HIERARCHICAL A POSTERIORI ERROR ESTIMATORS FOR MORTAR FINITE ELEMENT METHODS WITH LAGRANGE MULTIPLIERS

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Abstract. Hierarchical a posteriori error estimators are introduced and analyzed for mortar finite element methods. A weak continuity condition at the interfaces is enforced by means of Lagrange multipliers. The two proposed error estimators are based on a defect correction in higher order finite element spaces and an adequate hierarchical two-level splitting. The first provides upper and lower bounds for the discrete energy norm of the mortar finite element solution whereas the second also estimates the error for the Lagrange multiplier. It is shown that an appropriate measure for the nonconformity of the mortar finite element solution is the weighted L^2-norm of the jumps across the interfaces.

Key words. mortar finite elements, Lagrange multiplier, a posteriori error estimation, adaptive grid refinement

AMS subject classifications. 65N15, 65N30, 65N50, 65N55

1. Introduction.

In this paper, we consider a special nonoverlapping domain decomposition method for the discretization of the following model problem

$$\begin{aligned} Lu := -\mathrm{div}\,(a\nabla u) + b\,u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded, polygonal domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$. Furthermore, we assume $a = (a_{ij})_{i,j=1}^2$ to be a symmetric, uniformly positive definite matrix-valued function with $a_{ij} \in L^\infty(\Omega)$, $1 \le i,j \le 2$, and $0 \le b \in L^\infty(\Omega)$. The largest eigenvalue of a restricted to a subset $D \subset \Omega$ is denoted by α_D.

The decomposition of Ω into nonoverlapping subdomains allows a more flexible numerical approach including a discontinuous finite element solution and geometrical nonmatching triangulations across the interfaces of the subdomains. In particular, our approach is based on a hybrid formulation which gives rise to a saddle point problem. A good overview of more general mortar finite element methods including also the coupling of spectral elements with finite elements can be found in [15] (see also [5, 6, 11, 12, 13, 14, 18, 21, 22, 26]).
Recently, a lot of work has been done on the construction of efficient iterative solvers based on multilevel techniques [1, 3, 4, 25, 27, 28], but in contrast, there are only a few papers considering adaptive refinement techniques and a posteriori error estimators. Bernardi and Maday [16] have proved a priori estimates for the case that the gridsize of the different subdomains is either H or $2^{-k}H$. In [29], a residual based error indicator is presented but no lower bound for the true error is established. Finally in [33], a residual based error estimator is investigated.

This paper is organized as follows. In section 2, we briefly introduce a special mortar finite element discretization. We review well-known a priori estimates for the discretization with conforming finite elements in each subdomain, and we then consider the nonconformity of the discrete ansatz space in detail. A relation between the weighted L^2-norm of the jumps across the interfaces and the nonconformity is established.

In sections 3 and 4, we present two different hierarchical basis a posteriori error estimators. Both of them are based on a defect correction in a higher order space and a hierarchical two-level splitting. This type of error estimator is well known for standard conforming discretizations

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[10, 19, 20, 31]. An excellent overview of different techniques can be found in [31] (see also the references therein).
Section 3 is devoted to the construction of a hierarchical basis type error estimator for the energy norm of the error. We first consider the continuous defect problem with Dirichlet boundary conditions on each subdomain. In the next step, we replace the continuous problem by a discrete one in a higher order space, and the exact by adequate discrete boundary conditions. These simplifications are based on saturations assumptions which are motivated by certain a priori estimates. To obtain upper and lower bounds for the error, we additionally have to take the nonconformity of the mortar finite element solution into account.
In section 4, we introduce what we will call a fully hierarchical error estimator. Here, not only the energy norm of the error is measured but also that of the Lagrange multiplier. We note that a more general concept of hierarchical error estimators including saddle point formulations is presented in [9]. Error estimators for mixed finite element methods can be found in [2, 9, 24, 33]. In case of our fully hierarchical error estimator, the nonconformity of the mortar finite element solution enters in the error estimator in a natural way.

Finally in section 5, we present some numerical results illustrating the adaptive refinement process and the performance of the error estimators. In particular, we consider the influence of the choice of the Lagrange multiplier on the error and the adaptively generated mesh. We find that the Lagrange multiplier should preferably be defined on the side of the interface where the coefficient a of the partial differential equation is smaller.

2. A priori estimates. The initial domain Ω is decomposed into non-overlapping subdomains Ω_k, $1 \leq k \leq K$

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \quad \text{with } \Omega_l \cap \Omega_k = \emptyset, \ \ k \neq l.$$

We restrict ourselves to the geometrical conforming situation where the intersection between the boundary of any two different subdomains $\partial\Omega_l \cap \partial\Omega_k$, $k \neq l$, and $\partial\Omega_l \cap \partial\Omega$ is either empty, a vertex or a common edge, and we also assume that the subdomains are polygons. The interface between two subdomains is denoted by $\overline{\Gamma}_{lk} = \overline{\Gamma}_{kl} := \partial\Omega_l \cap \partial\Omega_k$ and $\mathbf{n}_{lk}$ stands for the unit normal from Ω_l towards Ω_k. The union of all interfaces is called the skeleton $\mathcal{S} := \bigcup_{k,l=1}^{K} \overline{\Gamma}_{lk}$. In each subdomain Ω_k we use conforming P_n finite elements associated with a shape regular simplicial triangulation $\mathcal{T}_{h_k}$. $\mathcal{E}_{h_k}$ and $\mathcal{P}_{h_k}$ stand for the sets of edges and vertices of the triangulation $\mathcal{T}_{h_k}$, respectively. The set of all triangles, vertices and edges is denoted by $\mathcal{T}_h$, $\mathcal{P}_h$ and $\mathcal{E}_h$, respectively. We do not require that the triangulations, $\mathcal{T}_{h_i}$ and $\mathcal{T}_{h_j}$, coincide on the interface Γ_{ij}.

In the following, $S_n(\Omega_k; \mathcal{T}_{h_k})$ is the standard conforming P_n space defined locally on Ω_k by

$$S_n(\Omega_k; \mathcal{T}_{h_k}) := \{v \in C(\Omega_k) \mid v|_T \in P_n(T), \ T \in \mathcal{T}_{h_k}, \ v|_{\partial\Omega \cap \partial\Omega_k} = 0\}.$$

Let $X_{n;h}^{-1}(\Omega)$ be the global product space given by

$$X_{n;h}^{-1}(\Omega) := \prod_{k=1}^{K} S_n(\Omega_k; \mathcal{T}_{h_k}).$$

In general, the mortar finite element solutions do not satisfy a pointwise continuity across the interfaces, but some weak continuity conditions have to be imposed in terms of Lagrange multipliers which are defined on the skeleton. Each interface $\Gamma_{ij} = \Gamma_{ji}$, inherits two 1D triangulations, one from $\mathcal{T}_{h_i}$ and one from $\mathcal{T}_{h_j}$. Without any restriction of generality, we will select $\mathcal{T}_{h_i}$ for the definition of the space of Lagrange multipliers. In the following, the side where the Lagrange multiplier is defined will be called the nonmortar side, and the opposite the mortar side. With this choice, the skeleton can be uniquely decomposed into the union of the edges of the nonmortar sides:

$$\mathcal{S} = \bigcup_{e \in \mathcal{E}_L} e, \quad \mathcal{E}_L := \{ e \in \mathcal{E}_h | \; \exists i, j \;\; 1 \le i \le K, \; j \in \mathcal{M}(i) \text{ such that } e \in \mathcal{E}_{h_i} \cap \Gamma_{ij} \}$$

where $\mathcal{M}(i) := \{1 \le j \le K | \text{ nonmortar side of } \Gamma_{ij} \text{ associated with } \Omega_i\}$ (see Fig. 2.1).

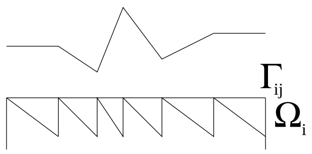


FIG. 2.1. *Lagrange multiplier space for $n = 1$*

The space $W_{n;h}(\mathcal{S})$ of Lagrange multiplier is now defined as

$$\begin{aligned} W_{n;h}(\mathcal{S}) &:= \prod_{i=1}^{K} \prod_{j \in \mathcal{M}(i)} W_n(\Gamma_{ij}; \mathcal{T}_{h_i}) \\ W_n(\Gamma_{ij}; \mathcal{T}_{h_i}) &:= \{ v \in C(\Gamma_{ij}) \mid v|_e \in P_n(e), \; e \in \mathcal{E}_{h_i} \cap \Gamma_{ij}, \\ &\qquad v|_e \in P_{n-1}(e), \text{ if } e \text{ contains an endpoint of } \Gamma_{ij} \} \,. \end{aligned}$$

Then, the mortar finite element solution $(u_n, \lambda_n) \in X_{n;h}^{-1}(\Omega) \times W_{n;h}(\mathcal{S})$ is defined as the unique solution of the saddle point problem

$$\begin{aligned} a\,(u_n, v) + b\,(\lambda_n, v) &= (f, v)_0, \qquad v \in X_{n;h}^{-1}(\Omega), \\ b\,(\mu, u_n) &= 0, \qquad \mu \in W_{n;h}(\mathcal{S}). \end{aligned} \tag{2.1}$$

The bilinear forms $a\,(\cdot,\cdot)$ and $b\,(\cdot,\cdot)$ are given by

$$\begin{aligned} a\,(v, w) &:= \sum_{i=1}^{K} a_i\,(v, w)\,, \quad a_i\,(v, w) := \int_{\Omega_i} a \, \nabla v \cdot \nabla w + b \, v \, w \, dx, \quad v, w \in \prod_{i=1}^{K} H^1(\Omega_i), \\ b\,(\mu, w) &:= \sum_{i=1}^{K} \sum_{j \in \mathcal{M}(i)} \int_{\Gamma_{ij}} \mu \, [w]_J \, d\sigma, \quad w \in \prod_{i=1}^{K} H^1(\Omega_i), \; \mu \in \prod_{i=1}^{K} \prod_{j \in \mathcal{M}(i)} H^{-\frac{1}{2}}(\Gamma_{ij}) \end{aligned}$$

with the jump across Γ_{ij} given by $[w]_J := w|_{\Omega_j} - w|_{\Omega_i}$. The broken energy norm is defined as $|\!|\!| v |\!|\!|^2 := a\,(v, v), \; v \in \prod_{i=1}^{k} H^1(\Omega_i)$. In the following, the constants $0 < c \le C < \infty$ are generic constants which depend on the local ratio of the eigenvalues of the coefficient matrix a, the local ratio of α_T and $h_T^2 b|_T$, the shape regularity of the initial triangulation and the order n of the finite element ansatz space but not on the refinement level.

If the solution (u, λ) of the continuous saddle point problem

$$\begin{aligned} a\,(u, v) + b\,(\lambda, v) &= (f, v)_0, \qquad v \in \prod_{i=1}^{k} H^1_{0;\partial\Omega_i \cap \partial\Omega}(\Omega_i), \\ b\,(\mu, u) &= 0, \qquad \mu \in \prod_{i=1}^{K} \prod_{j \in \mathcal{M}(i)} H^{-\frac{1}{2}}(\Gamma_{ij}), \end{aligned} \tag{2.2}$$

is smooth enough, we obtain the following a priori estimate

$$|\!|\!| u - u_n |\!|\!|^2 + \| \lambda - \lambda_n \|^2_{H_*^{-1/2}} \le C(n) \sum_{i=1}^{k} h_i^{2n} \| u \|^2_{n+1;\Omega_i} \tag{2.3}$$

(see, e.g., [11, 15, 16]). Here, $\|\cdot\|_{H_\star^{-1/2}}$ stands for the dual norm of $\prod_{i=1}^K \prod_{j\in\mathcal{M}(i)} H_{00}^{-1/2}(\Gamma_{ij})$, i.e.

$$\|\mu\|_{H_\star^{-1/2}} := \sup_{v\in \prod\limits_{i=1}^{K} \prod\limits_{j\in\mathcal{M}(i)} H_{00}^{1/2}(\Gamma_{ij})} \frac{(v,\mu)}{\|v\|_{H_{00}^{1/2}}},$$

where $\|v\|^2_{H_{00}^{1/2}} := \sum_{i=1}^K \sum_{j\in\mathcal{M}(i)} \alpha_{\Omega_i} \|v\|^2_{H_{00}^{1/2}(\Gamma_{ij})}$. Sometimes it is more convenient to use a mesh dependent norm $\|\cdot\|_L$ for the Lagrange multiplier

$$\|v\|_L^2 := \sum_{e\in\mathcal{E}_L} \frac{h_e}{\alpha_e}\|v\|^2_{0;e}, \quad v \in L^2(\mathcal{S}) \tag{2.4}$$

and the corresponding discrete dual norm $\|v\|^2_{L^{-1}} := \sum_{e\in\mathcal{E}_L} \frac{\alpha_e}{h_e}\|v\|^2_{0;e}$. The weight α_e, $e\subset\Gamma_{ij}$, is defined by $\alpha_e := \alpha_T$, where $T\in\mathcal{T}_{h_i}$ and $e\subset\partial T$. Note that the mesh dependent norm requires higher regularity of the Lagrange multiplier than the $H^{-1/2}$-norm.

LEMMA 2.1. *A discrete Babuška-Brezzi condition holds*

$$\|\mu\|_L \le \beta(n) \sup_{v\in X^{-1}_{n;h}(\Omega)} \frac{b\,(\mu,v)}{|\!|\!|v|\!|\!|}, \quad \mu\in W_{n;h}(\mathcal{S}) \tag{2.5}$$

for the mesh dependent norm $\|\cdot\|_L$. The same type of a priori estimate as in (2.3) holds if the solution (u,λ) is smooth enough:

$$|\!|\!|u-u_n|\!|\!|^2 + \|\lambda-\lambda_n\|_L^2 \le C(n)\sum_{i=1}^k h_i^{2n}\|u\|^2_{n+1;\Omega_i}. \tag{2.6}$$

Proof. In a first step, we define a positive, continuous, piecewise linear function $g\in W_{1;h}(\mathcal{S})$. We use that each element of $W_{1;h}(\mathcal{S})$ is uniquely determined by its values at the interior vertices of Γ_{ij}, $1\le i\le K$, $j\in\mathcal{M}(i)$. g reflects the weights of the mesh dependent norm and is given by

$$g(p) := \frac{1}{2}\left(\frac{h_{e_1}}{\alpha_{e_1}} + \frac{h_{e_2}}{\alpha_{e_2}}\right), \quad \bar{e}_1\cap\bar{e}_2 = p$$

where p is an interior vertex of Γ_{ij}, $1\le i\le K$, $j\in\mathcal{M}(i)$ and $e_1,e_2\in\mathcal{E}_L$. Let $P_{\Gamma_{ij}}$ be the L^2-projection onto $W_n^0(\Gamma_{ij};\mathcal{T}_{h_i}) := \{v\in C(\Gamma_{ij}) \mid v|_e\in P_n(e),\ e\in\mathcal{E}_{h_i}\cap\Gamma_{ij},\ v(p)=0 \text{ if } p \text{ is an endpoint of } \Gamma_{ij}\}$ defined by

$$\int_{\Gamma_{ij}} \mu\, v\, d\sigma = \int_{\Gamma_{ij}} \mu\, P_{\Gamma_{ij}} v\, d\sigma, \quad \mu\in W_n(\Gamma_{ij};\mathcal{T}_{h_i}).$$

Due to the stability of $P_{\Gamma_{ij}}$ with respect to the mesh dependent L^2-norm $\|\cdot\|_{L^{-1}}$ and the shape regularity of the triangulation, we obtain

$$\|P_{\Gamma_{ij}}(g\,\mu)\|_{L^{-1}} \le C\|g\,\mu\|_{L^{-1}} \le C\|\mu\|_L \tag{2.7}$$

and

$$\|\mu\|_L^2 \le C\sum_{i=1}^K\sum_{j\in\mathcal{M}(i)} (\mu, g\,\mu)_{0;\Gamma_{ij}} = C\sum_{i=1}^K\sum_{j\in\mathcal{M}(i)} (\mu, P_{\Gamma_{ij}}(g\,\mu))_{0;\Gamma_{ij}} \le C\|P_{\Gamma_{ij}}(g\,\mu)\|_{L^{-1}}\cdot\|\mu\|_L. \tag{2.8}$$

For each $\mu \in W_{n;h}(\mathcal{S})$ there exists a $v_\mu \in \tilde{X}^{-1}_{n;h}(\Omega)$ with $\tilde{X}^{-1}_{n;h}(\Omega) := \{v \in X^{-1}_{n;h}(\Omega) \,|\, v|_{\overline{\Omega}_j \cap \Gamma_{ij}} = 0,\ 1 \le i \le k, j \in \mathcal{M}(i)\}$, such that $c|||v_\mu||| \le \|[v_\mu]_J\|_{L^{-1}}$ holds and the jump of v_μ across Γ_{ij} is equal to $P_{\Gamma_{ij}}(g\,\mu)$, $1 \le i \le K$, $j \in \mathcal{M}(i)$. Then, (2.7) and (2.8) yield

$$\|\mu\|_L \le C\,\frac{b\,(\mu, v_\mu)}{\|\mu\|_L} \le C\,\frac{b\,(\mu, v_\mu)}{\|[v_\mu]_J\|_{L^{-1}}} \le C\,\frac{b\,(\mu, v_\mu)}{|||v_\mu|||}$$

and thus estimate (2.5) holds.

The proof of the a priori estimate (2.6) is based on the approximation property in the mesh dependent norm:

$$\inf_{\mu \in W_{n;h}(\mathcal{S})} \|\lambda - \mu\|_L \le C(n) \sum_{i=1}^{k} h_i^{2n} \|u\|^2_{n+1;\Omega_i}. \tag{2.9}$$

In our next step, we have to consider $\|\lambda_n - \mu\|_L$ in more detail. The first equation of the saddle point problems (2.1) and (2.2) yields

$$b\,(\lambda_n - \mu, v) = a\,(u - u_n, v) + b\,(\lambda - \mu, v)\,, \quad v \in X^{-1}_{n;h}(\Omega)$$

and thus

$$\|\lambda_n - \mu\|_L \le C \left(|||u - u_n||| + \|\lambda - \mu\|_L \sup_{v \in X^{-1}_{n;h}(\Omega)} \inf_{\substack{w \in X^{-1}_{n;h}(\Omega) \\ [w]_J = [v]_J}} \frac{\|[w]_J\|_{L^{-1}}}{|||w|||} \right) \tag{2.10}$$

which together with the approximation property (2.9) gives the a priori estimate (2.6). □

The nonconformity of a finite element $w \in X^{-1}_{n;h}(\Omega)$ can be measured by

$$\inf_{v \in H^1_0(\Omega)} \|w - v\|_1$$

where $\|\cdot\|_1$ denotes the broken H^1-norm on Ω. The following lemma shows a relation between the jumps of $w \in X^{-1}_{n;h}(\Omega)$ along the interfaces and the nonconformity.

LEMMA 2.2. *Let $w \in X^{-1}_{n;h}(\Omega)$ satisfy*

$$b\,(\mu, w) = 0, \quad \mu \in W_{1;h}(\mathcal{S}). \tag{2.11}$$

Then, there exists a constant $0 < C_J$ independent of the refinement level such that

$$\sum_{e \in \mathcal{E}_L} \frac{\alpha_e}{h_e} \|[w]_J\|^2_{0;e} \le C_J |||w - v|||^2, \quad v \in H^1_0(\Omega).$$

Proof. We consider the L^2-orthogonal projection operator $\Pi_{\Gamma_{ij}} : L^2(\Gamma_{ij}) \longrightarrow W_1(\Gamma_{ij}; \mathcal{T}_{h_i})$, defined by

$$\int_{\Gamma_{ij}} \Pi_{\Gamma_{ij}} v \mu \, d\sigma = \int_{\Gamma_{ij}} v \mu \, d\sigma, \quad \mu \in W_1(\Gamma_{ij}; \mathcal{T}_{h_i}).$$

Then, there exits a constant $\tilde{C}_J$ independent of i and j, $1 \le i \le K$, $j \in \mathcal{M}(i)$ such that

$$\sum_{e \in \mathcal{E}_L \cap \Gamma_{ij}} \frac{1}{h_e} \|v - \Pi_{\Gamma_{ij}} v\|^2_{0;e} \le \tilde{C}_J |v|^2_{1/2;\Gamma_{ij}}, \quad v \in H^{1/2}(\Gamma_{ij}). \tag{2.12}$$

This property can easily be seen by using

$$\sum_{e\in\mathcal{E}_L\cap\Gamma_{ij}} \frac{1}{h_e}\|v-\Pi_{\Gamma_{ij}}v\|_{0;e}^2 \le 2\left(\sum_{e\in\mathcal{E}_L\cap\Gamma_{ij}} \frac{1}{h_e}\|v-Qv\|_{0;e}^2 + \sum_{e\in\mathcal{E}_L\cap\Gamma_{ij}} \frac{1}{h_e}\|\Pi_{\Gamma_{ij}}(Qv-v)\|_{0;e}^2\right).$$

Here Q denotes a locally defined quasi-interpolant, and we also use the stability of $\Pi_{\Gamma_{ij}}$ with respect to the weighted norm. Assumption (2.11) guarantees that $\Pi_{\Gamma_{ij}}(w|_{\Omega_i}) = \Pi_{\Gamma_{ij}}(w|_{\Omega_j})$ and we can conclude that for $v \in H_0(\Omega)$

$$\begin{aligned}\sum_{e\in\mathcal{E}_L} \tfrac{\alpha_e}{h_e}\|\,[w]_J\,\|_{0;e}^2 \;&=\; \sum_{e\in\mathcal{E}_L} \tfrac{\alpha_e}{h_e}\|\,\left[w-v-\Pi_{\Gamma_{ij}}(w-v)\right]_J\|_{0;e}^2 \\ &\le\; 2\tilde{C}_J\sum_{i=1}^{K}\sum_{j\in\mathcal{M}(i)}\left(\alpha_{\Omega_i}|w|_{\Omega_i}-v|_{1/2;\Gamma_{ij}}^2+\alpha_{\Omega_j}|w|_{\Omega_j}-v|_{1/2;\Gamma_{ij}}^2\right)\;\le\; C_J|\!|\!|w-v|\!|\!|^2\end{aligned}$$

where $|\cdot|_{1/2;\Gamma_{ij}}$ is the standard $H^{1/2}$-seminorm on Γ_{ij}. □

Remark 2.1: *The constant C_J depends on the ratio of the eigenvalues of a along the interfaces. In case that the Lagrange multiplier is defined on the side where a is smaller, we obtain a smaller constant.*

Remark 2.2: *For Lemma 2.2 it is sufficient that the jump of $w \in X_{n;h}^{-1}(\Omega)$ is orthogonal to $W_{1;h}(\mathcal{S})$ with respect to the bilinear form $b(\cdot,\cdot)$.*

To introduce a posteriori error estimators, we restrict ourselves to the lowest order mortar finite elements, $n = 1$. However our results can be easily extended to the higher order case.

3. A Hierarchical Basis Error Estimator on Subdomains. In this section, we present a hierarchical basis error estimator which is based on a defect correction in an appropriate higher order space, a hierarchical splitting, as well as some localization techniques, cf., e.g., [9, 10, 19, 20, 31]. For standard conforming finite element discretizations there are basically two ways to obtain such an error estimator. One of them follows Bank and Weiser [10], where the defect problem is first localized and then discretized. Secondly, using the ideas of Deuflhard, Leinen, Yserentant [19], the resulting continuous defect problem is first discretized and then localized. These concepts have been generalized for nonconforming Crouzeix-Raviart, [23], and mixed Raviart-Thomas discretizations, [2, 24, 32]. Here, we will use a combination of both techniques.
In a first step, the continuous defect problem will be localized for each subdomain Ω_i, $1 \le i \le K$. We then use a higher order finite element discretization as well as an approximation of the exact Neumann data given by the Lagrange multipliers in the solution of a boundary value problem on each subdomain. Finally, the same decoupling techniques as in [19] are applied to obtain one scalar equation for each edge of the subdomains. To obtain an efficient and reliable error estimator, we additionally have to take the jump of the mortar finite element solution across the interfaces into account.

3.1. Saturation assumptions. A characteristic feature of hierarchical basis error estimators is that they are based on *adequate saturation assumptions*. Here, we need two different types of saturation assumptions. The first concerns the approximation of the normal derivative:

$$\inf_{\mu\in W_{1;h}(\mathcal{S})}\|\lambda-\mu\|_L + \inf_{\mu\in W_{1;h}(\mathcal{S})}\|\lambda-\mu\|_{H_\star^{-1/2}} \le \hat{C}_h|\!|\!|u-u_1|\!|\!|, \quad \hat{C}_h > 0 \tag{3.1}$$

with $\hat{C}_h \le \hat{C}_\infty < \infty$. This assumption and (2.10) at once yield

$$\inf_{\mu\in W_{1;h}(\mathcal{S})}\|\lambda-\lambda_1\|_L \le C_\infty|\!|\!|u-u_1|\!|\!|$$

with a constant $0 < C_\infty < \infty$. The same type of estimate as in (2.10) holds for the dual norm $\|\cdot\|_{H_*^{-1/2}}$ (see, e.g., [11]) and thus

$$\inf_{\mu \in W_{1;h}(\mathcal{S})} \|\lambda - \lambda_1\|_{H_*^{-1/2}} \le C_\infty \|\!|u - u_1|\!\|.$$

In addition to (3.1), we need a second saturation assumption which is related to higher order finite elements. We consider the space of conforming piecewise quadratic finite elements on each subdomain with homogeneous boundary conditions

$$S_{2;0}(\Omega_i; \mathcal{T}_{h_i}) := \{v \in S_2(\Omega_i; \mathcal{T}_{h_i}) \mid \ v|_{\partial\Omega_i} = 0\}.$$

If we assume that the weak solution u is continuous on Ω, we can define the Dirichlet boundary condition of a discrete finite element solution pointwise. Let $u_{2;i} \in S_2(\Omega_i; \mathcal{T}_{h_i})$, $1 \le i \le K$ be the conforming piecewise quadratic solution of the following variational problem on Ω_i:

$$a_i\,(u_{2;i}, v) = (f, v)_{0;\Omega_i}, \quad v \in S_{2;0}(\Omega_i; \mathcal{T}_{h_i}) \tag{3.2}$$

with $u_{2;i}$ given on the boundary $\partial\Omega_i$ by

$$u_{2;i}(x) := u(x). \tag{3.3}$$

Here x is either a vertex of $\mathcal{P}_i \cap \partial\Omega_i$ or the midpoint of an edge $e \in \mathcal{E}_i \cap \partial\Omega_i$. Then, unique solvability of the variational problem (3.2) is guaranteed and if the weak solution is smooth enough, the error $u - u_{2;i}$ in the energy norm $\|\!|u - u_{2;i}|\!\|_{\Omega_i}$ is of order $\mathcal{O}(h_i^2)$. This well known a priori estimate motivates the saturation assumption

$$\|\!|u - u_{2;i}|\!\|_{\Omega_i} \le \beta_{h_i} \|\!|u - u_1|\!\|_{\Omega_i} \tag{3.4}$$

with $\beta_{h_i} \le \beta_\infty$ where β_∞ has to be small enough.

3.2. Definition of the Error Estimator. The main problem with the variational problem (3.2) is that the boundary data are unknown. Therefore, $u_{2;i}$ cannot be used as an efficient and reliable error estimator. A first step towards the definition of such an error estimator is to use an adequate approximation of the boundary values. It is natural to select the Lagrange multiplier λ_1 as Neumann boundary condition. As we will see, this simplification is justified by the saturation assumption (3.1).

Let $\varepsilon_i \in S_2(\Omega_i; \mathcal{T}_{h_i})$, $1 \le i \le K$ be the conforming piecewise quadratic solution of the following variational problem on Ω_i:

$$a_i\,(\varepsilon_i, v) = r_i(v) := (f, v)_{0;\Omega_i} - a_i\,(u_1, v) - b\,(\lambda_1, \tilde{v}), \quad v \in S_2(\Omega_i; \mathcal{T}_{h_i}) \tag{3.5}$$

where $\tilde{v} \in L^2(\Omega)$, $\tilde{v}|_{\Omega_i} = v$, and $\tilde{v}|_{\Omega \setminus \Omega_i} = 0$. Let $\varepsilon \in \prod_{i=1}^K S_2(\Omega_i; \mathcal{T}_{h_i})$ with $\varepsilon|_{\Omega_i} = \varepsilon_i$. For each interior subdomain Ω_i, $\partial\Omega_i \cap \partial\Omega = \emptyset$ with $b \equiv 0$, ε_i is only defined up to a constant and we therefore require

$$\int_{\Omega_i} \varepsilon_i \, dx = 0.$$

Then, it is easy to verify that

$$a_i\,(\varepsilon_i, v) = 0, \quad v \in S_1(\Omega_i; \mathcal{T}_{h_i}). \tag{3.6}$$

We are now in the standard conforming situation on Ω_i with Neumann boundary data. It is well known, [19], that in (3.5), the coupling between $S_1(\Omega_i;\mathcal{T}_{h_i})$ and the hierarchical quadratic bubble surplus can be neglected. Therefore, there exist constants $C_{con} > c_{con} > 0$, independent of the refinement level, such that

$$c_{con} \sum_{e\in\mathcal{E}_{h_i}\setminus\partial\Omega} |||\gamma_e\Phi_e|||^2_{\Omega_i} \le |||\varepsilon_i|||^2_{\Omega_i} \le C_{con} \sum_{e\in\mathcal{E}_{h_i}\setminus\partial\Omega} |||\gamma_e\Phi_e|||^2_{\Omega_i}, \quad 1\le i\le K \tag{3.7}$$

where $\gamma_e \in \mathbb{R}$ is defined as

$$\gamma_e = \frac{r_i(\Phi_e)}{a_i\left(\Phi_e,\Phi_e\right)}, \quad e\in\mathcal{E}_{h_i}\setminus\partial\Omega.$$

Here the bubble function Φ_e living on Ω_i, is associated with the midpoint of the edge $e \in \mathcal{E}_{h_i}$. Unfortunately, we obtain only a lower bound for the error $|||u-u_1|||$ by means of $|||\varepsilon|||$. Therefore, we have to include a further term in the definition of a reliable and efficient error estimator. The hierarchical basis error estimator is defined locally by:

$$\begin{aligned}
\eta_H^2 &:= \sum_{T\in\mathcal{T}_h} \eta^2_{H;T},\\
\eta^2_{H;T} &:= \sum_{e\subset\partial T\setminus\partial\Omega} w_e |||\gamma_e\Phi_e|||^2_{\Omega_i} + \sum_{e\subset\partial T\cap\mathcal{E}_L} \tfrac{\alpha_e}{h_e}\|\,[u_1]_J\,\|^2_{0;e}, \quad T\in\mathcal{T}_h,
\end{aligned}$$

where $w_e = 0.5$ if e is an interior edge of a subdomain Ω_i and $w_e = 1$ otherwise.

THEOREM 3.1. *Under the weak saturation assumptions (3.1) and (3.4) there exist constants $0 < c_{hier} < C_{hier}$ independent of the refinement level, such that*

$$c_{hier}\eta_H^2 \le |||u-u_1|||^2 \le C_{hier}\eta_H^2.$$

Proof. The upper bound for the error estimator can be established easily. Due to Lemma 2.2 and (3.7) we need to consider only $|||e|||$ in detail. The Galerkin orthogonality (3.6) yields

$$|||\varepsilon|||^2 = \sum_{i=1}^K a_i\left(\varepsilon_i,\varepsilon_i - v\right) = \sum_{i=1}^K a_i\left(u-u_1,\varepsilon_i - v\right) - b\left(\lambda_1-\lambda,\varepsilon-v\right), \quad v\in X^{-1}_{1;h}(\Omega).$$

The special choice $v = I\varepsilon$, where I is the Lagrange interpolant defined locally on each subdomain, guarantees that $(\varepsilon - I\varepsilon)|_{\Gamma_{ij}} \in H^{1/2}_{00}(\Gamma_{ij})$. By means of the saturation assumption (3.1) and $\|[\varepsilon - I\varepsilon]_J\|_{H^{1/2}_{00}} \le C_I |||\varepsilon - I\varepsilon|||_{\Omega_i}$, we conclude that

$$\begin{aligned}
|||e|||^2 &\le |||u-u_1|||\,|||\varepsilon - I\varepsilon||| + \|\lambda_1-\lambda\|_{H_\star^{-1/2}}\|\,[\varepsilon - I\varepsilon]_J\,\|_{H^{1/2}_{00}}\\
&\le C_I |||u-u_1|||\cdot|||\varepsilon|||.
\end{aligned}$$

Hence, the upper bound for η_H is proved with a constant which also depends on the choice of the mortar side and on the ratio of the eigenvalues of a in the individual subdomains.
The saturation assumption (3.4) guarantees the following equivalence

$$\frac{1}{1+\beta_\infty}|||\hat{u}-u_1||| \le |||u-u_1||| \le \frac{1}{1-\beta_\infty}|||\hat{u}-u_1|||$$

where $\hat{u}|_{\Omega_i} = u_{2;i}$. Furthermore, we obtain on each subdomain Ω_i

$$|||u_{2;i}-u_1|||^2_{\Omega_i} = a_i\left(\varepsilon_i, u_{2;i}-u_1\right) + a_i\left(u_{2;i}-u,u_{2;i}-u_1\right) + b\left(\lambda_1-\lambda,u_{2;i}-u_1\right),$$

where $u_{2;i}$ is extended to zero on Ω_j, $j \neq i$. Since $(u - u_{2;i})|_{\Gamma_{ij}} \in H_{00}^{1/2}(\Gamma_{ij})$, we obtain

$$b(\lambda_1 - \lambda, \hat{u} - u_1) = b(\lambda_1 - \lambda, \hat{u} - u - u_1) \leq \|\lambda_1 - \lambda\|_{H_\star^{-1/2}} \|[\hat{u} - u]_J\|_{H_{00}^{1/2}} + \|\lambda_1 - \lambda\|_L + \|[u_1]_J\|_{L^{-1}}.$$

Summing over all subdomains, we finally get, by means of a trace theorem, (3.1), and (3.4),

$$|\!|\!|\hat{u} - u_1|\!|\!|^2 \leq (|\!|\!|\varepsilon|\!|\!| + C_\infty \|[u_1]_J\|_{L^{-1}} + (1 + cC_\infty)|\!|\!|u - \hat{u}|\!|\!|)|\!|\!|\hat{u} - u_1|\!|\!|.$$

If β_∞ is small enough, the error estimator provides an upper bound for energy norm of the error. □

Remark 3.1: *The constants in the upper and lower bound are better if the Lagrange multiplier is defined on the side where the eigenvalues of the diffusion coefficient a in (1.1) are smaller.*

4. A Fully Hierarchical Basis Error Estimator. In the previous section, we have studied a hierarchical basis error estimator η_H for the broken energy norm of the error $u - u_1$. Its design was mainly based on the decoupling of the subdomains and on the use of appropriate Neumann boundary conditions.
A more general framework for hierarchical based error estimators is presented in [9]. In particular, the results obtained for elliptic variational problems are applied on saddle point formulations. We also refer to [2, 24, 32] where hierarchical error estimators for mixed finite elements are considered. In this section, we consider a *fully hierarchical basis error estimator* where we will use the higher order mortar ansatz space $X_{2;h}^{-1}(\Omega)$ for the approximation of the weak solution u and the space $W_{2;h}(\mathcal{S})$ for a better approximation of the normal derivative.

Motivated by the a priori error estimate (2.6), we will use the following saturation assumption

$$|\!|\!|u - u_2|\!|\!|^2 + \|\lambda - \lambda_2\|_L^2 \leq \beta_h^2(|\!|\!|u - u_1|\!|\!|^2 + \|\lambda - \lambda_1\|_L^2) \tag{4.1}$$

where $0 < \beta_h^2 \leq \beta^2 < 1/2$. Under this assumption, $(u_2 - u_1, \lambda_2 - \lambda_1)$ yields upper and lower bounds of the error $(u - u_1, \lambda - \lambda_1)$

$$(1+\beta^2)\left(|\!|\!|u - u_1|\!|\!|^2 + \|\lambda - \lambda_1\|_L^2\right) \leq |\!|\!|u_2 - u_1|\!|\!|^2 + \|\lambda_2 - \lambda_1\|_L^2 \leq (1+2\beta^2)\left(|\!|\!|u - u_1|\!|\!|^2 + \|\lambda - \lambda_1\|_L^2\right). \tag{4.2}$$

In addition, $(u_2 - u_1, \lambda_2 - \lambda_1)$ satisfies the following discrete defect problem:

$$\begin{aligned} a(u_2 - u_1, v) + b(\lambda_2 - \lambda_1, v) &= r_1(v), \quad v \in X_{2;h}^{-1}(\Omega), \\ b(\mu, u_2 - u_1) &= r_2(v), \quad \mu \in W_{2;h}(\mathcal{S}) \end{aligned} \tag{4.3}$$

where $r_1(v) := (f; v)_0 - a(u_1, v) - b(\lambda_1, v)$ and $r_2(\mu) := -b(\mu, u_1)$. Then, it is obvious that $r_1(v) = 0$ for $v \in X_{1;h}^{-1}(\Omega)$ and that $r_2(\mu) = 0$ for $\mu \in W_{1;h}(\mathcal{S})$.
It is sufficient to consider the solution of the variational problem (4.3) in more detail. Unfortunately, it is a global saddle point problem and its exact solution cannot be obtained locally. We therefore need to construct an adequate approximation of $u_2 - u_1$ and $\lambda_2 - \lambda_1$ which can be easily computed, and that is, at the same time, equivalent to $(u_2 - u_1, \lambda_2 - \lambda_1)$.

A first step towards the definition of an efficient and reliable a posteriori error estimator is the introduction of a hierarchical splitting of the spaces

$$X_{2;h}^{-1}(\Omega) = X_{1;h}^{-1}(\Omega) \oplus \widehat{X}_{2;h}^{-1}(\Omega), \quad \widehat{X}_{2;h}^{-1}(\Omega) := \bigoplus_{k=1}^{K} \bigoplus_{e \in \mathcal{E}_k \setminus \partial\Omega} \operatorname{span}\{\Phi_e\}$$

where Φ_e denotes the quadratic bubble function associated with the midpoint of e. The support of Φ_e, $e \in \mathcal{E}_k$ is restricted to $\overline{\Omega}_k$. The ansatz space for the Lagrange multiplier is decomposed according to

$$W_{2;h}(\mathcal{S}) = W_{1;h}(\mathcal{S}) \oplus \widehat{W}_{2;h}(\mathcal{S}), \quad \widehat{W}_{2;h}(\mathcal{S}) := \bigoplus_{e\in\mathcal{E}_L} \operatorname{span}\{\Psi_e\}$$

where Ψ_e is the one dimensional quadratic bubble function associated with e, if e contains no endpoint of an interface Γ_{ij}, and Ψ_e is the linear hat function associated with p otherwise. In the case that e contains two endpoints of an interface, we select only one of the two hat functions (see Fig. 4.1).

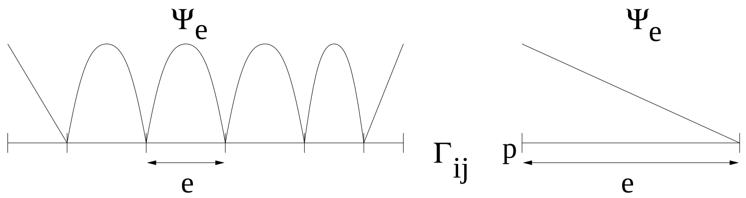


FIG. 4.1. *Basis functions of the hierarchical surplus* $\widehat{W}_{2;h}(\mathcal{S})$

Based on this splitting, we consider the following modified saddle point problem on the hierarchical surplus space: Find $(u_e, \lambda_e) \in \widehat{X}^{-1}_{2;h}(\Omega) \times \widehat{W}_{2;h}(\mathcal{S})$ such that

$$\begin{aligned} a\,(u_e, v) + b\,(\lambda_e, v) &= r_1(v), \quad v \in \widehat{X}^{-1}_{2;h}(\Omega), \\ b\,(\mu, u_e) &= r_2(v), \quad \mu \in \widehat{W}_{2;h}(\mathcal{S}). \end{aligned} \tag{4.4}$$

We will show that the solution of the variational problem (4.3) can be replaced by that of (4.4), and that (u_e, λ_e) still yields upper and lower bounds for the error. The equivalence of the saddle point problems (4.3) and (4.4) is obtained by a strengthened Cauchy-Schwarz inequality and the stability of the discrete saddle point problem (4.3) with respect to a mesh dependent norm. The natural norm for the Lagrange multiplier will be replaced by $\|\cdot\|_L$, see definition (2.4). Associated with $\|\cdot\|_L$ is the weighted bilinear form $(\cdot,\cdot)_L$ which is defined by

$$(\mu, \xi)_L := \sum_{e\in\mathcal{E}_L} \frac{h_e}{\alpha_e} \int_e \mu\,\xi\, d\sigma, \quad \mu,\, \xi \in W_{2;h}(\mathcal{S}).$$

The strengthened Cauchy-Schwarz inequalities have to be established between the functions of the original space and those of the hierarchical surplus spaces.

LEMMA 4.1. *There exist constants* $0 < \eta_1, \eta_2 < 1$ *independent of the refinement level such that the following strengthened Cauchy-Schwarz inequalities hold*

$$a\,(v, w)^2 \le \eta_1^2 \|v\|^2 \,\|w\|^2, \quad v \in X^{-1}_{1;h}(\Omega),\ w \in \widehat{X}^{-1}_{2;h}(\Omega) \tag{4.5}$$

$$(\mu, \xi)_L^2 \le \eta_2^2 \|\mu\|_L^2 \,\|\xi\|_L^2, \quad \mu \in W_{1;h}(\mathcal{S}),\ \xi \in \widehat{W}_{2;h}(\mathcal{S}). \tag{4.6}$$

Proof. The strengthened Cauchy-Schwarz inequality (4.5) is well known [19]. Taking the local structure of the ansatz space $\widehat{X}^{-1}_{2;h}(\Omega)$ into account, we easily get

$$\int_e \mu\,\xi\, d\sigma \le \sqrt{\frac{5}{6}} \|\mu\|_{0;e} \|\xi\|_{0;e}$$

and thus (4.6) holds with $\eta_2^2 = \frac{5}{6}$. ☐

For the stability, we need to consider the bilinear form $b(\cdot,\cdot)$ in more detail; the discrete Babuška-Brezzi condition is essential for the proof of the equivalence between (u_e,λ_e) and $(u_2-u_1,\lambda_2-\lambda_1)$.

LEMMA 4.2. *The bilinear form $b(\cdot,\cdot)$ is continuous on $\widehat{X}^{-1}_{2;h}(\Omega)\times W_{2;h}(\mathcal{S})$*

$$b(\mu,v)\le C_c\|\mu\|_L|||v|||,\quad v\in\widehat{X}^{-1}_{2;h}(\Omega),\ \mu\in W_{2;h}(\mathcal{S}) \tag{4.7}$$

with a constant C_c independent of h. In addition a discrete Babuška-Brezzi condition holds

$$\sup_{\substack{v\in X\\ |||v|||\neq 0}}\frac{b(\mu,v)}{|||v|||}\ge c_s\|\mu\|_L,\quad \mu\in V, \tag{4.8}$$

where $X\times V\in\{X^{-1}_{2;h}(\Omega)\times W_{2;h}(\mathcal{S}),\widehat{X}^{-1}_{2;h}(\Omega)\times\widehat{W}_{2;h}(\mathcal{S})\}$ and the constant c_s is independent of the refinement level.

Proof. By the definition of $\|\cdot\|_L$ and $\|\cdot\|_{L^{-1}}$, we obtain immediately

$$b(\mu,v)\le\|\mu\|_L\|\,[v]_J\,\|_{L^{-1}},\quad \mu,[v]_J\in L^2(\mathcal{S}),$$

and $\|\,[v]_J\,\|_{L^{-1}}\le C_c|||v|||$ for $v\in\widehat{X}^{-1}_{2;h}(\Omega)$. Thus, (4.7) is established.
The Babuška-Brezzi bound for $X\times V=X^{-1}_{2;h}(\Omega)\times W_{2;h}(\mathcal{S})$ has been established in Lemma 2.1. In case of $X\times V=\widehat{X}^{-1}_{2;h}(\Omega)\times\widehat{W}_{2;h}(\mathcal{S})$, we can choose $[v]_J|_e=h_e\alpha_e^{-1}\mu|_e$ on each edge $e\in\mathcal{E}_l$ which does not contain an endpoint of an interface. We get (4.8) by recalling that $\mu|_e\in P_1(e)$ on the edges e containing one endpoint of an interface, and by following the same lines of arguments as in the proof of Lemma 2.1, we get (4.8). ☐

The following theorem states the equivalence of the solution of the defect problem in the hierarchical surplus (u_e,λ_e) and the solution of the original discrete defect problem $(u_2-u_1,\lambda_2-\lambda_1)$.

THEOREM 4.3. *Under the saturation assumption (4.1), there exit constants $0<c_f\le C_f$, such that*

$$c_f(|||u_e|||^2+\|\lambda_e\|_L^2)\le|||u-u_1|||^2+\|\lambda-\lambda_1\|_L^2\le C_f(|||u_e|||^2+\|\lambda_e\|_L^2).$$

Proof. The proof is basically based on the discrete Babuška-Brezzi condition (4.8) and the strengthened Cauchy-Schwarz inequality (4.5). By means of the saturation assumption (4.1), it is sufficient to show the equivalence of $|||u_e|||^2+\|\lambda_e\|_L^2$ and $|||u_2-u_1|||^2+\|\lambda_2-\lambda_1\|_L^2$. In a first step, we establish the lower bound. Taking the stability of the saddle point problem (4.4), the continuity of the bilinear form $b(\cdot,\cdot)$ on $\widehat{X}^{-1}_{2;h}(\Omega)\times W_{2;h}(\mathcal{S})$, and Lemma 2.2 into account, we obtain

$$\begin{aligned}
c(|||u_e|||^2+\|\lambda_e\|_L^2)^{1/2} &\le \sup_{\substack{v\in\widehat{X}^{-1}_{2;h}(\Omega)\\ |||v|||\le 1}} a(u_e,v)+b(\lambda_e,v)+\sup_{\substack{\mu\in\widehat{W}_{2;h}(\mathcal{S})\\ \|\mu\|_L\le 1}} b(\mu,u_e)\\
&= \sup_{\substack{v\in\widehat{X}^{-1}_{2;h}(\Omega)\\ |||v|||\le 1}} r_1(v)+\sup_{\substack{\mu\in\widehat{W}_{2;h}(\mathcal{S})\\ \|\mu\|_L\le 1}} r_2(\mu)\\
&= \sup_{\substack{v\in\widehat{X}^{-1}_{2;h}(\Omega)\\ |||v|||\le 1}} a(u_2-u_1,v)+b(\lambda_2-\lambda_1,v)+\sup_{\substack{\mu\in\widehat{W}_{2;h}(\mathcal{S})\\ \|\mu\|_L\le 1}} b(\mu,u_2-u_1)\\
&\le (1+C_J)|||u_2-u_1|||+C_c\|\lambda_2-\lambda_1\|_L.
\end{aligned}$$

The upper bound follows from the stability of the saddle point problem (4.3), the continuity of the bilinear form $b(\cdot,\cdot)$ and Lemma 4.1. Let $v=v_1+v_2$ and $\mu=\mu_1+\mu_2$, with $v_1\in X^{-1}_{1;h}(\Omega)$,

$v_2 \in \widehat{X}_{2;h}^{-1}(\Omega)$ and $\mu_1 \in W_{1;h}(\mathcal{S})$, $\mu_2 \in \widehat{W}_{2;h}(\mathcal{S})$, respectively. Then,

$$\begin{aligned}
c(|\!|\!| u_2 - u_1 |\!|\!|^2 + \|\lambda_2 - \lambda_1\|_L^2)^{1/2} &\leq \sup_{\substack{v \in X_{2;h}^{-1}(\Omega) \\ |\!|\!| v |\!|\!| \leq 1}} a\,(u_2 - u_1, v) + b\,(\lambda_2 - \lambda_1, v) + \sup_{\substack{\mu \in W_{2;h}(\mathcal{S}) \\ \|\mu\|_L \leq 1}} b\,(\mu, u_2 - u_1) \\
&\leq \sup_{\substack{v \in X_{2;h}^{-1}(\Omega) \\ |\!|\!| v |\!|\!| \leq 1}} r_1(v) + \sup_{\substack{\mu \in W_{2;h}(\mathcal{S}) \\ \|\mu\|_L \leq 1}} r_2(\mu) = \sup_{\substack{v \in X_{2;h}^{-1}(\Omega) \\ |\!|\!| v |\!|\!| \leq 1}} r_1(v_2) + \sup_{\substack{\mu \in W_{2;h}(\mathcal{S}) \\ \|\mu\|_L \leq 1}} r_2(\mu_2) \\
&= \sup_{\substack{v \in X_{2;h}^{-1}(\Omega) \\ |\!|\!| v |\!|\!| \leq 1}} a\,(u_e, v_2) + b\,(\lambda_e, v_2) + \sup_{\substack{\mu \in W_{2;h}(\mathcal{S}) \\ \|\mu\|_L \leq 1}} b\,(\mu_2, u_e) \\
&\leq (|\!|\!| u_e |\!|\!| + C_c \|\lambda_e\|_L) \sup_{\substack{v \in X_{2;h}^{-1}(\Omega) \\ |\!|\!| v |\!|\!| \leq 1}} |\!|\!| v_2 |\!|\!| + C_c |\!|\!| u_e |\!|\!| \sup_{\substack{\mu \in W_{2;h}(\mathcal{S}) \\ \|\mu\|_L \leq 1}} \|\mu_2\|_L \\
&\leq \left(\tfrac{1}{\sqrt{1-\eta_1}} + \tfrac{C_c}{\sqrt{1-\eta_2}} \right) |\!|\!| u_e |\!|\!| + \tfrac{C_c}{\sqrt{1-\eta_1}} \|\lambda_e\|_L .
\end{aligned}$$

□

In the next step, we have to localize the saddle point problem (4.4) defined on the hierarchical surplus. As in the previous section, we neglect the coupling between the quadratic bubble functions and replace the bilinear form $a(\cdot,\cdot)$ by $\tilde{a}(\cdot,\cdot)$

$$\tilde{a}(\psi, \phi) := \sum_{i=1}^{K} \sum_{e \in \mathcal{E}_{h_i} \setminus \partial\Omega} \sigma_e \tau_e a(\Phi_e, \Phi_e), \quad \phi := \sum_{i=1}^{K} \sum_{e \in \mathcal{E}_{h_i} \setminus \partial\Omega} \sigma_e \Phi_e, \;\; \psi := \sum_{i=1}^{K} \sum_{e \in \mathcal{E}_{h_i} \setminus \partial\Omega} \tau_e \Phi_e .$$

The following simplified saddle point problem leads to the definition of the fully hierarchical basis error estimator: Find $(\tilde{u}_e, \tilde{\lambda}_e) \in \widehat{X}_{2;h}^{-1}(\Omega) \times \widehat{W}_{2;h}(\mathcal{S})$ such that

$$\begin{aligned}
\tilde{a}(\tilde{u}_e, v) + b(\tilde{\lambda}_e, v) &= r_1(v), \quad v \in \widehat{X}_{2;h}^{-1}(\Omega), \\
b\,(\mu, \tilde{u}_e) &= r_2(v), \quad \mu \in \widehat{W}_{2;h}(\mathcal{S}).
\end{aligned} \tag{4.9}$$

Then, the solution $\tilde{u}_e$ can be written in the form

$$\tilde{u}_e = \sum_{i=1}^{K} \sum_{e \in \mathcal{E}_{h_i} \setminus \partial\Omega} \tilde{\gamma}_e \Phi_e$$

and the coefficients $\tilde{\gamma}_e$ are given by (4.9). The coefficients associated with the interior edges of the subdomains can be obtained by the formula

$$\tilde{\gamma}_e = \frac{r_1(\Phi_e)}{a\,(\Phi_e, \Phi_e)} .$$

Remark 4.1: *In the interior of each subdomain, we obtain the same formula, as in the previous section for the definition of the error estimator.*

To get the coefficients $\tilde{\gamma}_e$ associated with an edge on the skeleton and $\tilde{\lambda}_e$, we have to solve one global system on each interface Γ_{ij}. These systems are not coupled and are, in contrast to the original saddle point problem, associated, with an 1D triangulation; the dimension is small compared with the original global system. Additionally, the condition number does not depend on the refinement level. Therefore, the system can be solved approximately in a few iteration steps.

We now define the fully hierarchical basis error estimator locally by:

$$\begin{aligned}
\eta_{FH}^2 &:= \sum_{T\in\mathcal{T}_h} \eta_{FH;T}^2 + \eta_{L;T}^2, \\
\eta_{FH;T}^2 &:= \sum_{e\subset\partial T\setminus\partial\Omega} w_e \|\gamma_e \Phi_e\|_{\Omega_i}^2, \\
\eta_{L;T}^2 &:= \sum_{e\subset\partial T\cap\mathcal{E}_L} \frac{h_e}{\alpha_e} \|\tilde{\lambda}_e\|_{0;e}^2, \quad T\in\mathcal{T}_h
\end{aligned}$$

where $w_e = 0.5$ if e is an interior edge of some subdomain Ω_i and $w_e = 1$ otherwise.

THEOREM 4.4. *Under the saturation assumption (4.1), there exit constants c_{fully}, $C_{fully} > 0$ independent of the refinement level, such that*

$$c_{fully}\eta_{FH}^2 \le \|u-u_1\|^2 + \|\lambda-\lambda_1\|_L^2 \le C_{fully}\eta_{FH}^2.$$

Proof. The proof is an easy consequence of Theorem 4.3 and the equivalence of the bilinear forms $a(\cdot,\cdot)$ and $\tilde{a}(\cdot,\cdot)$. ☐

Considering the 1D subproblems on the interfaces in detail, we obtain the following algebraic system on the interfaces

$$\begin{pmatrix} D_M & 0 & B_\lambda \\ 0 & D_L & D_\lambda \\ B_\lambda^T & D_\lambda & 0 \end{pmatrix} \begin{pmatrix} u_M \\ u_L \\ \lambda_L \end{pmatrix} = \begin{pmatrix} r_M \\ r_L \\ r_\lambda \end{pmatrix},$$

where the stiffness matrices starting with capital D stand for diagonal matrices. The index L refers to the nodes on the nonmortar side whereas the index M refers to those on the opposite mortar side. We recall, that the Lagrange multiplier is defined on the nonmortar side. Then, elimination of u_M and u_L gives

$$\left(D_\lambda D_L^{-1} D_\lambda + B_\lambda^T D_M^{-1} B_\lambda\right)\lambda_L = \tilde{r}. \tag{4.10}$$

The first part of the Schur complement matrix $D_\lambda D_L^{-1} D_\lambda$ is a diagonal matrix whereas the second matrix, generally, is not.

We call an edge $e \subset \Gamma_{ij}$ a macroedge if it can be written as

$$e = \bigcup_{\hat{e}\in\mathcal{E}_{h_i},\ \hat{e}\subset e} \hat{e}, \quad e\in\mathcal{E}_{h_j} \quad \text{or} \quad e = \bigcup_{\hat{e}\in\mathcal{E}_{h_j},\ \hat{e}\subset e} \hat{e}, \quad e\in\mathcal{E}_{h_i}.$$

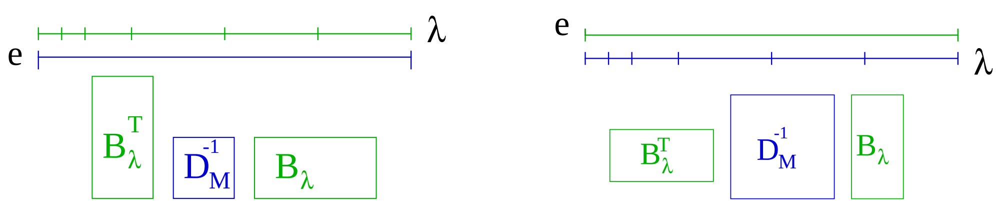


FIG. 4.2. *Macroedge e on mortar side (left) or on nonmortar side (right)*

Figure 4.2 shows the two different types of macroedges depending on the choice of the Lagrange multiplier. If two edges $e_i \in \mathcal{E}_{h_i}$ and $e_j \in \mathcal{E}_{h_j}$ are the same, $e_i = e_j$, we will designate only as a macroedge; the choice is arbitrary.

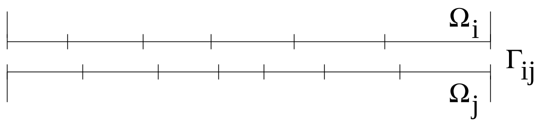


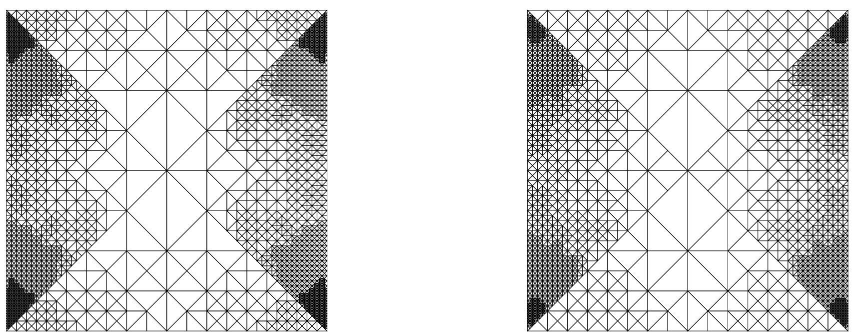
FIG. 4.3. *Nonconforming initial triangulation*

In the special case where we start with a geometrical conforming macrotriangulation and use some standard refinement techniques based on bisection of the edges [7, 8] to obtain the initial coarse triangulation, each interface is the union of its macroedges. This is not true if we start with a nonconforming initial triangulation (see Fig. 4.3) or if we use sliding meshes.

In case that the skeleton can be written as the union of macroedges, the second part of the Schur complement $B_\lambda^T D_M^{-1} B_\lambda$ is blockdiagonal. The number of blocks of the Schur complement is given by the number of macroedges whereas the dimension of a single block, associated with the macroedge e, is given by n_e, where n_e is the number of its subedges. A closer look at the structure of the block systems shows that it can be easily solved explicitly. Thus, in this special case the use of the error estimator only requires the solution of local problems on the macroedges.

5. Numerical Results. In this section, we will present some numerical results illustrating the adaptive refinement process and the efficiency of the different error estimators. We consider the error estimators analyzed in section 3 and 4.
Starting from a coarse triangulation, the discretized saddle point problems are solved on each refinement level by a preconditioned iteration scheme [3, 25, 28]. For the adaptive refinement process, we use the bisection strategy proposed by Bänsch [7].

We apply the error estimators on two examples with discontinuous coefficients. We consider the diffusion equation $-\text{div}\, a\nabla u = f$, on $(0,1)^2$, where the coefficient a is discontinuous. The unit square Ω is decomposed into four subdomains $\Omega_1 := \{(x,y) \in \Omega|\ x < y < 1-x\}$, $\Omega_2 := \{(x,y) \in \Omega|\ y < x < 1-y\}$, $\Omega_3 := \{(x,y) \in \Omega|\ x > y > 1-x\}$ and $\Omega_4 := \{(x,y) \in \Omega|\ 1-y < x < y\}$ and the coefficient a restricted to the subdomains Ω_i is given by a constant a_i, $1 \le i \le 4$. The right hand side f and the Dirichlet boundary conditions are chosen to match a given exact solution.
In the first example, the solution is $u(x,y) = (y-x)(1-x-y)(x-0.5)^2(y-0.5)^2/a_i$ and $a_1 = a_3 = 1$, $a_2 = a_4 = 100$.
Figure 5.1 shows the adaptively generated triangulations for example 1. No matching between



Hierarchical Error Estimator | Fully Hierarchical Error Estimator

FIG. 5.1. *Adaptive refined triangulations (Example 1)*

the triangulations at the interfaces is required. Strongly nonconforming global triangulations are generated by the adaptive process. This illustrates the advantage of the mortar method compared with a standard approach where only conforming triangulations are allowed.

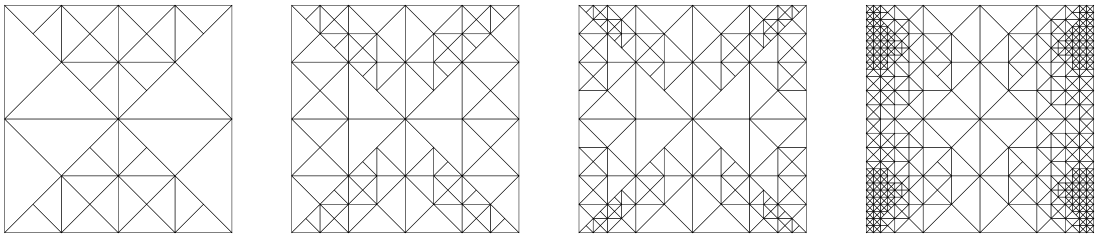


FIG. 5.2. *Nonmortar side where a is larger (Fully Hierarchical EE)*

Figure 5.2 and Figure 5.3 show the influence of the choice of the Lagrange multiplier on the adaptively generated triangulations. We observe a completely different behavior during the first refinement steps. In the first case, we obtain a strong adaptive refinement along the nonmortar side at the beginning of the refinement process. However, asymptotically the adaptive refinement generates almost the same triangulations on the subdomains where the coefficient is smaller. In

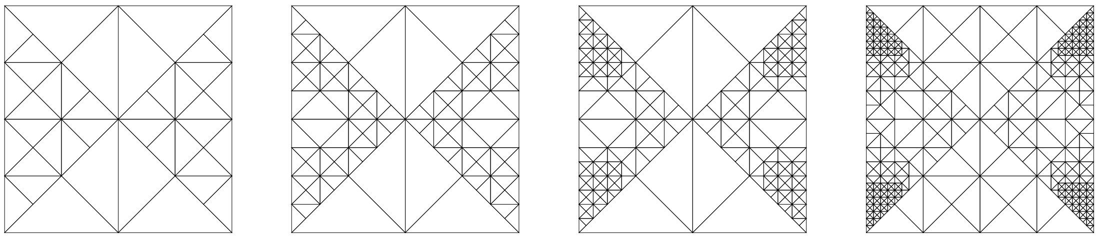


FIG. 5.3. *Nonmortar side where a is smaller (Fully Hierarchical EE)*

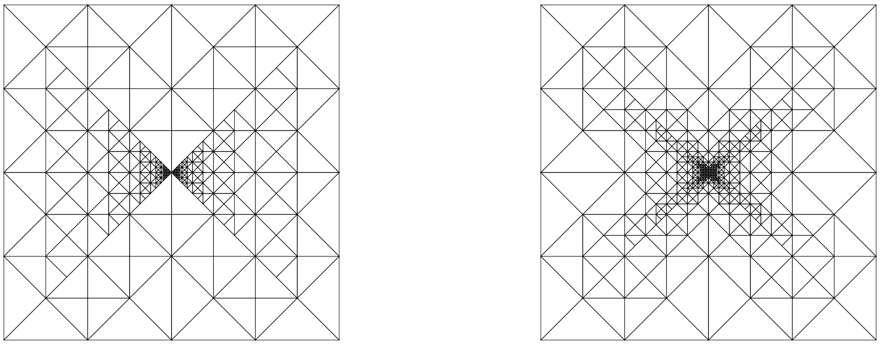
Figures 5.1 and 5.3 the Lagrange multiplier are defined on the side where a is smaller. If the nonmortar side is chosen where a is larger, we obtain a triangulation which tends to be more conforming at the interfaces. In this case, the Lagrange multiplier is associated with the subdomain where the triangulation is coarser. This leads to poor approximation properties for the Neumann boundary condition, and we have to adapt the triangulation along the interface on the side where a is larger, see Fig. 5.2.

TABLE 5.1
Efficiency error for the fully hierarchical error estimator (Example 1)

	Lagrange multiplier on $a = 100$				Lagrange multiplier on $a = 1$			
level	nodes	est. err.	real err.	eff. in.	nodes	est. err.	real err.	eff. in.
0	24	0.173	$0.287{\cdot}10^{-1}$	6.03	24	0.171	$0.287{\cdot}10^{-1}$	5.96
1	42	$0.338{\cdot}10^{-1}$	$0.156{\cdot}10^{-1}$	2.17	42	$0.169{\cdot}10^{-1}$	$0.132{\cdot}10^{-1}$	1.28
2	88	$0.224{\cdot}10^{-1}$	$0.101{\cdot}10^{-1}$	2.23	92	$0.604{\cdot}10^{-2}$	$0.615{\cdot}10^{-2}$	0.981
3	144	$0.916{\cdot}10^{-2}$	$0.528{\cdot}10^{-2}$	1.73	172	$0.303{\cdot}10^{-2}$	$0.321{\cdot}10^{-2}$	0.945
4	492	$0.148{\cdot}10^{-2}$	$0.143{\cdot}10^{-2}$	1.03	364	$0.189{\cdot}10^{-2}$	$0.178{\cdot}10^{-2}$	1.06
5	1242	$0.117{\cdot}10^{-2}$	$0.846{\cdot}10^{-3}$	1.39	862	$0.116{\cdot}10^{-2}$	$0.107{\cdot}10^{-2}$	1.09
6	3128	$0.546{\cdot}10^{-3}$	$0.508{\cdot}10^{-3}$	1.07	2246	$0.636{\cdot}10^{-3}$	$0.619{\cdot}10^{-3}$	1.03
7	8374	$0.307{\cdot}10^{-3}$	$0.301{\cdot}10^{-3}$	1.02	5922	$0.375{\cdot}10^{-3}$	$0.371{\cdot}10^{-3}$	1.01

In Table 5.1, the efficiency index as well as the norm of the error is given for the different choices of the Lagrange multiplier. At the beginning the error is smaller in case where the nonmortar side is associated with $a = 1$. In both cases, the efficiency index tends to one within the adaptive refinement process.

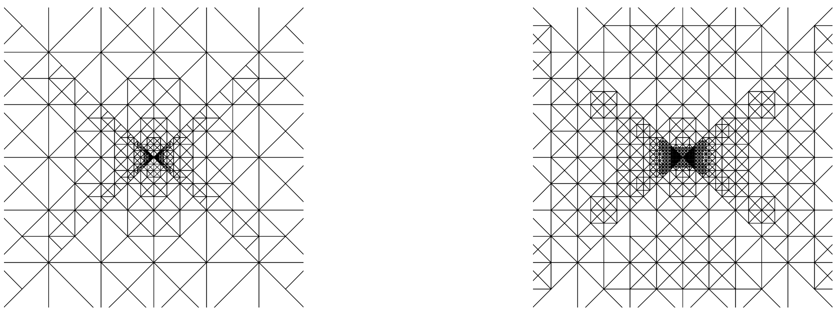
The second example is more complicate (see [30]) and the solution $u(x,y) = \alpha_i r^{0.1} \sin(0.1\phi + \theta_i)$ has a singularity at $(x,y) = (0.5, 0.5)$. Here $x = r \cdot \cos\phi + 0.5$, $y = r \cdot \sin\phi + 0.5$ and $\phi \in [\frac{\pi}{4}, \frac{9\pi}{4})$. The parameters of the subdomains Ω_i are given by $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = \alpha_4 = \sin(2.1\frac{\pi}{4})(\sin(0.1\frac{\pi}{4}))^{-1}$ and $a_1 = a_3 = 1$, $a_2 = a_4 = \alpha_2^2$ and $\theta_1 = 0.9\pi$, $\theta_2 = 1.35\pi$, $\theta_3 = 1.8\pi$, $\theta_4 = 0.45\pi$. Then, the solution is continuous and $[a_i \nabla u \mathbf{n}]_J$ is equal to zero on the interfaces.



Nonmortar side where a is smaller Nonmortar side where a is larger

FIG. 5.4. *Hierarchical Error Estimator (Example 2)*

Figures 5.4 and 5.5 show the influence of the choice of the Lagrange multiplier on the adaptive refinement process for the second example. Here, we use the hierarchical basis error estimator defined in section 3.



Nonmortar side where a is smaller Nonmortar side where a is larger

FIG. 5.5. *Hierarchical Error Estimator - Zoom of final triangulation (Example 2)*

If we compare the true error in Table 5.2, we find that in case where the nonmortar side is associated with the smaller coefficient a the number of nodes to obtain a given accuracy is considerably smaller. Therefore, to obtain better numerical results the Lagrange multiplier should be chosen on the side where the coefficient a is smaller. This can be also seen by a detailed a priori analysis; the constants in the estimates depend on the ratio of the coefficients restricted to the subdomains. We recall that the constants in the upper and lower a posteriori bounds are also

TABLE 5.2
Efficiency error for the hierarchical error estimator (Example 2)

	Lagrange multiplier on $a = 161.44$				Lagrange multiplier on $a = 1$			
level	nodes	est. err.	real err.	eff. in.	nodes	est. err.	real err.	eff. in.
0	60	16.4	8.43	1.95	60	5.97	8.43	0.708
1	108	10.2	7.28	1.41	132	4.55	6.39	0.712
2	162	9.95	6.41	1.55	168	4.06	5.69	0.715
3	252	8.03	5.73	1.40	220	3.64	5.09	0.715
4	366	4.09	5.14	0.794	274	3.29	4.60	0.716
5	444	3.42	4.65	0.737	344	2.99	4.17	0.718
6	522	2.99	4.23	0.708	416	2.74	3.80	0.721
7	600	2.68	3.86	0.695	488	2.52	3.48	0.726
8	696	2.43	3.54	0.687	560	2.34	3.20	0.731
9	780	2.23	3.25	0.686	632	2.17	2.95	0.737
10	872	2.06	3.00	0.686	710	2.02	2.73	0.742
11	944	1.91	2.78	0.690	814	1.88	2.52	0.747
12	1032	1.11	2.72	0.409	869	1.77	2.35	0.754
13	1808	0.908	2.48	0.366	1006	1.66	2.18	0.760

better if the nonmortar side is defined on the side where the eigenvalues of a are smaller, and the constants do not depend on the ratio of the eigenvalues on two subdomains sharing one interface.

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