POINCARÉ AND FRIEDRICHS INEQUALITIES FOR MORTAR FINITE ELEMENT METHODS

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Abstract. Mortar finite elements are nonconforming finite elements that allow for a geometrically nonconforming decomposition of the computational domain and, at the same time, for the optimal coupling of different variational approximations in different subregions. Poincaré and Friedrichs inequalities for mortar finite elements are derived. Using these inequalities, it is shown that the condition number for self-adjoint elliptic problems discretized using mortars is comparable to that of the conforming finite element case. Geometrically non-conforming mortars of the second generation are considered, i.e. no continuity conditions are imposed at the vertices of the subregions.

Key words. mortar finite elements, Poincaré and Friedrichs inequalities, elliptic finite element methods, condition number

AMS(MOS) subject classifications. 65N30, 65N55

1. Introduction.

Mortar finite elements are nonconforming finite elements that allow for nonconforming decomposition of the computational domain into subregions and for the optimal coupling of different variational approximations in different subregions. Here, by optimality we mean that the global error is bounded by the sum of the local approximation errors on each subregion. Because of these features, the mortar elements can be used effectively in solving large classes of problems.

The mortar finite element methods were first introduced by Bernardi, Maday, and Patera in [9], for low-order and spectral finite elements. A three dimensional version was developed by Ben Belgacem and Maday in [7], and was further analyzed for three dimensional spectral elements in [6].

Mortar finite elements have been shown to perform as well as the conforming finite elements in many numerical algorithms. For domain decomposition methods, see Achdou, Maday, and Widlund [3] and Dryja [12, 13] for iterative substructuring methods, and Widlund [15] for additive Schwarz algorithms; for other studies of preconditioners for the mortar method see Casarin and Widlund [11] for a hierarchical preconditioner, Achdou, Kuznetsov, and Pironneau [2], and Achdou, and Kuznetsov [1] for substructuring preconditioners. For the use of mortars for the Navier-Stokes equations, see Achdou and Pironneau [4, 5], and for multigrid methods for mortars, see Braess, Dahmen, and Wieners [10].

We first briefly describe the mortar finite element space V^h, restricting our discussion to the two dimensional case. The computational domain Ω is decomposed into a nonoverlapping polygonal partition $\{\Omega_k\}_{k=\overline{1:K}}$. The partition is geometrically conforming if the intersection between the closure of any two subregions is either empty, a vertex, or an entire edge, and it is nonconforming otherwise. We may also use subregions with curved boundaries, but here we discuss only the polygonal case, since the extension is straightforward. The restriction of V^h to any subregion Ω_k is a conforming finite element space. Across the interface Γ, i.e. the set of points that belong to the boundaries of at least two subregions, pointwise continuity is not required. We partition Γ into a union of nonoverlapping edges of the subregions $\{\Omega_k\}_{k=\overline{1:K}}$, called

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nonmortars. The edges not chosen to be nonmortars, are called mortars. On the two sides of an edge which coincides with a nonmortar, there are two distinct traces of the mortar functions. We only require that the difference of these two traces is L^2-orthogonal to spaces of test functions defined on the nonmortar.

In this paper, we prove Poincaré and Friedrichs inequalities for mortar finite elements. The constants in these inequalities depend only on the diameter of Ω, and neither on the properties of the partition $\{\Omega_k\}_{k=\overline{1:K}}$, nor on those of the mortar finite element. Using this result, we prove, in section 6, that the condition number of the unpreconditioned mortar finite element method has the same upper bound as in the continuous finite element case, and does not depend on the number of the subregions in the partition of Ω. This is a refinement of a result of Bernardi, Maday, and Patera [9]. There, a Friedrichs inequality is proven using the Rellich compactness theorem, which leads to an estimate of the condition number which depends on the number of the subregions in the partition $\{\Omega_k\}_{k=\overline{1:K}}$ and their diameters.

In the geometrically conforming case, a variant of the Friedrichs inequality for mortars was proven by Bernardi and Maday [8].

The rest of the paper is structured as follows. In the next section, we introduce the elliptic problem and describe the mortar finite element method in greater detail. In Section 3, we present some technical tools. Our most important auxiliary result is proved in Section 4. In Section 5, we prove Poincaré and Friedrichs inequalities for the mortar functions, and in Section 6, we prove an estimate of the the condition number of the mortar stiffness matrices. We conclude our paper by extending, in Section 7 our results for the case of general polygonal partition.

2. The Elliptic Problem and the Mortar Finite Element Method. To keep the presentation simple, our model problem will be Poisson's equation with Dirichlet boundary conditions on Ω, a bounded open polygon in R^2. Our results can also be obtained, using the same methods, for any second order self-adjoint elliptic problems with mixed boundary conditions, and for the three dimensional case.

2.1. Partition of the region into subregions. We partition Ω into K open, nonoverlapping, shape regular, polygonal subregions:

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k, \qquad \Omega_i \bigcap \Omega_j = \emptyset \quad \text{if} \quad 1 \le i \ne j \le K.$$

This partition may be geometrically nonconforming. If an edge of the polygons $\{\Omega_k\}_{k=\overline{1:K}}$ intersects the boundary $\partial\Omega$ at an interior point of the edge, we require that the entire edge belongs to $\partial\Omega$. Let H_k be the diameter of Ω_k, h_k the smallest diameter of any of the elements of Ω_k, and $h = \min h_k$, $k = \overline{1:K}$. We do not require that all the H_k are of the same order of magnitude, but only require that the diameters of any two adjacent subregions Ω_r and Ω_s (i.e. $\overline{\Omega}_r \cap \overline{\Omega}_s \ne \emptyset$) are comparable, i.e. $c \le H_r/H_s \le C$, where c and C are positive constants independent of the subregions considered. We assume that all the subregions are generated from a reference domain $\widehat{\Omega}$ by mappings F_k, such that $\Omega_k = F_k(\widehat{\Omega})$, and

$$||\partial F_k|| \le CH_k, \quad \forall k = \overline{1:K}; \quad ||\partial F_k^{-1}|| \le CH_k^{-1}, \quad \forall k = \overline{1:K}.$$

As a consequence, we note that the length of every side of Ω_k is bounded from below by a uniform fraction of H_k.

2.2. The mortar finite elements. To introduce the space V^h of mortar finite element functions over Ω, we need some additional definitions and notations. Let $V^h(S)$ be the restriction of V^h to a set S. The interface between the subregions $\{\Omega_k\}_{k=\overline{1:K}}$, denoted by Γ, is defined as the closure of the union of the parts of $\{\partial\Omega_k\}_{k=\overline{1:K}}$ that are interior to Ω:

$$\Gamma = \overline{\cup_{k=1}^{K}(\partial\Omega_k \setminus \partial\Omega)}.$$

A set of mortars $\{\zeta_m\}_{m=1}^{M}$ is obtained by selecting open edges of the subregions $\{\Omega_k\}_{k=\overline{1:K}}$, such that

$$\Gamma = \bigcup_{m=1}^{M} \overline{\zeta}_m, \quad \zeta_m \cap \zeta_n = \emptyset \text{ if } 1 \le m \ne n \le M.$$

This partition is not unique, but any choice can be treated the same from a theoretical point of view. The edges of $\{\Omega_k\}_{k=\overline{1:K}}$ that are part of Γ and not chosen to be mortars are called nonmortars and are denoted by $(\gamma_l)_{l=\overline{1:L}}$. We note that the nonmortars also cover the interface:

$$\Gamma = \bigcup_{l=1}^{L} \overline{\gamma}_l, \quad \gamma_l \cap \gamma_n = \emptyset \text{ if } 1 \le l \ne n \le L,$$

and that each nonmortar γ_l belongs to exactly one subregion, denoted by $\Omega_{i(l)}$. Let Γ_l be the union of the parts of the mortars that coincides geometrically with $\overline{\gamma}_l$:

$$\Gamma_l = \bigcup_{i=1}^{q(\gamma_l)} (\overline{\zeta}_{l,i} \cap \overline{\gamma}_l). \tag{1}$$

For each γ_l, we construct a space of test functions $\Psi^h(\gamma_l)$, which is a subspace of codimension two of $V^h(\gamma_l)$, the restriction of $V^h(\Omega_{i(l)})$ to $\overline{\gamma}_l$. Thus, when the space $V^h(\Omega_{i(l)})$ is piecewise linear, i.e. P_1 or Q_1, $\Psi^h(\gamma_l)$ is given by the restriction of $V^h(\Omega_{i(l)})$ to $\overline{\gamma}_l$, subject to the constraints that these continuous, piecewise linear functions are constant in the first and last mesh intervals of $\overline{\gamma}_l$.

The mortar projection on γ_l is defined on $L^2(\Gamma_l)$ and takes values in $V^h(\gamma_l)$. For two arbitrary values q_1 and q_2, and for $u_l \in L^2(\Gamma_l)$, the function $\pi_{q_1,q_2}(u_l) \in V^h(\gamma_l)$ equals q_1 and q_2 at the two end points of γ_l, and satisfies

$$\int_{\gamma_l} (u_l - \pi_{q_1,q_2}(u_l))\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma_l). \tag{2}$$

We are now able to define the mortar finite element space V^h fully. The restriction of V^h to Ω_k is a conforming finite element space. For any mortar function $v \in V^h$, we impose $v_{|\partial\Omega} = 0$, since we have assumed zero Dirichlet boundary conditions for our Poisson problem. We also require v to satisfy the mortar conditions for each nonmortar γ_l, i.e. $v_{|\overline{\gamma}_l}$ is equal to the mortar projection of $v_{|\Gamma_l}$. This allows the values of v at the end points of $\overline{\gamma}_l$ (denoted by A_l and B_l) to be genuine degrees of freedom:

$$v_{|\overline{\gamma}_l} = \pi_{v_l(A_l),v_l(B_l)}(v_{|\Gamma_l}).$$

2.3. The discrete mortar problem. As in other nonconforming methods, we work with a bilinear form $a^\Gamma(\cdot,\cdot)$ defined as the sum of contributions from the individual subregions:

$$a^\Gamma(v_h, w_h) = \sum_{k=1}^{K} a_k(v_h, w_h),$$

where $a_k(v_h, w_h) = \int_{\Omega_k} \nabla v_h \cdot \nabla w_h$. For $w_h = v_h$, we obtain the square of what is often called a broken norm: $a^\Gamma(v_h, v_h) = \sum_{k=1}^{K} |v_h|^2_{H^1(\Omega_k)}$.

The discrete problem is then:

$$\text{Find } u_h \in V^h \text{ such that } a^\Gamma(u_h, v_h) = f^\Gamma(v_h), \quad \forall\, v_h \in V^h, \tag{3}$$

where $f^\Gamma(v_h)$ is a sum of contributions from all the subregions $\{\Omega_k\}_{k=\overline{1:K}}$:

$$f^\Gamma(v_h) = \sum_{k=1}^{K} f_k(v_h), \quad f_k(v_h) = \int\limits_{\Omega_k} f\ v_h\ dx.$$

The existence and uniqueness of the solution of problem (3) is a consequence of the Lax-Milgram lemma, as soon as we have proven the coercivity of the broken norm with respect to the L^2 norm:

$$c||v_h||^2_{L^2(\Omega)} \;\le\; a^\Gamma(v_h, v_h), \quad \forall v_h \in V^h,$$

with a constant $c > 0$. This bound is a consequence of the Friedrichs inequality; cf. Theorem 5.1:

$$||v||^2_{L^2(\Omega)} \;\le\; C(\mathrm{diam}(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)}, \quad \forall v \in V^h \cap H^1_0(\Omega).$$

3. Technical tools. We begin with a version of the Friedrichs inequality on a reference subregion $\widehat{\Omega}$.

Lemma 1. *Let $\widehat{\Omega} \subset R^2$ be a fixed, open, bounded domain with Lipschitz boundary. Let $c_0 > 0$ and let $\widehat{\Lambda} \subset \partial\widehat{\Omega}$ be a part of the boundary of $\widehat{\Omega}$ such that*

$$c_0 \mu(\partial\widehat{\Omega}) \;\le\; \mu(\widehat{\Lambda}),$$

where μ is the Lebesgue measure. Then,

$$||w||^2_{L^2(\widehat{\Omega})} \le C\Big(\ |w|^2_{H^1(\widehat{\Omega})} \ +\ \frac{1}{c_0}|\int_{\widehat{\Lambda}} w d\sigma|^2\ \Big), \quad \forall\ w \in H^1(\widehat{\Omega}),$$

where C is a constant that depends only on $\widehat{\Omega}$, and not on w, $\widehat{\Lambda}$, or c_0.

We also need a generalized version of the Friedrichs inequality. We note that Lemma 1 follows from Lemma 2.

Lemma 2. *Let $\widehat{\Omega} \subset R^2$ be a fixed, open, bounded domain with a Lipschitz boundary, let $c_0 \in (0,1)$ be a constant, and let $\widehat{\Lambda} \subset \partial\widehat{\Omega}$ such that*

$$c_0 \mu(\partial\widehat{\Omega}) \;\le\; \mu(\widehat{\Lambda}). \tag{4}$$

Let $\widehat{\psi}$ be a bounded positive function defined on $\widehat{\Lambda}$ with the following properties:

$$(5) \qquad 0 \le \widehat{\psi} \le 2 \quad \text{and} \quad \frac{1}{2}\mu(\widehat{\Lambda}) \;\le\; \mu(\{x \in \widehat{\Lambda} \;:\; \widehat{\psi}(x) \ge 1\}).$$

Then, the following inequality holds,

$$(6) \quad ||w||^2_{L^2(\widehat{\Omega})} \;\le\; C\,\Big(\;(diam(\widehat{\Omega}))^2 |w|^2_{H^1(\widehat{\Omega})} \;+\; \frac{1}{c_0^2}\Big|\int_{\widehat{\Lambda}} w\widehat{\psi}d\sigma\Big|^2\;\Big),\quad \forall w \in H^1(\widehat{\Omega}),$$

where C is a constant independent of c_0, w, $\widehat{\Lambda}$, and $\widehat{\psi}$.

Proof. We may assume that $\text{diam}(\widehat{\Omega}) = 1$. The general inequality is obtained easily by a scaling argument.

Suppose that Lemma 2 is not true. Then there exists a sequence $\{w_n\}_{n=\overline{1:\infty}}$ of functions in $H^1(\widehat{\Omega})$, a sequence of boundary parts $(\widehat{\Lambda}_n)_{n=\overline{1:\infty}}$ satisfying (4), and a sequence of functions $(\widehat{\psi}_n)_{n=\overline{1:\infty}}$ defined on $(\widehat{\Lambda}_n)_{n=\overline{1:\infty}}$ and satisfying property (5), such that

$$(7) \qquad ||w_n||_{L^2(\widehat{\Omega})} \;=\; 1,\;\; \forall\, n = \overline{1:\infty};$$

$$(8) \qquad |w_n|^2_{H^1(\widehat{\Omega})} \;+\; \Big(\frac{1}{c_0^2}\int_{\widehat{\Lambda}_n} w_n\widehat{\psi}_n d\sigma\Big)^2 \;\le\; \frac{1}{n},\;\; \forall\, n = \overline{1:\infty}.$$

For $n \to \infty$, w_n converges to 0 in the $H^1(\widehat{\Omega})$-seminorm. Therefore, the sequence $\{w_n\}$ is bounded in $H^1(\widehat{\Omega})$, and we obtain, from Rellich's theorem, the existence of a subsequence of $\{w_n\}$ that converges in the $L^2(\widehat{\Omega})$ norm. For simplicity, we also denote this subsequence by $\{w_n\}$. Since $|w_n|^2_{H^1(\widehat{\Omega})} \to 0$, we also have convergence of $\{w_n\}$ in the $H^1(\widehat{\Omega})$ norm. The limit function is a constant function, $\tilde{c}$.

From (8), we know that $|\int_{\widehat{\Lambda}_n} w_n\widehat{\psi}_n d\sigma| \to 0$. Since $w_n \to \tilde{c}$ in $H^1(\widehat{\Omega})$, and the functions $\widehat{\psi}_n$ are uniformly bounded by assumption, we obtain, using a trace theorem, that

$$(9) \qquad \Big|\int_{\widehat{\Lambda}_n} \widehat{\psi}_n d\sigma\Big|\;|\tilde{c}| \;\to\; 0.$$

From (4) and (5):

$$(10) \qquad \Big|\int_{\widehat{\Lambda}_n} \widehat{\psi}_n d\sigma\Big| \;\ge\; \frac{1}{2}\mu(\widehat{\Lambda}) \;\ge\; \frac{c_0}{2}\mu(\partial\widehat{\Omega}).$$

Finally, from (9) and (10), we obtain that $\tilde{c} = 0$, which implies $w_n \to 0$ in $L^2(\widehat{\Omega})$. This contradicts assumption (7), and the proof is completed. □

The next lemma is purely geometrical, and is a straightforward generalization of a result of Bernardi and Maday [8].

LEMMA 3. *Let Ω be a bounded domain in the plane and let $\{\Omega_k\}_{k=\overline{1:K}}$ be a shape regular partition of Ω, where Ω_k is a polygon of diameter H_k. Let ℓ be a line passing through Ω and let $(\Omega_{i,\ell})_{i=\overline{1:n(\ell)}}$ be the subregions with interiors intersecting ℓ. Then,*

$$\sum_{i=1}^{n(\ell)} H_{i,\ell} \;\le\; C\,diam(\Omega),$$

where C is a constant which depends only on the minimal angle of the polygonal subregions $\{\Omega_k\}_{k=\overline{1:K}}$, and not on their diameters $(H_k)_{k=\overline{1:K}}$.

4. An estimate of the L^2 norm of jumps across nonmortars. The lack of continuity of the mortar finite element functions prevents us from applying the standard Poincaré and Friedrichs inequalities on the domain Ω. As we will see in the next section, the most important step in the proofs of these inequalities is an estimate of the L^2 norm of the jump of the mortar finite element function v over the nonmortars.

We restrict the technical discussion of this section to the case when all the subregions $\{\Omega_k\}_{k=\overline{1:K}}$ are rectangles. In the last section, we explain how our results can be extend ed for a polygonal partition.

We need to introduce additional notations. Let γ be a nonmortar side of the subregion Ω_l, and let v_l be the restriction of the mortar function v to Ω_l. Since the partition of Ω can be geometrically nonconforming, we can have a union of parts of several mortars $\zeta_{l,i}$, $i = \overline{1 : q(\gamma)}$ opposite γ across the interface Γ; cf. (1). We order these segments from left to right. Let $\Omega_{l,i}$ be the subregion which has $\zeta_{l,i}$ as a side, and let

$$\delta_{i,i+1} = \partial\Omega_{l,i} \cap \partial\Omega_{l,i+1}, \quad \forall\ i = \overline{1 : (q(\gamma) - 1)}.$$

Since every subregion $\Omega_{l,i}$ has a diameter on the order of H_l, $q(\gamma)$ is uniformly bounded by a constant C which depends only on the lower and upper bounds of the ratios of the diameters of adjacent subregions.

Let $\tilde{v}$ be the function that is equal to $v_{l,i}$ (the restriction of v to $\Omega_{l,i}$) on $\zeta_{l,i}$. Note that $\tilde{v}$ can have two values at the vertices on the interface Γ that are interior to γ.

LEMMA 4. *Let $[v] = v_l - \tilde{v}$ be the jump of v across the nonmortar γ. Then,*

$$\int_\gamma [v]^2 d\sigma \ \le\ C\ \mu\ (\gamma)\Big(\ |v_l|^2_{H^1(\Omega_l)}\ +\ \sum_{i=1}^{q(\gamma)} |v_{l,i}|^2_{H^1(\Omega_{l,i})}\ \Big), \tag{11}$$

where C is a constant that does not depend on γ.

Proof. By definition:

$$\int_\gamma [v]^2\, d\sigma = \int_\gamma |v_l - \tilde{v}|^2\, d\sigma. \tag{12}$$

The functions v_l and $\tilde{v}$ satisfy the mortar conditions (2).

Since any space of test functions $\Psi^h(\gamma)$ contains the constant functions, and since the functions v_l and $\tilde{v}$ satisfy the mortar conditions (2), the averages of v_l and $\tilde{v}$ over γ are equal:

$$\int_\gamma v_l d\sigma = \int_\gamma \tilde{v} d\sigma = \overline{v}_\gamma \mu(\gamma).$$

Opposite Ω_l, we construct a rectangle Ω_{new} with one side equal to γ and with sides of length $\min\{\mu(\delta_{i,i+1})\}$, $i = 1 : (q(\gamma) - 1)$, perpendicular to γ.

Let v_{new} be an extension of $\tilde{v}$ from γ to Ω_{new} with the following properties:

$$v_{new} \in H^1(\Omega_{new}), \tag{13}$$

$$\int_\gamma v_{new} d\sigma\ =\ \int_\gamma \tilde{v} d\sigma\ =\ \overline{v}_\gamma \mu(\gamma). \tag{14}$$

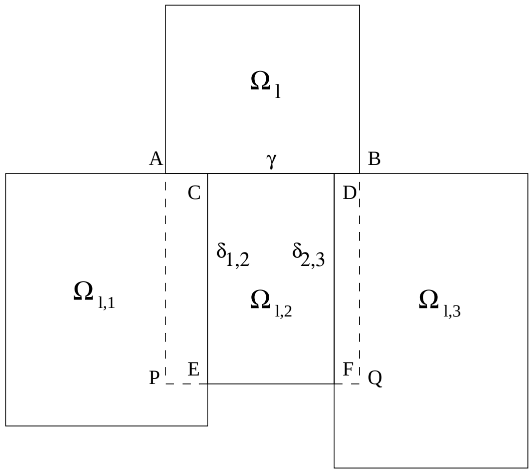

FIG. 1. *Local nonconforming situation*

Note that v_{new} need not be a finite element function, and its trace on γ need not be equal to $\tilde{v}$. We just require that the average of v_{new} over γ is equal to that of $\tilde{v}$.

We now provide the details of the construction of v_{new}. Without loss of generality, we can consider the case of only three rectangles $\Omega_{l,1}$, $\Omega_{l,2}$, $\Omega_{l,3}$. Let $v_{l,1}$, $v_{l,2}$, and $v_{l,3}$ be the restrictions of v to $\Omega_{l,1}$, $\Omega_{l,2}$, $\Omega_{l,3}$, respectively. These functions may be discontinuous across $\delta_{1,2}$ and $\delta_{2,3}$; cf. Figure 1, where $\Omega_{new} = ABQP$, $\delta_{1,2} = CE$ and $\delta_{2,3} = DF$.

For every segment $\delta_{i,i+1}$, we construct in one of the rectangles $\Omega_{l,i}$ or $\Omega_{l,i+1}$ (say on $\Omega_{l,i}$) a function $\chi_i \in H^1(\Omega_{l,i})$ such that χ_i is equal to $v_{l,i+1} - v_{l,i}$ on $\delta_{i,i+1}$ and vanishes on the side opposite $\delta_{i,i+1}$

The choice of whether to construct the function χ_i on $\Omega_{l,i}$ or $\Omega_{l,i+1}$ is made according to which of $\partial\Omega_{l,i} \cap \gamma$ and $\partial\Omega_{l,i+1} \cap \gamma$ is the largest. If $\mu(\partial\Omega_{l,i} \cap \gamma) \geq \mu(\partial\Omega_{l,i+1} \cap \gamma)$, we choose subregion $\Omega_{l,i}$; otherwise $\Omega_{l,i+1}$. As a consequence, since the subregions $\Omega_{l,i}$ and $\Omega_{l,i+1}$ have diameters on the order of H_l, we find that the length of the intersection of γ with the boundary of the chosen domain is on the order of H_l.

This choice avoids one of the problems that occur in the study of the geometrically nonconforming case, namely the existence of small intersections of the boundaries of two subregions from the partition $\{\Omega_k\}_{k=\overline{1:K}}$. Such a configuration can appear naturally, e.g. from a small perturbation of a geometrically conforming partition.

In our case, we construct the functions χ_2 and χ_3 as follows:

$$
\begin{aligned}
\chi_2 : \Omega_{l,2} \to R; \quad \chi_{2|\Omega_{new}\cap\delta_{1,2}} &= (v_{l,1} - v_{l,2})_{|\Omega_{new}\cap\delta_{1,2}};\\
\chi_{2|\Omega_{new}\cap\delta_{2,3}} &= 0;\\
\int_{CD} \chi_2 \, d\sigma &= 0;\\
\chi_3 : \Omega_{l,2} \to R; \quad \chi_{3|\Omega_{new}\cap\delta_{2,3}} &= (v_{l,3} - v_{l,2})_{|\Omega_{new}\cap\delta_{2,3}};\\
\chi_{3|\Omega_{new}\cap\delta_{1,2}} &= 0;\\
\int_{CD} \chi_3 \, d\sigma &= 0.
\end{aligned}
$$

For this purpose, we use extension and trace theorems on the unit square $\widehat{\Omega}$; see, e.g. Nečas [14].

Let $\hat{s}$, $\hat{s}_1$, $\hat{s}_2$, and $\hat{s}_3$ be the sides of $\widehat{\Omega}$, in consecutive order. If $\hat{\psi} \in H^{\frac{1}{2}}(\hat{s})$, we can, using several reflections, extend it to a function $E(\hat{\psi}) \in H^{\frac{1}{2}}(\partial\widehat{\Omega})$ such that

$$||E(\hat{\psi})||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \leq C||\hat{\psi}||_{H^{\frac{1}{2}}(\hat{s})}.$$

Let ϕ_1 and ϕ_2 be positive $C^\infty(\partial\widehat{\Omega})$ functions with the following properties: ϕ_1 is 1 on $\hat{s}$ and 0 on $\hat{s}_2$ (the side opposite to $\hat{s}$ in the rectangle $\widehat{\Omega}$) and is bounded from above by 1 on $\partial\widehat{\Omega}$; ϕ_2 is supported in $\hat{s}_1$, and $\int_{\hat{s}_1} \phi_2 dx = 1$. The function

$$E_0(\hat{\psi}) \;=\; \phi_1 E(\hat{\psi}) - \phi_2 \int_{\hat{s}_1} \phi_1 E(\hat{\psi}) dx$$

is an extension of $\hat{\psi}$ satisfying

$$\begin{aligned} E_0(\hat{\psi})_{|\hat{s}} = \hat{\psi}; \qquad & E_0(\hat{\psi})_{|\hat{s}_2} = 0; \\ ||E_0(\hat{\psi})||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \;&\leq\; C||\hat{\psi}||_{H^{\frac{1}{2}}(\hat{s})}; \\ \int_{\hat{s}_1} E_0(\hat{\psi}) d\sigma \;&=\; 0. \end{aligned}$$

Since $E_0(\hat{\psi}) \in H^{\frac{1}{2}}(\partial\widehat{\Omega})$, there exists a harmonic extension of $E_0(\hat{\psi})$ to the unit square $\widehat{\Omega}$, $\hat{u} \in H^1(\widehat{\Omega})$, which does not have to be a finite element function, such that

$$|\hat{u}|_{H^1(\widehat{\Omega})} \leq C||E_0(\hat{\psi})||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \leq C||\hat{\psi}||_{H^{\frac{1}{2}}(\hat{s})}.$$

There exists a diffeomorphism $F : \widehat{\Omega} \to \Omega_{l,2}$ that induces a natural mapping from the functions defined on $\Omega_{l,2}$ into the functions defined on $\widehat{\Omega}$. We construct χ_2 in the following steps:

1. let $\hat{\psi}_2 := (v_{l,1} - v_{l,2})_{|\Omega_{new} \cap \delta_{1,2}} \circ F$;
2. let $\hat{u}_2$ be the extension of $E_0(\hat{\psi}_2)$ from $\partial\widehat{\Omega}$ to $\widehat{\Omega}$ described above;
3. let $\chi_2 := \hat{u}_2 \circ F^{-1}$.

From the properties of F and the extensions on $\widehat{\Omega}$, we obtain the following estimates for χ_2:

$$\begin{aligned} |\chi_2|_{H^1(\Omega_{l,2})} \;&\leq\; C|\hat{u}_2|_{H^1(\widehat{\Omega})} \;\leq\; C||E_0(\hat{\psi}_2)||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \;\leq\; C||\hat{\psi}_2||_{H^{\frac{1}{2}}(\hat{s})} \\ &=\; C||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}, \\ ||\chi_2||_{L^2(CD)} \;&\leq\; C\mu(CD)||\hat{u}_2||_{L^2(\partial\widehat{\Omega})} \;\leq\; C\mu(\gamma)||\hat{u}_2||_{L^2(\partial\widehat{\Omega})} \;\leq\; C\mu(\gamma)||\hat{u}_2||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \\ &=\; C\mu(\gamma)||E_0(\hat{\psi}_2)||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \;\leq\; C\mu(\gamma)||\hat{\psi}_2||_{H^{\frac{1}{2}}(\hat{s})} \\ &=\; C\mu(\gamma)||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}. \end{aligned}$$

Therefore the function χ_2 has the following properties:

$$\begin{aligned} \chi_{2|\Omega_{new} \cap \delta_{1,2}} \;&=\; (v_{l,1} - v_{l,2})_{|\Omega_{new} \cap \delta_{1,2}}; \\ \chi_{2|\Omega_{new} \cap \delta_{2,3}} \;&=\; 0; \end{aligned}$$

$$\int_{CD} \chi_2 d\sigma \;=\; 0; \tag{15}$$

$$|\chi_2|_{H^1(\Omega_{l,2})} \;\leq\; C||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}; \tag{16}$$

$$||\chi_2||_{L^2(CD)} \;\leq\; C\mu(\gamma)||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}. \tag{17}$$

The function χ_3 is constructed similarly.

Finally, v_{new} is defined as follows:

$$v_{new} = \begin{cases} v_{l,1} & \text{on } \overline{\Omega}_{l,1} \cap \overline{\Omega}_{new}; \\ v_{l,2} + \chi_2 + \chi_3 & \text{on } \overline{\Omega}_{l,2} \cap \overline{\Omega}_{new}; \\ v_{l,3} & \text{on } \overline{\Omega}_{l,3} \cap \overline{\Omega}_{new}. \end{cases} \tag{18}$$

Note that v_{new} is continuous by construction.

We now check that v_{new} satisfies (13) and (14). Since v_{new} is piecewise H^1 and continuous, we obtain that $v_{new} \in H^1(\Omega_{new})$, which means that v_{new} satisfies (13). For (14):

$$\begin{aligned} \int_\gamma v_{new} d\sigma &= \int_{AB} v_{new} d\sigma = \int_{AB} \tilde{v} d\sigma + \int_{CD} \chi_2 + \int_{CD} \chi_3 dx \\ &= \int_{AB} \tilde{v} d\sigma = \overline{v}_\gamma \mu(\gamma), \end{aligned}$$

where we have used (15), and the fact that χ_3 has properties similar to those of χ_2.

Using the construction of v_{new} and Ω_{new}, we begin the proof of (11):

$$\begin{aligned} \int_\gamma [v]^2 d\sigma &= \int_\gamma |v_l - \tilde{v}|^2 d\sigma \leq 2\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma + 2\int_\gamma |\tilde{v} - \overline{v}_\gamma|^2 d\sigma \\ &\leq 2\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma + 4\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma + 4\int_\gamma |\tilde{v} - v_{new}|^2 d\sigma. \end{aligned} \tag{19}$$

We now estimate the three terms of (19). For the first two, we use Lemma 1 and Lemma 2, both for the unit square $\hat{\Omega}$, which is the reference subregion. We will apply the Friedrichs inequality, trace theorems, and inverse inequalities only on the reference unit square $\hat{\Omega}$, since we look for results independent of the partition $\{\Omega_k\}_{k=\overline{1:K}}$ of Ω.

Estimate of $\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma$:

$$\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma = \mu(\gamma) \int_{\hat{s}} |\hat{v}_l - \overline{v}_\gamma|^2 d\sigma \leq \mu(\gamma) ||\hat{v}_l - \overline{v}_\gamma||^2_{L^2(\partial\hat{\Omega})} \leq C\mu(\gamma) ||\hat{v}_l - \overline{v}_\gamma||^2_{H^1(\hat{\Omega})},$$

where we have used a trace theorem for the last inequality. Since,

$$\int_{\hat{s}} (\hat{v}_l - \overline{v}_\gamma) d\sigma = 0,$$

we obtain from Lemma 1:

$$||\hat{v}_l - \overline{v}_\gamma||_{H^1(\hat{\Omega})} \leq C|\hat{v}_l|^2_{H^1(\hat{\Omega})}.$$

Combining the last two relations, we can bound the first term of (19) by the seminorm of the restriction of the mortar function v to Ω_l:

$$\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma \leq C\mu(\gamma)|\hat{v}_l|^2_{H^1(\hat{\Omega})} \leq C\mu(\gamma)|v_l|^2_{H^1(\Omega_l)}. \tag{20}$$

Estimate of $\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma$:

From (14) we obtain that $\int_\gamma (v_{new} - \overline{v}_\gamma) d\sigma = 0$. Applying the same method as for the first term, we get

$$\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma \leq C\mu(\gamma)|v_{new}|^2_{H^1(\Omega_{new})}. \tag{21}$$

The last inequality holds since, by assumption, the diameters of adjacent subregions are uniformly comparable. From the construction of v_{new}, we find:

$$(22) \quad |v_{new}|^2_{H^1(\Omega_{new})} \;\le\; 3\Big(\sum_{i=1}^{3} |v_{l,i}|^2_{H^1(\Omega_{l,i})}\Big) \;+\; 3|\chi_2|^2_{H^1(\Omega_{l,2})} \;+\; 3|\chi_3|^2_{H^1(\Omega_{l,2})}$$

The estimates of $|\chi_2|^2_{H^1(\Omega_{l,2})}$ and $|\chi_3|^2_{H^1(\Omega_{l,2})}$ are similar, so we derive only one of them. One of the sides of the subregions $\Omega_{l,1}$ and $\Omega_{l,2}$ that intersect $\Omega_{new} \cap \delta_{1,2}$ is a nonmortar side. For our proof it does not make any difference which it is, and we can assume that it is a side of $\Omega_{l,1}$. To this nonmortar will correspond a space of test functions $\Psi_{1,2}$. Denote by $\psi_{1,2} \in \Psi_{1,2}$ the test function that is equal to 1 at all the nodes of the nonmortar side that are also in $\Omega_{new} \cap \delta_{1,2}$ except for the last one, and is equal to 0 at all the other nodes. We replace $\Omega_{new} \cap \delta_{1,2}$ with CE. From the mortar condition (2), we obtain:

$$\int_{CE} v_{l,1}\psi_{1,2}\, d\sigma \;=\; \int_{CE} v_{l,2}\psi_{1,2}\, d\sigma.$$

Let α denote the following congruent terms:

$$\alpha \;=\; \frac{\int_{CE} v_{l,1}\psi_{1,2}\, d\sigma}{\int_{CE} \psi_{1,2}\, d\sigma} \;=\; \frac{\int_{CE} v_{l,2}\psi_{1,2}\, d\sigma}{\int_{CE} \psi_{1,2}\, d\sigma}.$$

Then it is easy to see that:

$$(23) \qquad \int_{CE} (v_{l,1} - \alpha)\psi_{1,2}\, d\sigma \;=\; \int_{CE} (v_{l,2} - \alpha)\psi_{1,2}\, d\sigma \;=\; 0.$$

Since the rectangles $(\Omega_{l,i})_{i=\overline{1:3}}$ are neighbors of Ω_l, their diameters are of the same order as H_l, the diameter of Ω_l. Moreover, CE has a length on the order of H_l since it is a side of the rectangle $\Omega_{l,2}$, and the following estimate holds,

$$c_0\mu(\partial\Omega_{l,1}) \;\le\; \tilde{c}_0 H_l \;\le\; \mu(CE).$$

Here, c_0 is a constant that does not depend on the subregion $\Omega_{l,1}$. It is easy to see, from the definition of $\psi_{1,2}$, that

$$\frac{1}{2}\mu(CE) \;\le\; \mu(\{x \in CE \;:\; \psi_{1,2}(x) \ge 1\}).$$

Therefore, conditions (4) and (5) of Lemma 2 are satisfied. Let $F : \widehat{\Omega} \to \Omega_{l,2}$ be the diffeomorphism between $\Omega_{l,2}$ and the reference unit square $\widehat{\Omega}$. The induced mapping takes a function u defined on $\Omega_{l,2}$ into the function $\hat{u} = u \circ F$ defined on $\widehat{\Omega}$. It is easy to see that conditions (4) and (5) are also satisfied on $\widehat{\Omega}$. From Lemma 2, we obtain:

$$(24)\ ||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^1(\widehat{\Omega})} \le C\left(|\hat{v}_{l,1}|^2_{H^1(\widehat{\Omega})} \;+\; \frac{1}{c_0^2}\left|\int_{\hat{s}_1} (\hat{v}_{l,1} - \hat{\alpha})\hat{\psi}_{1,2}\, d\sigma\right|^2\right) = C|\hat{v}_{l,1}|^2_{H^1(\widehat{\Omega})},$$

since, by (23), the integral over $\hat{s}_1$ vanishes. Using a trace theorem, inequalities of Sobolev norms on affine equivalent domains, and (24), we obtain:

$$\begin{aligned} ||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} \;&\le\; C||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \;\le\; C||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^1(\widehat{\Omega})} \;\le\; C|\hat{v}_{l,1}|^2_{H^1(\widehat{\Omega})} \\ &\le\; C|v_{l,1}|^2_{H^1(\Omega_{l,1})}. \end{aligned}$$

Once again, C is a constant that does not depend on the subregion $\Omega_{l,1}$. A similar estimate holds for $v_{l,2}-\alpha$. Therefore,

$$
\begin{aligned}
(25)\quad ||\hat{v}_{l,1}-\hat{v}_{l,2}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} &\leq 2||\hat{v}_{l,1}-\hat{\alpha}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} + 2||\hat{v}_{l,2}-\hat{\alpha}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} \\
&\leq C\Big(|v_{l,1}|^2_{H^1(\Omega_{l,1})}+|v_{l,2}|^2_{H^1(\Omega_{l,2})}\Big).
\end{aligned}
$$

From (16) and (25), we obtain,

$$
|\chi_2|^2_{H^1(\Omega_{l,2})} \leq C\Big(|v_{l,1}|^2_{H^1(\Omega_{l,1})}+|v_{l,2}|^2_{H^1(\Omega_{l,2})}\Big).
$$

Since the estimate of χ_3 is similar, we obtain, using (22), an estimate of v_{new},

$$
|v_{new}|^2_{H^1(\Omega_{new})} \leq C\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})}. \tag{26}
$$

From (21) and (26),

$$
\int_\gamma |v_{new}-\overline{v}_\gamma|^2 d\sigma \leq C\mu(\gamma)\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})}. \tag{27}
$$

Estimate of $\int_\gamma |\tilde{v}-v_{new}|^2 d\sigma$:
From the definitions of $\tilde{v}$ and v_{new}, we find

$$
\int_\gamma |\tilde{v}-v_{new}|^2 d\sigma = \int_{CD}|\chi_2+\chi_3|^2 d\sigma \leq 2||\chi_2||^2_{L^2(CD)}+2||\chi_3||^2_{L^2(CD)}.
$$

From (17) and (25), we obtain,

$$
||\chi_2||^2_{L^2(CD)} \leq C\mu(\gamma)||\hat{v}_{l,1}-\hat{v}_{l,2}||^2_{H^{\frac{1}{2}}(\Omega_{new}\cap\delta_{1,2})} \leq C\Big(|v_{l,1}|^2_{H^1(\Omega_{l,1})}+|v_{l,2}|^2_{H^1(\Omega_{l,2})}\Big),
$$

and therefore,

$$
\int_\gamma |\tilde{v}-v_{new}|^2 d\sigma \leq C\mu(\gamma)\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})}. \tag{28}
$$

We now complete the proof of our lemma. Substituting the estimates (20), (27), and (28) into (19), we get:

$$
\begin{aligned}
\int_\gamma [v]^2 d\sigma &\leq 3\int_\gamma |v_l-\overline{v}_\gamma|^2 d\sigma + 3\int_\gamma |v_{new}-\overline{v}_\gamma|^2 d\sigma + 3\int_\gamma |\tilde{v}-v_{new}|^2 d\sigma \\
&\leq C\mu(\gamma)|v_l|^2_{H^1(\Omega_l)} + C\mu(\gamma)\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})} + C\mu(\gamma)\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})} \\
&\leq C\mu(\gamma)\Big(|v_l|^2_{H^1(\Omega_l)} + \sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})}\Big),
\end{aligned}
$$

where C is a constant not depending on the length of γ. $\square$

5. Poincaré and Friedrichs inequalities. In this section, we prove two inequalities for mortar finite element functions.

THEOREM 5.1. *(Friedrichs inequality) For every* $v \in V^h$,

$$||v||^2_{L^2(\Omega)} \;\leq\; C(diam(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)},$$

where C *is a constant independent of* $(H_k)_{k=\overline{1:K}}$, $(h_k)_{k=\overline{1:K}}$, *and* K.

Proof. We may assume that no edge of the subregions $\{\Omega_k\}_{k=\overline{1:K}}$ is parallel to the x- or y-axis. Otherwise, since the number of support lines for the edges of $\{\Omega_k\}_{k=\overline{1:K}}$ is finite for any partition of Ω, we can rotate Ω to obtain the desired property.

Let ℓ_0 be a parallel to the x-axis passing through Ω. The intersection of ℓ_0 and the interface Γ consists of a finite number, $n(y_0)-1$, of points, denoted by P_1, P_2, ..., $P_{n(y_0)-1}$ in increasing order of their x-coordinates. Let $\{P_0, P_{n(y_0)}\} = \ell \cap \partial\Omega$, such that P_0 is the leftmost of the two points, and let $(\alpha_i(y_0), y_0)$ be the coordinates of P_i, for $i = \overline{0:n(y_0)}$. Then,

$$\begin{aligned}
\{(\alpha_0(y_0), y_0),\ (\alpha_{n(y_0)}(y_0), y_0)\} \;&=\; \ell \cap \partial\Omega, \\
\Big(\alpha_i(y_0), y_0\Big) \in \Gamma \cap \ell_0, \quad \forall \quad & i = \overline{1:n(y_0)-1}, \\
\alpha_i(y_0) \leq \alpha_{i+1}(y_0), \quad \forall \quad & i = \overline{0:n(y_0)-1}.
\end{aligned}$$

Let $\gamma_{\ell,i}$, for $i = \overline{1:n(y_0)-1}$, be the nonmortar to which P_i belongs; if P_i is vertex of a subregion, and there are several nonmortars ending at P_i, we can choose $\gamma_{\ell,i}$ arbitrarily among them.

Let $(x, y_0) \in \ell_0$ be an arbitrary point on ℓ_0. Denote by $n(x, y_0)$ the well-defined index with the property:

$$\alpha_{n(x,y_0)}(y_0) \;\leq\; x \;<\; \alpha_{n(x,y_0)+1}(y_0).$$

By integrating $\frac{\partial v}{\partial x}$ along ℓ_0 from $(\alpha_0(y_0), y_0)$ to (x, y_0), we obtain:

$$|v(x,y_0) - v(\alpha_0(y_0), y_0)| \;\leq\; \sum_{i=0}^{n(x,y_0)} \Big| \int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)} \frac{\partial v}{\partial x}(t, y_0) dt \Big| \;+\; \sum_{i=1}^{n(x,y_0)} \Big| [v]\Big(\alpha_i(y_0), y_0\Big) \Big|.$$

Since $v \in V^h$ and $(\alpha_0(y_0), y_0) \in \partial\Omega$, we find that $v(\alpha_0(y_0), y_0) = 0$. Using the Schwarz inequality and the previous formula, we obtain:

$$\begin{aligned}
|v(x,y_0)| \;&\leq\; \sum_{i=0}^{n(x,y_0)} \Big(\alpha_{i+1}(y_0) - \alpha_i(y_0)\Big)^{\frac{1}{2}} \Big(\int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)} \Big|\frac{\partial v}{\partial x}\Big|^2 dt \Big)^{\frac{1}{2}} \\
&+\; \Big(\sum_{i=1}^{n(x,y_0)} \mu(\gamma_{l,i}) \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n(x,y_0)} \frac{1}{\mu(\gamma_{l,i})} \Big| [v]\Big(\alpha_i(y_0), y_0\Big) \Big|^2 \Big)^{\frac{1}{2}}.
\end{aligned} \tag{29}$$

Squaring both sides of (29) and apply Schwarz inequality, the inequality becomes:

$$\begin{aligned}
|v(x,y_0)|^2 \;&\leq\; 2 \Big(\sum_{i=0}^{n(x,y_0)} \Big(\alpha_{i+1}(y_0) - \alpha_i(y_0)\Big) \Big) \Big(\sum_{i=0}^{n(x,y_0)} \int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)} \Big|\frac{\partial v}{\partial x}(t, y_0)\Big|^2 dt \Big) \\
&+\; 2 \Big(\sum_{i=1}^{n(x,y_0)} \mu(\gamma_{l,i}) \Big) \Big(\sum_{i=1}^{n(x,y_0)} \frac{1}{\mu(\gamma_{l,i})} \Big| [v](\alpha_i(y_0), y_0) \Big|^2 \Big).
\end{aligned}$$

Since

$$\sum_{i=0}^{n(x,y_0)} \Big(\alpha_{i+1}(y_0) - \alpha_i(y_0) \Big) \;=\; \alpha_{n(x,y_0)+1}(y_0) - \alpha_0(y_0) \;\le\; \mathrm{diam}(\Omega),$$

and, from Lemma 3,

$$\sum_{i=1}^{n(x,y_0)} \mu(\gamma_{l,i}) \;\le\; \sum_{i=1}^{n(y_0)} \mu(\gamma_{l,i}) \;\le\; \mathrm{diam}(\Omega),$$

we obtain,

$$\begin{aligned} (30) \qquad |v(x,y_0)|^2 \;&\le\; 2\ \mathrm{diam}(\Omega) \sum_{i=0}^{n(x,y_0)} \int\limits_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)} \Big| \frac{\partial v}{\partial x}(t,y_0) \Big|^2 dt \\ &+\; C\ \mathrm{diam}(\Omega) \sum_{i=1}^{n(x,y_0)} \frac{1}{\mu(\gamma_{l,i})} \Big| [v](\alpha_i(y_0), y_0) \Big|^2 . \end{aligned}$$

Integrate (30) over Ω. The first term is then bounded from above by

$$2(\mathrm{diam}(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)}.$$

Let γ be a nonmortar with endpoints of coordinates (x_1, y_1) and (x_2, y_2), and slope λ. Since no edge of the subregions $\{\Omega_k\}_{k=\overline{1:K}}$ is parallel to the x- or y-axis, then $\lambda \neq 0$ and $\lambda \neq \infty$ and we can write the equation for γ as $x = \frac{y}{\lambda} + b$. When the second term of (30) is integrated, the jump of v across γ is integrated over $y_1 \le y \le y_2$ and $x \ge \frac{y}{\lambda} + b$. Its contribution is:

$$\begin{aligned} \frac{1}{\mu(\gamma)} \int\limits_{y_1}^{y_2} \int\limits_{x \ge \frac{y}{\lambda}+b} \Big| [v](\frac{y}{\lambda} + b, y) \Big|^2 \, dx dy \;&\le\; \frac{1}{\mu(\gamma)}\ \mathrm{diam}(\Omega) \int_{y_1}^{y_2} |[v](\frac{y}{\lambda} + b, y)|^2 dy \\ &=\; \frac{1}{\mu(\gamma)}\ \mathrm{diam}(\Omega) \sqrt{\frac{\lambda^2}{1+\lambda^2}} \int_\gamma [v]^2 d\sigma \\ &\le\; \mathrm{diam}(\Omega)\ \frac{1}{\mu(\gamma)} \int_\gamma [v]^2 d\sigma. \end{aligned}$$

As a consequence, after integrating (30) over Ω we obtain:

$$\begin{aligned} ||v||^2_{L^2(\Omega)} \;&\le\; 2\ (\mathrm{diam}(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)} \\ (31) \qquad &+\; C\ (\mathrm{diam}(\Omega))^2 \sum_{\gamma\ nonmortar} \frac{1}{\mu(\gamma)} \int_\gamma [v]^2 d\sigma. \end{aligned}$$

If γ is a side of the subregion Ω_l, then, from Lemma 4 and using the notations therein, we have

$$(32) \qquad \frac{1}{\mu(\gamma)} \int_\gamma [v]^2 d\sigma \;\le\; C\Big(|v_l|^2_{H^1(\Omega_l)} \;+\; \sum_{i=1}^{q(l)} |v_{l,i}|^2_{H^1(\Omega_{l,i})} \Big).$$

Recall that $\Omega_{l,i}$ are the subregions with a side opposite γ. When we add (32) over all nonmortar sides γ, every term $|v_{l,i}|^2_{H^1(\Omega_{l,i})}$ appears a finite number of times N, where N is bounded from above independently of $(H_k)_{k=\overline{1:K}}$, and $(h_k)_{k=\overline{1:K}}$. Then,

$$\sum_{s\ nonmortar} \frac{1}{\mu(\gamma)} \int_\gamma [v]^2 d\sigma \ \le\ C \sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)}. \tag{33}$$

Substituting (33) into (31), we obtain

$$||v||^2_{L^2(\Omega)} \ \le\ C\ (\mathrm{diam}(\Omega))^2\ \sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)}.$$

□

The next theorem is a variant of the Poincaré inequality. The proof is similar to that of Theorem 5.1 and it is based on Lemma 4.

THEOREM 5.2. *(Poincaré inequality) For every $v \in V^h$,*

$$||v||^2_{L^2(\Omega)} \ \le\ C(diam(\Omega))^2 \sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)} \ +\ C\frac{1}{\sigma(\Omega)}| \int_\Omega v dx|^2,$$

where $\sigma(\Omega)$ is the area of the region Ω and C is a constant independent of $(H_k)_{k=\overline{1:K}}$, $(h_k)_{k=\overline{1:K}}$, and K.

Proof. The proof follows the steps of the proof of the Friedrichs inequality. We can again assume that no edge of the subregions $\{\Omega_k\}_{k=\overline{1:K}}$ is parallel to the x- or y-axis.

Let (x_1, y_1) and (x_2, y_2) be arbitrary points in Ω. We evaluate $v(x_1, y_1) - v(x_2, y_2)$ by adding the integral of $\frac{\partial v}{\partial y}$ from (x_1, y_1) to (x_1, y_2) and the integral of $\frac{\partial v}{\partial x}$ from (x_1, y_2) to (x_2, y_2), taking the jumps of v across the interface Γ into account.

We square both sides of the resulting inequality and integrate them twice over Ω, once with respect to (x_1, y_1) and once with respect to (x_2, y_2).

The left hand side becomes:

$$\int_\Omega \int_\Omega |v(x_1, y_1) - v(x_2, y_2)|^2 dx_1 dy_1\ dx_2 dy_2 \ =\ 2\sigma(\Omega)||v||^2_{L^2(\Omega)} \ -\ 2\Big|\int_\Omega v dx\Big|^2, \tag{34}$$

with $\sigma(\Omega)$ on the order of $(\mathrm{diam}(\Omega))^2$.

The right hand side is bounded from above by the sum of the H^1-seminorms of the restrictions of the finite element function v to the subregions $\{\Omega_k\}_{k=\overline{1:K}}$ and the result of the integration of the squares of all the jumps of v over nonmortar sides. Reasoning as in the previous proof, we obtain a bound for the right hand side,

$$4(\mathrm{diam}(\Omega))^2\sigma(\Omega) \sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)} \ +\ C\ (\mathrm{diam}(\Omega))^4 \sum_{\gamma\ nonmortar} \frac{1}{\mu(\gamma)} \int_\gamma [v]^2 d\sigma. \tag{35}$$

We use Lemma 4 to estimate the second term of (35). As in the previous proof, the number of appearances for any term $|v_k|^2_{H^1(\Omega_k)}$ does not depend on $(H_k)_{k=\overline{1:K}}$, $(h_k)_{k=\overline{1:K}}$, and K,

$$\sum_{\gamma\ nonmortar} \frac{1}{\mu(\gamma)} \int_\gamma [v]^2 d\sigma \ \le\ C \sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)}. \tag{36}$$

From (34), (35), and (36), we obtain,

$$
\begin{aligned}
2\sigma(\Omega)||v||^2_{L^2(\Omega)} - 2\Big|\int_\Omega v dx\Big|^2 \;&\leq\; 4(\mathrm{diam}(\Omega))^2\sigma(\Omega)\sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)} \\
&+\; C(\mathrm{diam}(\Omega))^4\sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)}.
\end{aligned}
$$

Dividing both sides by $\sigma(\Omega) \cong (\mathrm{diam}(\Omega))^2$, we find,

$$
||v||^2_{L^2(\Omega)} \;\leq\; C(\mathrm{diam}(\Omega))^2\sum_{k=1}^K |v_k|^2_{H^1(\Omega_k)} \;+\; C\frac{1}{\sigma(\Omega)}|\int_\Omega v dx|^2.
$$

□

6. Condition number estimate. A consequence of the Friedrichs inequality is that the condition number of the Poisson problem solved using mortar finite elements has the same form as in the conforming case.

THEOREM 6.1. *For any $u \in V^h$,*

$$
c\sum_{k=1}^K ||u||^2_{L^2(\Omega_k)} \;\leq\; a^\Gamma(u,u) \;\leq\; C\frac{1}{h^2}\sum_{k=1}^K ||u||^2_{L^2(\Omega_k)}, \tag{37}
$$

where c and C are constants that do not depend on K, $(H_k)_{k=\overline{1:K}}$, and $(h_k)_{k=\overline{1:K}}$. Thus, the condition number of the stiffness matrix K_{mortar} corresponding to the discrete problem (3) is bounded by:

$$
\kappa(K_{mortar}) \;\leq\; \frac{C}{h^2},
$$

where C is independent of the partition of Ω.

Proof. From the definition of the broken norm, $a^\Gamma(u,u) = \sum_{k=1}^K |u_k|^2_{H^1(\Omega_k)}$. The right inequality of (37) follows from the inverse inequality. The left inequality follows from Theorem 5.1, since $||u||^2_{L^2(\Omega)} = \sum_{k=1}^K ||u||^2_{L^2(\Omega_k)}$. The estimate of the condition number is a direct consequence of (37). □

7. Extensions to more general geometries. In this section, we extend the construction from Section 4 to a general partition. The assumption that all subregions are rectangles was only used in our estimate of the L^2-norm of the jumps of a mortar function across nonmortars, in particular, in the construction of the function v_{new}.

We use the same notations, and make a similar construction, as in Section 4. We now require that the partition of Ω has all the properties required in Section 2.1. Thus, the length of every side of Ω_k is bounded from below by a uniform fraction of H_k, each subregion is obtained from a reference domain by a uniformly bounded mapping, and the ratio of the diameters of any two adjacent subregions is uniformly bounded. For each partition, a (finite) number of different reference domains might be required.

Opposite the nonmortar γ, we construct a polygon by cutting off part of the union of the subregions, the boundaries of which intersect γ, by a line parallel to γ. Because of the properties just reviewed, we can choose that line such that the length of the side parallel to γ of $\Omega_{l,i} \cap \Omega_{new}$ is bounded from below by a uniform fraction

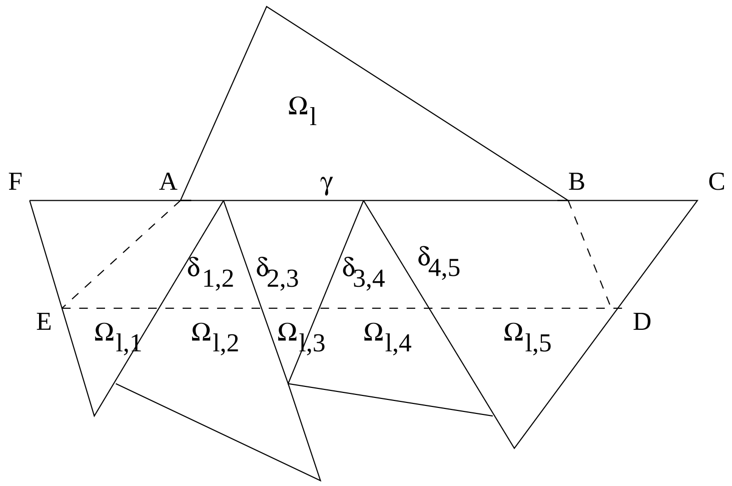

FIG. 2. *Triangular subregions case*

of $H_{l,i}$. As before, we extend the jump of v from $\delta_{i,i+1} \cap \Omega_{new}$ to $\Omega_{l,i}$ or $\Omega_{l,i+1}$ (say to $\Omega_{l,i}$), according to which of $\partial\Omega_{l,i} \cap \gamma$ and $\partial\Omega_{l,i+1} \cap \gamma$ is the largest. We can do this uniformly, using the corresponding reference domain. We obtain a function χ_i vanishing on the sides opposite $\delta_{i,i+1}$, the average of which over $\gamma \cap \Omega_{l,i}$ is 0, and which satisfies properties similar to (16) and (17); cf. Section 4. After this step, the proof can be completed as before.

The construction presented above must be changed slightly, if there are triangles among the subregions $\{\Omega_k\}_{k=\overline{1:K}}$. For a triangle with only one vertex on γ, we can not uniformly extend a function defined on one side so that it vanishes on an opposite side; cf. Figure 2, for $\Omega_{l,2} \cap \Omega_{new}$. Instead, we can construct an extension χ_2 of $v_{l,2} - v_{l,1}$ from $\delta_{1,2} \cap \Omega_{new}$ to $\Omega_{l,2} \cap \Omega_{new}$ satisfying

$$||\chi_2||_{H^{\frac{1}{2}}(\delta_{2,3} \cap \Omega_{new})} \leq ||\chi_2||_{H^{\frac{1}{2}}(\delta_{1,2} \cap \Omega_{new})}.$$

Then, we extend $v_{l,3} - v_{l,2} + \chi_2$ from $\delta_{2,3} \cap \Omega_{new}$ to $\Omega_{l,3} \cap \Omega_{new}$, resulting in a function, χ_3, which vanishes on $\delta_{3,4}$, by using the usual construction. We can do this since $\Omega_{l,3} \cap \Omega_{new}$ is a quadrilateral, and the length of the side parallel to $\gamma \cap \Omega_{l,3}$ is, by construction, uniformly bounded from below by $H_{l,3}$. A similar extension, χ_4, is made for the jump of v across $\delta_{4,5}$ on $\Omega_{l,4} \cap \Omega_{new}$. Thereafter, $v_{l,4} - v_{l,3} + \chi_4$ is extended from $\delta_{3,4} \cap \Omega_{new}$ to $\Omega_{l,3} \cap \Omega_{new}$, resulting in a function which vanishes on $\delta_{2,3}$. Finally, the function v_n is obtained by adding all the auxiliary functions χ_i to $v\,|_{\Omega_{new}}$.

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