ON THE L^2 STABILITY OF THE 1-D MORTAR PROJECTION

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Abstract. It is previously known that the one dimensional mortar finite element projection is stable in the L^2 norm, provided that the ratio of any two neighboring mesh intervals is uniformly bounded, but with the constant in the bound depending on the maximum value of that ratio. In this paper, we show that this projection is stable in the L^2 norm, independently of the properties of the nonmortar mesh. The 1D trace of the mortar space considered here is a piecewise polynomial space of arbitrary degree; therefore, our result can be used for both the h and the hp version of the mortar finite element.

Key words. mortar finite elements, mortar projection.

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1. Introduction. Mortar finite elements are nonconforming finite elements that allow for nonconforming decomposition of the computational domain and for the optimal coupling of different variational approximations in different subregions. Here, by optimality we mean that the global error is bounded by the sum of the local best approximation errors on each subregion. Because of these features, the mortar elements are quite general and they are used effectively in solving large classes of problems.

The mortar finite element methods were first introduced by Bernardi, Maday, and Patera in [4]. A three dimensional version was developed by Ben Belgacem and Maday in [3], and was further analyzed for three dimensional spectral elements in [2].

Let us briefly describe the mortar finite element space V^h underlying the mortar method. The computational domain Ω is decomposed into a nonoverlapping polygonal partition $\{\Omega_k\}_{k=\overline{1:K}}$. Since we are working with geometrically nonconforming mortars, we do not require that the intersection of the boundaries of two different subregions be either empty, or a vertex, or an entire edge. The restriction of the mortar space to any subregion Ω_i is a conforming P_{m_i} or Q_{m_i} finite element space. In other words, Ω_i is partitioned in a geometrically conforming fashion into triangles or quadrilaterals, and the restriction of V^h to each element of this partition is a polynomial of total degree m_i (for P_{m_i}), or of degree m_i in each variable (for Q_{m_i}). Since our arguments will be local, the degrees m_i are completely arbitrary.

Across the interface Γ, i.e. the set of points that belong to the boundaries of at least two subregions, pointwise continuity is not required. We partition Γ into a union of nonoverlapping edges of the subregions $\{\Omega_k\}_{k=\overline{1:K}}$, called nonmortars; on the two sides of the edge which coincides with a nonmortar we find two distinct traces of the mortar function, and we will require only that the difference of these two traces be orthogonal, in the L^2 inner product, to a space of test functions.

More formally, if γ is a nonmortar side, let $V^h(\gamma)$ be the continuous piecewise polynomial space which is the restriction of V^h to γ. We then define the mortar projection operator $\pi_\gamma : L^2(\gamma) \to V^h(\gamma) \cap H_0^1(\gamma)$ by the following L^2-orthogonality

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condition:

$$\int_{\overline{\gamma}} \Big(\chi - \pi_\gamma(\chi)\Big)\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma).$$

Here, $\chi \in L^2(\gamma)$ and the test function space $\Psi^h(\gamma)$ is the subspace of $V^h(\gamma)$, whose restriction to the first and last mesh intervals are polynomials of degree 1 less than the corresponding degree from $V^h(\gamma)$.

In this paper, we address the issue of the L^2 stability of the mortar projection, which arises in the study of the discretization error, as well as when various domain decomposition methods are used to solve mortar finite element problems.

It has previously been proven that the mortar projection is stable for different particular meshes; see Bernardi, Maday, and Patera [4] for uniform meshes, Ben Belgacem [1] and Braess, Dahmen, Wieners [5] for quasiuniform meshes. In a more general case, Seshaiyer and Suri [7] give a proof of the L^2 stability of the mortar projection, if the ratio of any two neighboring mesh intervals over γ is uniformly bounded. The constant in their bound depends on the maximum value of that ratio and on m, the polynomial degree.

We prove that the mortar projection is uniformly stable in L^2 for arbitrary meshes, with the constant in the bound depending only on m. Our result is obtained by refining a method used in Ben Belgacem and Maday [3] for 2-D mortar projections with uniform meshes.

Let us also make some comments about the H^1_0 stability of the mortar projection. In [7], Seshaiyer and Suri prove that the mortar projection is stable in the H^1_0 norm, if the ratio of any two neighboring mesh intervals is uniformly bounded. In [6], Crouzeix and Thomée prove a similar result for the L_2-projection from $L^2(\gamma)$ onto $V^h(\gamma) \cap H^1_0(\gamma)$. They also show that the projection is not stable in the H^1_0 norm for arbitrary meshes. Therefore, it is reasonable to believe that some condition on the mesh of γ is necessary in order to obtain the H^1_0 stability of the mortar projection π_γ.

The rest of the paper is structured as follows. In section 2, we present some technical results about minimizing L^2 norms of polynomials, and in section 3 we use an idea from Ben Belgacem and Maday [3] to prove the L^2 stability of the mortar projection.

2. Technical Tools. The main result of this section gives a good L^2 approximation (see Lemma 2.3 for the precise result) of a polynomial from $V^h(\gamma)$ which vanishes at the end points of γ by another polynomial from $\Psi^h(\gamma)$. To do so, we need some results about minimizing the L^2 norm of polynomials satisfying certain constraints. The main idea of the proofs is to use Legendre polynomial expansions and Lagrange multipliers methods.

For simplicity, we only work with odd degree polynomials. Similar estimates and results can also be derived for even degree polynomials.

LEMMA 2.1. *Let P be a polynomial of degree $2n+1$ on $[-1,\ 1]$, such that $P(-1) = c_1$ and $P(1) = c_2$. Then*

$$\inf_P ||P||^2_{L^2(-1,1)} = \frac{2(c_1^2+c_2^2)(n+1)+2c_1c_2}{(n+1)(2n+1)(2n+3)}.$$

Proof. We write P in the basis of Legendre polynomials,

$$P(x) = \sum_{k=0}^{2n+1} a_k L_k(x). \tag{1}$$

Since $L_k(1) = 1$ and $L_k(-1) = (-1)^k$, the conditions $P(-1) = c_1$ and $P(1) = c_2$ can be expressed as

$$\sum_{k=0}^{2n+1} a_k = c_2, \quad \text{and} \quad \sum_{k=0}^{2n+1} (-1)^k a_k = c_1,$$

or, equivalently:

$$\sum_{k=0}^{n} a_{2k} = \frac{c_1 + c_2}{2} \tag{2}$$

$$\sum_{k=0}^{n} a_{2k+1} = \frac{c_2 - c_1}{2}. \tag{3}$$

The Legendre polynomials are orthogonal in the L^2 inner product, and

$$||L_k||^2_{L^2(-1,1)} = \frac{1}{k + 1/2}.$$

Therefore

$$||P||^2_{L^2(-1,1)} = \sum_{k=0}^{2n+1} \frac{a_k^2}{k + 1/2}. \tag{4}$$

Therefore, we can split our minimization problem into two subproblems, corresponding to the even and odd degree coefficients, respectively, which will be solved in a similar fashion.

For the subproblem corresponding to $\{a_{2k}\}_{k=\overline{0:n}}$, we want to minimize

$$\sum_{k=0}^{n} \frac{a_{2k}^2}{2k + 1/2},$$

subject to the constraint (2). Using a Lagrange multipliers method, which in this case is equivalent to using Schwarz's inequality, we obtain

$$\begin{aligned} \left(\sum_{k=0}^{n} \frac{a_{2k}^2}{2k + 1/2}\right) \left(\sum_{k=0}^{n} 2k + 1/2\right) &= \left(\sum_{k=0}^{n} \frac{a_{2k}^2}{2k + 1/2}\right) \frac{(n+1)(2n+1)}{2} \\ &\geq \left(\sum_{k=0}^{n} a_{2k}\right)^2 = \frac{(c_1 + c_2)^2}{4}. \end{aligned}$$

Therefore,

$$\min_{\sum a_{2k} = (c_1 + c_2)/2} \sum_{k=0}^{n} \frac{a_{2k}^2}{2k + 1/2} = \frac{(c_1 + c_2)^2}{2(n+1)(2n+1)}. \tag{5}$$

Similarly, for the problem corresponding to $\{a_{2k+1}\}_{k=\overline{0:n}}$, we obtain:

$$(6) \qquad \min_{\sum a_{2k+1}=(c_2-c_1)/2} \sum_{k=0}^{n} \frac{a_{2k+1}^2}{2k+3/2} \;=\; \frac{(c_2-c_1)^2}{2(n+1)(2n+3)}.$$

Adding (5) and (6), we obtain:

$$\inf_P ||P||^2_{L^2(-1,1)} \;=\; \frac{2(c_1^2+c_2^2)(n+1)+2c_1c_2}{(n+1)(2n+1)(2n+3)}.$$

□

LEMMA 2.2. *Let P be a polynomial of degree $2n+1$ on $[-1,\ 1]$, such that $P(-1) = 0$ and $P(1) = c_2$. Expand P in the Legendre polynomials basis and assume that a_{2n+1}, the coefficient of the highest degree term in the expansions, is given. Then*

$$(7) \quad \inf_P ||P||^2_{L^2(-1,1)} \;=\; a_{2n+1}^2\frac{2(2n+3)(n+1)}{n(2n+1)(4n+3)} \;-\; \frac{2c_2a_{2n+1}}{n(2n+1)} \;+\; \frac{c_2^2}{2n(2n+1)}.$$

If the value of P at -1 is no longer required to be 0, then

$$(8) \quad \inf_P ||P||^2_{L^2(-1,1)} \;=\; a_{2n+1}^2\frac{8(n+1)^2}{(4n+3)(2n+1)^2} \;-\; \frac{4c_2a_{2n+1}}{(2n+1)^2} \;+\; \frac{2c_2^2}{(2n+1)^2}.$$

Proof. Writing P in the Legendre basis as in (1), and imposing $P(-1) = 0$ and $P(1) = c_2$, we obtain

$$a_{2n+1} + \sum_{k=0}^{2n} a_k \;=\; c_2, \quad \text{and} \quad -a_{2n+1} + \sum_{k=0}^{2n}(-1)^k a_k \;=\; 0.$$

Since a_{2n+1} is fixed, we can solve for $\sum a_{2k}$ and $\sum a_{2k+1}$:

$$\sum_{k=0}^{n} a_{2k} \;=\; \frac{c_2}{2}, \quad \text{and} \quad \sum_{k=0}^{n-1} a_{2k+1} \;=\; \frac{c_2-2a_{2n+1}}{2}.$$

In Lemma 2.1, we have solved the problem of maximizing $\sum_{k=1}^{2n}\frac{a_k^2}{k+1/2}$ when the sums of the odd and even terms, respectively, are kept constant. Using that result and (4), we obtain

$$\begin{aligned}\inf_P ||P||^2_{L^2(-1,1)} \;&=\; a_{2n+1}^2\frac{1}{(2n+1)+1/2} \;+\; \frac{c_2^2}{2(n+1)(2n+1)} \;+\; \frac{(c_2-2a_{2n+1})^2}{2n(2n+1)}\\ &=\; a_{2n+1}^2\frac{2(n+1)(2n+3)}{n(2n+1)(4n+3)} \;-\; \frac{2c_2a_{2n+1}}{n(2n+1)} \;+\; \frac{c_2^2}{2n(n+1)}.\end{aligned}$$

If the value of P at -1 is no longer fixed, then the only condition that the coefficients $\{a_k\}_{k=\overline{0:2n}}$ must satisfy is

$$\sum_{k=0}^{2n} a_k \;=\; c_2 - a_{2n+1}.$$

Using once again Schwarz's inequality, and without splitting the problem into two cases, we obtain:

$$\Big(\sum_{k=0}^{2n} \frac{a_k^2}{k+1/2}\Big)\,\Big(\sum_{k=0}^{2n} k+1/2\Big) \;=\; \Big(\sum_{k=0}^{2n} \frac{a_k^2}{k+1/2}\Big)\,\frac{(2n+1)^2}{2} \;\geq\; (\sum_{k=0}^{2n} a_k)^2,$$

which can be written as:

$$\sum_{k=0}^{2n} \frac{a_k^2}{k+1/2} \;\geq\; \frac{2}{(2n+1)^2}\,(\sum_{k=0}^{2n} a_k)^2 \;=\; \frac{2(c_2-a_{2n+1})^2}{(2n+1)^2}.$$

Therefore:

$$||P||^2_{L^2(-1,1)} \;=\; \sum_{k=0}^{2n+1} \frac{a_k^2}{k+1/2} \;\geq\; \frac{2a_{2n+1}^2}{4n+3} \;+\; \frac{2(c_2-a_{2n+1})^2}{(2n+1)^2}.$$

Since in Schwarz's inequality there exist coefficients a_k, $k=\overline{0:2n}$ such that the equality is realized, we conclude that there exists a polynomial P such that

$$\begin{aligned}\inf_P ||P||^2_{L^2(-1,1)} \;&=\; \frac{2a_{2n+1}^2}{4n+3} \;+\; \frac{2(c_2-a_{2n+1})^2}{(2n+1)^2}\\ &=\; a_{2n+1}^2\frac{8(n+1)^2}{(4n+3)(2n+1)^2} \;-\; \frac{4c_2a_{2n+1}}{(2n+1)^2} \;+\; \frac{2c_2^2}{(2n+1)^2}.\end{aligned}$$

$\square$

The next lemma is the main result of this section. We introduce the following notations. Let $\gamma=[a,b]$ be a segment partitioned into intervals $\{I_j\}_{j=\overline{1:(N+1)}}$, $I_j=\overline{(x_{j-1},x_j)}$, with $a=x_0<x_1<\ldots<x_{N+1}=b$, and let $h_j=x_j-x_{j-1}$, for $j=\overline{1:N+1}$. Let $\{m_j\}_{j=\overline{1:(N+1)}}$ be a set of positive integers. We define the piecewise polynomial spaces $V^h(\gamma)$ and $\Psi^h(\gamma)$ as follows:

$$V^h(\gamma)=\{v\in C(0,1);\;\; v|_{I_j}\in P_{m_j}(I_j),\ \forall j=\overline{1:N+1}\},$$

and

$$\Psi^h(\gamma)=\{v\in C(0,1);\;\; v|_{I_j}\in P_{m_j}(I_j),\ \forall j=\overline{2:N};\;\; v|_{I_j}\in P_{m_j-1}(I_j),\ j\in\{1,N+1\}\}.$$

LEMMA 2.3. *Let $\pi\chi\in V^h(\gamma)\cap H_0^1(\gamma)$. Then there exists a function $\chi_h\in\Psi^h(\gamma)$ and a constant $0<C(m)<1$, depending only on m_1 and m_{N+1}, and not on the partition of γ, such that*

$$||\pi\chi-\chi_h||_{L^2(\gamma)} \;\leq\; C(m)||\pi\chi||_{L^2(\gamma)}. \tag{9}$$

Proof. We will choose χ_h to be equal to $\pi\chi$ on all of the partition intervals except for the first two and the last two intervals. If we look for χ_h equal to $\pi\chi$ on all the intervals except the first and last ones, it can be proven that the best constant M in (9) would depend on h_0 and h_1, which we want to avoid. Since χ_h will be defined in a similar way at both ends of γ, we only present the construction of χ_h on $I_1=(x_0,x_1)$ and $I_2=(x_1,x_2)$.

We may assume, without any loss of generality, that m_1 and m_2 are odd, i.e. $m_1 = 2n_1 + 1$ and $m_2 = 2n_2 + 1$; similar results can be obtained for even degrees. Let $\beta_1 = \pi\chi(x_1)$ and $\beta_2 = \pi\chi(x_2)$. Note that $\pi\chi(x_0) = 0$, since $\pi\chi$ vanishes at the end points of γ. We require that $\chi_h(x_2) = \beta_2$, and denote the value of χ_h at x_1 by α_1, which will be different than β_1: $\chi_h(x_1) = \alpha_1 \neq \beta_1$. We will look for $\chi_h \in \Psi^h(\gamma)$ such that $||\pi\chi - \chi_h||_{L^2(\gamma)}$ is minimal, and then choose α_1 such that relation (9) will hold on the two intervals I_1 and I_2.

On I_2, $\pi\chi - \chi_h$ is a polynomial of degree $2n_2+1$ which takes the values $\beta_1 - \alpha_1$ and 0, respectively, at the left and right end points. After a suitable change of variables, which maps I_2 into $(-1, 1)$, and using Lemma 2.1, we can find χ_h on I_2 such that

$$\frac{8}{h_2}||\pi\chi - \chi_h||^2_{L^2(I_2)} = \frac{2(\beta_1 - \alpha_1)^2}{(2n_2+1)(2n_2+3)}. \tag{10}$$

On I_1, $\pi\chi - \chi_h$ is a polynomial of degree $2n_1 + 1$ which takes the value $\beta_1 - \alpha_1$ at x_1, the left end point of I_1. Let a be the coefficient of L_{2n_1+1} in the Legendre expansion of $\pi\chi$ over I_1. Since χ_h is a polynomial of degree $2n_1$ over I_1, a is also the coefficient of L_{2n_1+1} in the Legendre expansion of $\pi\chi - \chi_h$. After a suitable change of variables, which maps I_1 into $(-1, 1)$, and using (8) from Lemma 2.2, there exists χ_h, satisfying all the above mentioned properties, such that

$$\frac{8}{h_1}||\pi\chi - \chi_h||^2_{L^2(I_1)} = a^2\frac{8(n_1+1)^2}{(4n_1+3)(2n_1+1)^2} - \frac{4a(\beta_1 - \alpha_1)}{(2n_1+1)^2} + \frac{2(\beta_1 - \alpha_1)^2}{(2n_1+1)^2}. \tag{11}$$

We now find lower bounds for $||\pi\chi||_{L^2(I_2)}$ and $||\pi\chi||_{L^2(I_1)}$. On I_2, $\pi\chi$ takes the values β_1 and β_2 at the end points. After a change of variables and using Lemma 2.1, we obtain

$$\frac{8}{h_2}||\pi\chi||^2_{L^2(I_2)} \geq \frac{2(\beta_1^2 + \beta_2^2)(n_2+1) + 2\beta_1\beta_2}{(n_2+1)(2n_2+1)(2n_2+3)}. \tag{12}$$

The minimal value of the right hand side of (12) is obtained for $\beta_2 = -\beta_1/2(n_2+1)$, and therefore:

$$\frac{8}{h_2}||\pi\chi||^2_{L^2(I_2)} \geq \left(1 - \frac{1}{4(n_2+1)^2}\right)\frac{2\beta_1^2}{(2n_2+1)(2n_2+3)}. \tag{13}$$

On I_1, $\pi\chi$ takes the values 0 and β_1 at the end points, and in its Legendre expansion the coefficient of L_{2n_1+1} is a. After a change of variables, and using (7) from Lemma 2.2, we obtain

$$\frac{8}{h_1}||\pi\chi||^2_{L^2(I_1)} \geq a^2\frac{2(2n_1+3)(n_1+1)}{n_1(2n_1+1)(4n_1+3)} - \frac{2\beta_1 a}{n_1(2n_1+1)} + \frac{\beta_1^2}{2n_1(2n_1+1)}. \tag{14}$$

We choose $\alpha_1 = \beta_1/2$, and compare the L^2 norms of $\pi\chi - \chi_h$ and $\pi\chi$ separately on I_1 and I_2.

On I_2, we obtain by using (10) and (13), that

$$||\pi\chi - \chi_h||^2_{L^2(I_2)} \leq \frac{2}{3}\,||\pi\chi||^2_{L^2(I_2)}. \tag{15}$$

On I_1, we obtain by using (11) and (14), that

$$||\pi\chi - \chi_h||^2_{L^2(I_1)} \leq (1 - 1/(2n_1+1)^2)\,||\pi\chi||^2_{L^2(I_1)}. \tag{16}$$

We make a similar construction for χ_h on I_{N-1} and I_N. Since $\pi\chi - \chi_h$ vanishes outside the first and last two mesh intervals of γ, we can conclude, by using (15) and (16), that

$$||\pi\chi - \chi_h||_{L^2(\gamma)} \ \leq \ C(m)||\pi\chi||_{L^2(\gamma)},$$

where $C(m)$ depends only on m_1 and m_{N+1}, and not on the particular properties of the partition of γ. □

3. Stability property of the mortar projection. In this section, we prove the main result of our paper, namely the uniform stability of the mortar projection onto $V^h(\gamma) \cap H_0^1(\gamma)$, i.e. that the bound is independent of the mesh. In Theorem 3.1, the spaces $V^h(\gamma)$ and $\Psi^h(\gamma)$ are those defined in Sections 1 and 2 .

THEOREM 3.1. *Let γ be a nonmortar side, and let m be the degree of the piecewise polynomial restriction of the mortar function to γ. Let π_h be the mortar projection of $L^2(\gamma)$ into $V^h(\gamma) \cap H_0^1(\gamma)$, which satisfies*

$$\int_{\overline{\gamma}} \Big(\chi - \pi_h(\chi)\Big)\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma).$$

Then there exists a constant $\tilde{C}(m)$ depending only on m such that

$$||\pi_h(\chi)||_{L^2(\gamma)} \ \leq \ \tilde{C}(m)||\chi||_{L^2(\gamma)}, \quad \forall \chi \in L^2(\gamma).$$

Proof. Let $p_h : L^2(\gamma) \to \Psi^h(\gamma)$ be the L^2 projection into the space $\Psi^h(\gamma)$:

$$\int_{\overline{\gamma}} \Big(\chi - p_h(\chi)\Big)\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma),$$

where $p_h(\chi) \in \Psi^h(\gamma)$. Then

$$\int_{\overline{\gamma}} \Big(\pi_h(\chi) - p_h(\chi)\Big)\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma),$$

and therefore $p_h(\chi)$ is the projection of $\pi_h(\chi)$ into $\Psi^h(\gamma)$. Then:

$$||\pi_h(\chi) - p_h(\chi)||_{L^2(\gamma)} \ = \ \inf_{\chi_h \in \Psi^h(\gamma)} ||\pi_h(\chi) - \chi_h||_{L^2(\gamma)} \ \leq \ C(m)||\pi_h(\chi)||_{L^2(\gamma)},$$

with $M < 1$, according to Lemma 2.3, applied for the case when all the degrees m_j are equal to m.

A simple computation will lead us to the desired conclusion:

$$\begin{aligned}
||\pi_h(\chi)||_{L^2(\gamma)}^2 \ &= \ \int_{\overline{\gamma}} \Big(\pi_h(\chi) - p_h(\chi)\Big)\pi_h(\chi) ds \ + \ \int_{\overline{\gamma}} p_h(\chi)\pi_h(\chi) ds \\
&= \ \int_{\overline{\gamma}} \Big(\pi_h(\chi) - p_h(\chi)\Big)^2 ds \ + \ \int_{\overline{\gamma}} p_h(\chi)\pi_h(\chi) ds \\
&\leq \ C(m)^2||\pi_h(\chi)||_{L^2(\gamma)}^2 \ + \ ||\pi_h(\chi)||_{L^2(\gamma)}||p_h(\chi)||_{L^2(\gamma)}.
\end{aligned}$$

Since $C(m) < 1$, and since $p_h(\chi)$ is an L^2 projection of χ:

$$||\pi_h(\chi)||_{L^2(\gamma)} \ \leq \ \frac{1}{1 - C(m)^2}||p_h(\chi)||_{L^2(\gamma)} \ \leq \ \frac{1}{1 - C(m)^2}||\chi||_{L^2(\gamma)}.$$

□

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