	11	10	16	16	33	28
1.e-04	11	12	19	19	46	42
1.e-05	11	11	20	20	50	47
1.e-06	11	11	20	20	50	48

TABLE 3

Thermal boundary layer. *Robin–Robin method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for the one–level (first columns) and the two–level algorithm (second columns). Cases of* 16*,* 64*, and* 256 *substructures, with meshes consisting of* 1792*,* 1920*, and* 1984 *elements, respectively.*

4.2. Variable flow field. We next consider a more complicated flow; see [36] for a similar test case. We choose $\Omega = (-1,1)^2$ and a discontinuous Dirichlet data that is advected inside Ω by the flow field

$$\mathbf{a} = \frac{1}{2}\left((1-x^2)(1+y), -(4-(1+y)^2)x\right);$$

see Figure 2. The Dirichlet conditions are defined by

$$\begin{aligned} u &= 1, \quad y=-1,\ -1<x<0, \\ u &= 0, \quad \text{elsewhere.} \end{aligned}$$

We have chosen the values $f = 0$ and $c = 10^{-4}$. The solution for $\nu = 0.01$ is shown in Figure 2. We note that there are boundary layers along $\partial\Omega$.

In Table 4, we show the number of iterations for FETI–1 and FETI–2 for different choices of Q, versus the viscosity ν. We consider the same partitions into 4×4, 8×8, and 16×16 substructures.

We first note that for the diffusion–dominated case, we observe a similar behavior as in the previous test case, where a coarse space is quite useful if the number of

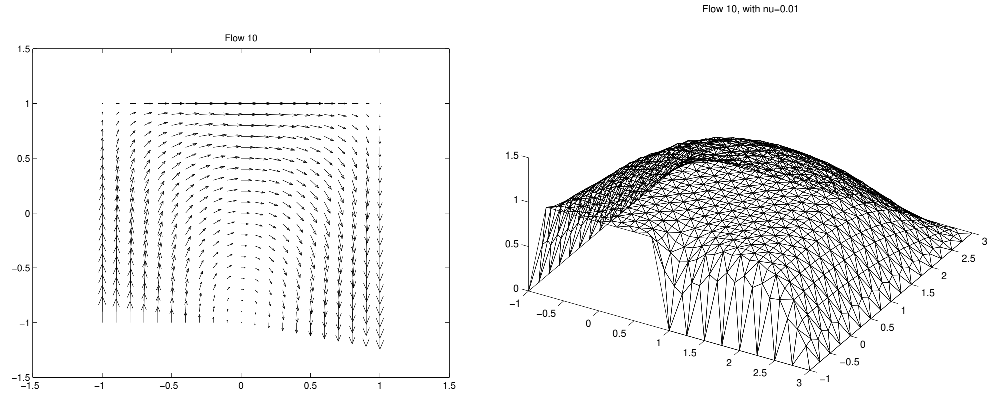

FIG. 2. **Variable flow field**. *The flow field (left) and the solution for* $\nu = 0.01$ *(right).*

subdomains is large; see Table 1. For values of ν between 10^{-2} and 10^{-4}, the numbers of iterations for the one– and the two–level algorithms are very similar, as in the case of the thermal boundary layer problem. On the other hand, for smaller values of ν, the FETI–2 algorithm with $Q = I$ shows a relatively larger improvement with respect to FETI–1. We remark that considerable worse results are obtained using Q_1 or Q_2 for convection–dominated cases. In this case as well, the numbers of iterations appear to be bounded as ν goes to zero. We note that, for convection–dominated problems, this is the same behavior exhibited by some preconditioned iterative methods based on incomplete factorizations for similar flows, see [36].

In Table 5, we present some results for the same meshes considered in the previous test case. For the case of 64 substructures and FETI–1 and FETI–2 with $Q = I$, we show the numbers of iterations versus the viscosity ν, for four different meshes. The results in Table 5 are similar to those in Table 2 and, in this case as well, our iterations counts appear to be bounded as the mesh–size becomes smaller.

We finally report some results obtained with the Robin–Robin algorithms in [2]. For the same test problem, Table 6 shows the number of iterations versus the viscosity ν. By comparing the values in Tables 4 and 6, we observe that for this test problem as well, the FETI and the Robin–Robin methods give very similar iteration counts, except for the case of 256 substructures and extremely high Reynolds numbers, where the two–level FETI method with $Q = I$ gives better results.

We conclude this section with some remarks on the two test cases considered so far. Our FETI algorithms perform very well for convection–dominated problems and, when the mesh–size goes to zero, the number of iterations appears to be bounded. Moreover, for parallel flows, a coarse space does not appear to be necessary. We have found a similar behavior for different types of parallel flows and, more generally, for flows with no closed streamlines (other results are not presented here).

4.3. Rotating flow field. We now consider a difficult test case with a flow with some closed streamlines. We choose $\Omega = (-1, 1)^2$ and a discontinuous Dirichlet data that is advected inside Ω by the rotating flow field

$$\mathbf{a} = (y, -x)\,;$$

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	14	11	11	11
0.1	14	12	11	11
0.01	10	10	10	10
0.001	12	12	12	11
1e-04	14	14	14	14
1e-05	14	14	14	14
1e-06	14	14	14	14
1e-07	14	14	14	14

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	44	9	9	9
0.1	29	12	11	11
0.01	18	17	17	16
0.001	22	21	20	22
1e-04	28	23	26	28
1e-05	35	25	33	33
1e-06	44	26	30	31
1e-07	40	25	30	32

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	125	8	8	7
0.1	66	12	12	10
0.01	40	31	30	26
0.001	49	45	40	43
1e-04	73	60	69	68
1e-05	87	71	81	85
1e-06	87	68	72	89
1e-07	88	69	78	89

TABLE 4

Variable flow field. *FETI method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for FETI–1 (first columns) and FETI–2 with different choices of the scaling matrix* Q*. Cases of* 16 *(first table),* 64 *(second table), and* 256 *(third table) substructures, with meshes consisting of* 1792*,* 1920*, and* 1984 *elements, respectively.*

see Figure 3. We consider the following Dirichlet conditions on $\partial\Omega$:

$$
\begin{aligned}
u = 1 \quad & \begin{cases} y = -1,\ 0 < x \leq 1, \\ y = 1,\ 0 < x \leq 1, \\ x = 1,\ -1 \leq y \leq 1, \end{cases} \\
u = 0, \quad & \text{elsewhere,}
\end{aligned}
$$

and the values $f = 0$ and $c = 10^{-4}$. The solution for $\nu = 0.01$ is shown in Figure 3. As also noted in [2], this case is quite difficult, since even in the convection–dominated regime, diffusion is the only mechanism that propagates the information from the boundary of the domain into its center.

In Table 7, we show the number of iterations for FETI–1 and FETI–2 for different choices of Q, versus the viscosity ν. As before, we consider partitions into 4×4, 8×8, and 16×16 substructures.

elem.	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 1e-04$	$\nu = 1e-05$	$\nu = 1e-06$
480	43	29	22	34	44	42	42
1920	44	29	18	22	28	35	44
7680	46	30	19	19	22	28	30
30720	48	32	21	17	20	24	25

elem.	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 1e-04$	$\nu = 1e-05$	$\nu = 1e-06$
480	8	12	20	27	37	37	37
1920	9	12	17	21	23	24	25
7680	11	13	17	18	20	25	25
30720	12	15	18	16	20	23	25

TABLE 5

Variable flow field. *FETI method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for different meshes, for FETI–1 (first table) and FETI–2 with* $Q = I$ *(second table). Case of* 64 *substructures.*

	16 substr.		64 substr.		256 substr.	
ν	1 lev.	2 lev.	1 lev.	2 lev.	1 lev.	2 lev.
1	14	11	45	9	127	7
0.1	14	12	29	12	66	9
0.01	10	10	19	17	41	23
0.001	12	12	23	21	50	42
1.e-04	14	14	28	24	73	60
1.e-05	15	15	34	27	86	74
1.e-06	15	15	38	27	88	93
1.e-07	15	15	36	27	88	85

TABLE 6

Variable flow field. *Robin–Robin method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for the one–level (first columns) and the two–level algorithm (second columns). Cases of* 16*,* 64*, and* 256 *substructures, with meshes consisting of* 1792*,* 1920*, and* 1984 *elements, respectively.*

For the diffusion–dominated case, we observe a similar behavior as in the previous test cases, where a coarse space is quite useful if the number of subdomains is large; see Tables 1 and 4. For the 16 and 64 subdomains cases, the numbers of iterations still appear to be bounded as the viscosity becomes small, but the numbers in this case are larger than previously. In addition, when the number of subdomains becomes quite large, our methods exhibit poor convergence properties for convection–dominated cases; cf. the 256 subdomains case.

We also note that the FETI–2 methods present some improvement with respect to FETI–1, and that the best choice for the scaling matrix Q appears to be Q_2 for this test case. We recall that the best choice for the previous test case is $Q = I$.

In Table 8, we present some results for the same meshes considered in the previous test cases. For the case of 64 substructures and FETI–1 and FETI–2 with $Q = Q_2$, we show the number of iterations versus the viscosity ν, for four different meshes. The iteration counts still appear to be bounded as the mesh–size becomes smaller, for $\nu \geq 0.001$. However, they do not appear to be uniformly bounded, when the viscosity

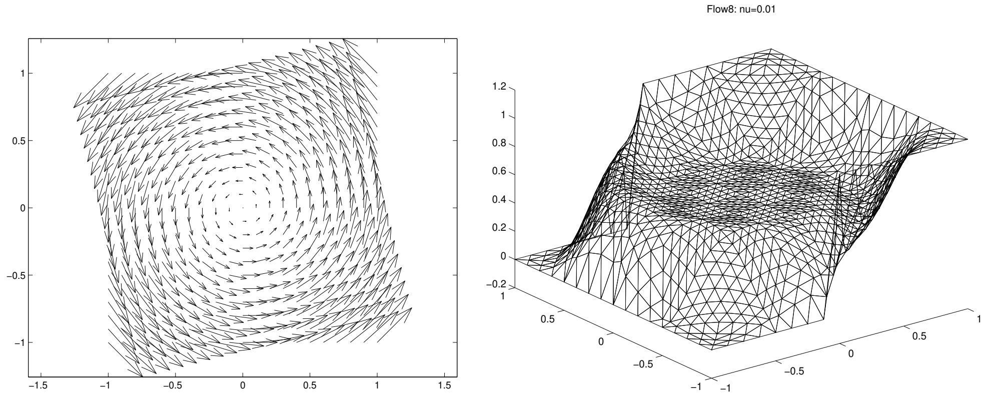

FIG. 3. **Rotating flow field**. *The flow field (left) and the solution for* $\nu = 0.01$ *(right).*

decreases further.

We finally report some results obtained with the Robin–Robin algorithms in [2]. For the same test problem, Table 9 shows the number of iterations versus the viscosity ν. By comparing the values in Tables 7 and 9, we observe that also for this test problem, the FETI and the Robin–Robin methods give similar iteration counts, except for the two–level algorithms for the cases of 256 substructures and $\nu \leq 10^{-4}$, where the two–level FETI algorithms require more than 200 iterations.

5. Conclusions. We have considered one– and two–level FETI methods for the solution of linear systems arising from the finite element approximation of scalar advection–diffusion problems. Using ideas similar to those of a recently developed Neumann–Neumann algorithm, we have modified the local problems by adding certain boundary terms, which have the effect of making the local problems stable.

We have presented some numerical arguments for three significant test problems and offer the following conclusions.

- For flows without closed streamlines, both the one– and two–level methods are optimal and the number of iterations tends to a constant value as the viscosity becomes small. The number of iterations increases with the number of subdomains, but our numerical results remain very good even for very large numbers of subdomains. In addition, for parallel flows, a coarse space correction does not appear to be necessary.
- For rotating flows, our methods appear to be optimal if the viscosity is not too small. For a fixed mesh, the number of iterations tends to a constant value as the viscosity becomes small, if the number of subdomains is not too large. A coarse space correction improves convergence for convection–dominated problems.
- In our experiments, we obtain iteration counts that are very similar to those for the corresponding Robin–Robin methods in [2], and the two families of methods have similar computational costs.
- For the two–level FETI method, a suitable scaling matrix can improve the convergence in some cases, and the best choice of such a matrix appears to depend on the particular flow considered.

Acknowledgments. The author is grateful to Axel Klawonn and Olof Widlund for enlightening discussions of his work.

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	14	11	11	11
0.1	16	12	12	12
0.01	28	23	22	22
0.001	50	45	45	50
1e-04	75	69	72	68
1e-05	83	77	80	76
1e-06	85	79	81	77

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	44	10	9	9
0.1	34	11	11	10
0.01	45	27	24	23
0.001	82	68	62	60
1e-04	124	111	105	99
1e-05	135	121	113	109
1e-06	138	123	115	111

ν	no cs	$Q = I$	$Q = Q_1$	$Q = Q_2$
1	119	9	8	8
0.1	67	16	13	12
0.01	95	63	51	32
0.001	>200	176	174	144
1e-04	>200	>200	>200	>200
1e-05	>200	>200	>200	>200
1e-06	>200	>200	>200	>200

TABLE 7

Rotating flow field. *FETI method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for FETI–1 (first columns) and FETI–2 with different choices of the scaling matrix* Q*. Cases of* 16 *(first table),* 64 *(second table), and* 256 *(third table) substructures, with meshes consisting of* 1792*,* 1920*, and* 1984 *elements, respectively.*

elem.	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 1e-04$	$\nu = 1e-05$	$\nu = 1e-06$
480	43	33	39	73	86	92	92
1920	44	34	45	82	124	135	138
7680	46	36	50	93	163	>200	>200
30720	48	38	53	93	>200	>200	>200

elem.	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 1e-04$	$\nu = 1e-05$	$\nu = 1e-06$
480	7	10	25	59	77	82	84
1920	9	10	23	60	99	109	111
7680	11	12	24	63	134	191	195
30720	12	14	26	70	182	>200	>200

TABLE 8

Rotating flow field. *FETI method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}*, versus the viscosity* ν*, for different meshes, for FETI–1 (first table) and FETI–2 with* $Q = Q_2$ *(second table). Case of* 64 *substructures.*

	16 substr.		64 substr.		256 substr.	
ν	1 lev.	2 lev.	1 lev.	2 lev.	1 lev.	2 lev.
1	14	10	44	9	122	7
0.1	15	12	34	11	66	10
0.01	25	22	44	24	93	33
0.001	48	43	80	58	>200	106
1.e-04	74	65	121	93	>200	183
1.e-05	82	74	134	103	>200	193
1.e-06	83	75	136	105	>200	194

TABLE 9

Rotating flow field. *Robin–Robin method. Number of GMRES iterations to decrease the residual norm by a factor* 10^{-6}, *versus the viscosity* ν, *for the one–level (first columns) and the two–level algorithm (second columns). Cases of* 16, 64, *and* 256 *substructures, with meshes consisting of* 1792, 1920, *and* 1984 *elements, respectively.*

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