OVERLAPPING SCHWARZ METHODS FOR VECTOR VALUED ELLIPTIC PROBLEMS IN THREE DIMENSIONS

RALF HIPTMAIR* AND ANDREA TOSELLI†

Abstract. This paper is intended as a survey of current results on algorithmic and theoretical aspects of overlapping Schwarz methods for discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problems set in suitable finite element spaces. The emphasis is on a unified framework for the motivation and theoretical study of the various approaches developed in recent years.

Generalized Helmholtz decompositions – orthogonal decompositions into the null space of the relevant differential operator and its complement – are crucial in our considerations. It turns out that the decompositions the Schwarz methods are based upon have to be designed separately for both components. In the case of the null space, the construction has to rely on liftings into spaces of discrete potentials.

Taking the cue from well-known Schwarz schemes for second order elliptic problems, we devise uniformly stable splittings of both parts of the Helmholtz decomposition. They immediately give rise to powerful preconditioners and iterative solvers.

Key words. Schwarz methods, domain decomposition, multilevel methods, multigrid, Raviart–Thomas finite elements, Nédélec's finite elements

AMS subject classifications. 65N55, 65N30

1. Introduction.

Schwarz methods offer highly efficient iterative solvers for discrete second order elliptic problems. In a finite element setting, the guiding principle is to provide a splitting of the finite element approximation space into subspaces and to seek corrections of an approximate solution in these subspaces [27, 54]. Considerable research has in recent years been devoted to Schwarz methods for second order elliptic problems. Prominent are multigrid methods [33], which were relatively recently revealed to be Schwarz methods [61, 32, 12]. Multigrid methods belong to the larger class of multilevel Schwarz methods, which includes multilevel preconditioners [15] and hierarchical basis type methods [62, 63, 5], as well. In all these methods, the basic subspace decomposition arises from a sequence of finite element spaces associated with a hierarchy of meshes generated by, possibly local, refinements. Other important Schwarz methods are overlapping domain decomposition methods [27, 54] and iterative substructuring algorithms [13, 14, 27, 26]. They base the subspace splitting on the decomposition of the computational domain as the union of smaller subregions. In addition, a global space, defined on a coarse mesh, is often indispensable [59]. First conceived for standard h–version conforming finite elements, Schwarz methods have been successfully applied to spectral methods [49, 51, 50] and nonconforming schemes [47]. They have also proved to be a valuable tool for the fast solution of fourth order problems [46, 66].

We point out that this presentation is confined to overlapping Schwarz methods in a broad sense. This means that the support of the functions belonging to different subspaces of the decomposition have some overlap. We will consider overlapping and multigrid methods, but important schemes, like hierarchical basis methods and iterative substructuring algorithms, will not be addressed.

* Institut für Mathematik, Universität Augsburg, e-mail: hiptmair@math.uni-augsburg.de

† Courant Institute, New York University, e-mail: toselli@cims.nyu.edu. This work was supported in part by the National Science Foundation under Grant NSF-CCR-9503408, and Grant NSF-ODURF-354151.

The algorithmic developments in the field of Schwarz methods have been paralleled by the emergence of a rather comprehensive convergence theory, which permits us to assess the performance of a scheme based on a few estimates characterizing the stability of the subspace decomposition [27, 54, 61, 64, 12]. Thus for many of the schemes *asymptotic optimality* can be established, which means that the rate of convergence does not deteriorate as we proceed to finer and finer approximating spaces.

Many physical models, when cast into variational form, lead to problems posed in the vector-valued function spaces $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$. Here $\Omega \subset \mathbb{R}^3$ is a bounded connected domain. These spaces are defined by

$$\begin{aligned} \boldsymbol{H}(\mathbf{curl};\Omega) &:= \{\boldsymbol{\xi} \in \boldsymbol{L}^2(\Omega);\ \mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{L}^2(\Omega)\} \\ \boldsymbol{H}(\mathrm{div};\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\}\ , \end{aligned}$$

and are endowed with the natural Hilbert space graph norm. Appropriate essential boundary conditions prescribe the normal component for $\boldsymbol{H}(\mathrm{div};\Omega)$ and the tangential component for $\boldsymbol{H}(\mathbf{curl};\Omega)$, respectively, [31]. In this presentation, spaces with homogeneous essential boundary conditions imposed on the whole boundary $\partial\Omega$ will be tagged with a subscript 0.

These spaces are of considerable physical relevance: $\boldsymbol{H}(\mathrm{div};\Omega)$ is the ideal space for quantities that obey flux conservation, whereas $\boldsymbol{H}(\mathbf{curl};\Omega)$ is a the natural choice for electric and magnetic fields and certain stream functions in fluid mechanics. Moreover $\boldsymbol{H}(\mathrm{div};\Omega)$ is also encountered in the mixed formulation of second order elliptic problems [53, 19].

Usually, when these spaces come into play, we have to deal with variational problems and bilinear forms, respectively, that are elliptic with respect to the corresponding norm of the function space. The prototypes of such bilinear forms are given by

$$a(\boldsymbol{j},\boldsymbol{v}) := (\boldsymbol{j},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \eta\cdot(\mathrm{div}\,\boldsymbol{j},\mathrm{div}\,\boldsymbol{v})_{L^2(\Omega)} \tag{1}$$

and

$$a(\boldsymbol{\xi},\boldsymbol{\eta}) := (\boldsymbol{\xi},\boldsymbol{\eta})_{\boldsymbol{L}^2(\Omega)} + \eta\cdot(\mathbf{curl}\,\boldsymbol{\xi},\mathbf{curl}\,\boldsymbol{\eta})_{\boldsymbol{L}^2(\Omega)}\ , \tag{2}$$

respectively, where η is a real, positive parameter. From each of these selfadjoint bilinear forms we can derive a symmetric linear operator from the function space into its dual. We adopt the notation A for this operator in the remainder of the paper.

Commensurate with the importance of the spaces $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ there is a need for efficient iterative solvers for discrete variational problems arising from the above bilinear forms. They can be used as preconditioners for the minimal residual method to obtain a fast solver for mixed schemes for second order elliptic equations [41, 3]. Also penalty methods and augmented Lagrangian techniques for the same class of problems require an efficient solution of $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problems [40, 57]. In this case the parameter η occurring in the definition of the bilinear form (1) is linked to the penalization parameter and the solver must not deteriorate for large values of η. Furthermore, efficient methods for $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problems are also the key to the fast solution of the linear systems arising from some first order system least squares methods, since the least square functional is elliptic in $\boldsymbol{H}(\mathrm{div};\Omega)\times H^1(\Omega)$ [21, 52]. We should also mention that the sequential regularization methods for the incompressible Navier–Stokes equations involves the solution of an $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problem [42].

There are also numerous problems in computational electromagnetism that would benefit greatly from a fast solver for $\boldsymbol{H}(\mathbf{curl};\Omega)$–elliptic problems. Examples are eddy current simulations [8] and time domain simulations of Maxwell's equations with implicit time stepping [38, 43]. Here, η is related to the size of the time step, so that convergence of the iterative solver should not deteriorate for small η. Another promising application of such methods might be the stream function–vorticity formulation of Stokes' problem [31, 30].

Compared to the $H^1(\Omega)$–elliptic case, progress toward efficient preconditioners in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ has been rather slow. This has to be attributed to the presence of large null spaces of the operators div and **curl**, that replace the more benign differential operator **grad** related to $H^1(\Omega)$. These null spaces destroy what may be called the proper ellipticity of the corresponding differential operators $\mathbf{grad}\,\mathrm{div} + Id$ and $\mathbf{curl}\,\mathbf{curl} + Id$. We use this term to refer to the uniform amplification of functions of a certain "frequency", regardless of their "direction". In a sense, this corresponds to the classical concept of ellipticity based on the symbol of a differential operator. It turns out that most Schwarz methods for selfadjoint problems owe much of their clout to the proper ellipticity of the differential operators: it is prerequisite for the effectiveness of the coarse space correction. Thus naive approaches to $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ might run into difficulties.

These considerations highlight the need to treat the kernels of the differential operators and their orthogonal complements separately. In a slight generalization of the term [31, Chapter 1], we call such L^2–orthogonal splittings of the function spaces induced by the differential operator, *Helmholtz–decompositions*. Thus the design of Schwarz methods or $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ follows one idea: find viable decompositions for both parts of the Helmholtz decomposition separately and merge them into an overall Schwarz scheme. It might not be conspicuous in the final algorithm, but many successful Schwarz methods are based on this rule.

The bulk of earlier contributions in the field of Schwarz methods for elliptic problems in $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$ is confined to the 2D case. Nevertheless, we point out that investigations of viable subspace decomposition of $\boldsymbol{H}(\mathrm{div};\Omega)$–conforming Raviart–Thomas finite element spaces in two dimensions were germinal in the development of the theory: the principal idea that solenoidal vector fields have to be targeted separately was first pursued in [28] to construct asymptotically optimal multilevel preconditioners and domain decomposition methods for mixed saddle point problems in 2D. Based on these techniques, a $\boldsymbol{H}(\mathrm{div};\Omega)$–stable multilevel decomposition was proposed in [58]. Simultaneously, nonoverlapping Schwarz schemes based on a hierarchical basis multilevel decomposition, were introduced in [20, 60]. In [3, 4] Helmholtz–decompositions were exploited in a study of overlapping Schwarz methods for $\boldsymbol{H}(\mathrm{div};\Omega)$ in two dimensions. The application of these ideas to $\boldsymbol{H}(\mathbf{curl};\Omega)$ in two dimensions is discussed in [55].

The principles guiding the design of Schwarz methods are basically the same in any dimension. Yet the technical devices employed in the proofs in two and three dimensions differ significantly. The reason is that there is no genuine analogue for the **curl**–operator in 2D; thus in three dimensions we have to grapple with a more complicated representation of solenoidal vector fields by vector potentials instead of scalar stream functions and we encounter a new class of $\boldsymbol{H}(\mathbf{curl};\Omega)$–elliptic variational problems.

This paper provides a survey of the state of the art of the algorithmic and theoretical development of Schwarz methods for elliptic problems in vector valued functions spaces in three dimensions. It summarizes results published in a series of papers in recent years. In [22], an overlapping domain decomposition method for $\boldsymbol{H}(\mathrm{div};\Omega)$ was studied, while in [36, 39], a multilevel splitting of $\boldsymbol{H}(\mathbf{curl};\Omega)$, which was shown to be stable with respect to the $\|\mathbf{curl}\,\cdot\|_{\boldsymbol{L}^2(\Omega)}$–seminorm, was instrumental in the construction of an efficient preconditioner in the space of divergence free vector fields. Based on these results, a proof of the optimality of a multilevel decomposition for $\boldsymbol{H}(\mathrm{div};\Omega)$ was given in [37]. This work also paved the way to a fast multigrid method in $\boldsymbol{H}(\mathbf{curl};\Omega)$ [38] and provided the theoretical underpinning for an overlapping domain decomposition of $\boldsymbol{H}(\mathbf{curl};\Omega)$ [56].

In this paper, we develop a unified view, suggested by some glaring parallels between the spaces $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. If we write $\mathbf{D}$ for a generic differential operator, which may stand for div and **curl**, the definitions can be lumped together into

$$\boldsymbol{H}(\boldsymbol{D},\Omega) := \{v \in \boldsymbol{L}^2(\Omega);\ \mathbf{D}\,v \in L^2(\Omega)\}\ .$$

Thus, we will use our unified framework as long as possible, studying the bilinear form

$$a(u,v) := (u,v)_{\boldsymbol{L}^2(\Omega)} + \eta\cdot(\mathbf{D}\,u,\mathbf{D}\,v)_{L^2(\Omega)}\ . \tag{3}$$

It is our objective to illustrate the common rationale behind the construction of the subspace decompositions and point out the relationship to the standard $H^1(\Omega)$–elliptic case, which is, in fact, also covered by setting $\mathbf{D} = \mathbf{grad}$. We want to stress that it takes nothing but the smart application of totally standard techniques to master Schwarz methods for $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$, both theoretically and from the point of view of implementation.

The outline of the paper is as follows. In the next section, we provide a brief description of the finite element spaces used for discretizing the bilinear forms (1) and (2). These are the $\boldsymbol{H}(\mathrm{div};\Omega)$–conforming Raviart–Thomas spaces and $\boldsymbol{H}(\mathbf{curl};\Omega)$–conforming Nédélec spaces. We list their relevant properties and discuss the close relationship between them.

In the third section, we specify the subspace decompositions underlying both the multigrid method and the overlapping domain decomposition algorithm. Prior to that we try to give a sound motivation of their construction by studying the properties of the bilinear forms. We then describe the actual implementations of the algorithms that arise from these decompositions.

The fourth section provides the technical tools employed for establishing the stability of the decomposition. These are mainly discrete and continuous *Helmholtz–decompositions* and various projection operators affiliated with them.

In the fifth section, we investigate the convergence properties of the Schwarz methods relying on the algebraic theory of Schwarz methods for selfadjoint bilinear forms. Under certain assumptions on the computational domain, we show that the performance of the preconditioners is independent of the problem size and the number of subspaces involved in the splitting, which amounts to asymptotic optimality.

2. Finite element spaces. Let $\mathcal{T}_h := \{T_i\}_i$ denote a quasiuniform simplicial or hexahedral triangulation of the polyhedral domain $\Omega \subset \mathbb{R}^3$ with meshwidth $h := \max\{\mathrm{diam}\,T_i\}$. We require that the elements are uniformly shape–regular in the sense of [23] and we introduce the following conforming finite element spaces on this mesh:

$\mathcal{V}_d(\mathbf{grad}, \mathcal{T}_h) \subset H^1(\Omega)$ stands for the space of continuous finite element functions, piecewise polynomial of degree $d+1$ over $\mathcal{T}_h$, the conventional Lagrangian finite elements (see [23]).

$\mathcal{V}_d(\mathbf{curl}, \mathcal{T}_h) \subset \boldsymbol{H}(\mathbf{curl}; \Omega)$ denotes the so-called Nédélec finite element space of order $d \in \mathbb{N}$ introduced in [44]. For a tetrahedron $T \in \mathcal{T}_h$, these finite element functions have a local representation given by

$$\mathcal{V}_d(\mathbf{curl}, T) := (\mathcal{P}_d(T))^3 + \left\{ \boldsymbol{p} \in (\mathcal{P}_{d+1}(T))^3 \, ; \, \langle \boldsymbol{p}(\boldsymbol{x}), \boldsymbol{x} \rangle = 0 \right\} \ ,$$

where $\mathcal{P}_d(T)$ designates the space of polynomials of degree $\leq d$ over T. For the lowest order case $d = 0$, this leads to the representation $\mathcal{V}_0(\mathbf{curl}, T) = \{\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}, \, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$. On a hexahedron T, aligned with the coordinate axes, the local finite element spaces are

$$\mathcal{V}_d(\mathbf{curl}, T) := Q_{d-1,d,d}(T) \times Q_{d,d-1,d}(T) \times Q_{d,d,d-1}(T) \ ,$$

where $Q_{k_1,k_2,k_3}(T)$ is the space of polynomials of degree $\leq k_i$ in the ith coordinate direction, $i = 1, 2, 3$.

$\mathcal{V}_d(\mathrm{div}, \mathcal{T}_h) \subset \boldsymbol{H}(\mathrm{div}; \Omega)$ denotes the Raviart–Thomas finite element space of order $d \in \mathbb{N}_0$ (see [53, 44, 19]). On tetrahedral meshes they locally agree with the space

$$\mathcal{V}_d(\mathrm{div}, T) := (\mathcal{P}_d(T))^3 + \boldsymbol{x} \cdot (\mathcal{P}_d(T))^3$$

for each tetrahedron $T \in \mathcal{T}_h$. On a hexahedral grid we obtain instead

$$\mathcal{V}_d(\mathrm{div}, T) := Q_{d+1,d,d}(T) \times Q_{d,d+1,d}(T) \times Q_{d,d,d+1}(T) \ ,$$

for each hexahedron $T \in \mathcal{T}_h$.

The same notations, supplemented by a subscript 0, will denote the spaces equipped with homogeneous boundary conditions (in the sense of an appropriate trace operator, as explained in the introduction):

$$\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_h) := \mathcal{V}_d(\mathbf{D}, \mathcal{T}_h) \cap \boldsymbol{H}_0(\boldsymbol{D}, \Omega) \ ,$$

A few alternative choices for $\boldsymbol{H}(\mathrm{div}; \Omega)$– and $\boldsymbol{H}(\mathbf{curl}; \Omega)$–conforming spaces are conceivable, e.g., the spaces introduced in [18] and [45], respectively. With slight modifications, the approach presented in this paper carries over to discretizations based on them, as well.

Despite the differences in their definitions these finite element spaces are closely related. As discussed in [10, 35, 11], they all can be viewed as spaces of *discrete differential forms*, for which they offer very natural approximations. Our unusual notation for the finite element spaces is meant to underscore this common pattern.

All finite element spaces $\mathcal{V}_d(\mathbf{D}, \mathcal{T}_h)$ are equipped with sets of degrees of freedom (d.o.f.), denoted by $\Xi_d(\mathbf{D}, \mathcal{T}_h)$, which ensure conformity. In the lowest order case $d = 0$, they are given, for $\mathcal{V}_0(\mathbf{grad}, \mathcal{T}_h)$ by point values at the vertices, for $\mathcal{V}_0(\mathbf{curl}, \mathcal{T}_h)$ by path integrals along the edges, and for $\mathcal{V}_0(\mathrm{div}, \mathcal{T}_h)$ by normal fluxes through the faces. We refer to [44, 35] for a comprehensive exposition, also covering higher order finite elements. All degrees of freedom remain invariant under the respective canonical transformations of the finite element functions (see e.g. [35]). Consequently, all finite element spaces form affine families in the sense of [23].

Based on the degrees of freedom, sets of *canonical nodal basis functions* can be introduced as bases dual to $\Xi_d(\mathbf{D}, \mathcal{T}_h)$. They are locally supported and form an L^2–*frame*: we can find generic constants $\underline{C}, \overline{C} > 0$, independent of the meshwidth h and depending only on the type of finite element space and the shape regularity of $\mathcal{T}_h$, such that for all $\boldsymbol{\xi}_h \in \mathcal{V}_d(\mathbf{D}, \mathcal{T}_h)$

$$\underline{C}\|\boldsymbol{\xi}_h\|^2_{\boldsymbol{L}^2(\Omega)} \leq \sum_\kappa \kappa(\boldsymbol{\xi}_h)^2 \|\psi_\kappa\|^2_{\boldsymbol{L}^2(\Omega)} \leq \overline{C}\,\|\boldsymbol{\xi}_h\|^2_{\boldsymbol{L}^2(\Omega)}\,, \tag{4}$$

where κ runs through all degrees of freedom of the respective finite element space and $\boldsymbol{\psi}_\kappa$ stands for the canonical basis function of $\mathcal{V}_d(\mathbf{D}, \mathcal{T}_h)$ belonging to the d.o.f. $\kappa \in \Xi_d(\mathbf{D}, \mathcal{T}_h)$. In the following, a capital C will denote a generic constant. Its value can vary between different occurrences, but we will always specify what it must not depend on.

Now, given the degrees of freedom, for sufficiently smooth argument functions the nodal projections (nodal interpolation operators) $\Pi^{\mathbf{D}}_{d,\mathcal{T}_h}$ onto the finite element space $\mathcal{V}_d(\mathbf{D}, \mathcal{T}_h)$ can be introduced as in [23]. First of all we stress a particular algebraic property of these operators, expressed by the following *commuting diagram property* [25, 30, 19]: For $d \in \mathbb{N}_0$ the diagram

$$\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega) \\
\Big\downarrow \Pi^{\mathbf{grad}}_{d,\mathcal{T}_h} & & \Big\downarrow \Pi^{\mathbf{curl}}_{d,\mathcal{T}_h} & & \Big\downarrow \Pi^{\mathrm{div}}_{d,\mathcal{T}_h} & & \Big\downarrow \Pi^{0}_{d,\mathcal{T}_h} \\
\mathcal{V}_d(\mathbf{grad}, \mathcal{T}_h) & \xrightarrow{\mathbf{grad}} & \mathcal{V}_d(\mathbf{curl}, \mathcal{T}_h) & \xrightarrow{\mathbf{curl}} & \mathcal{V}_d(\mathrm{div}, \mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{V}_d(0, \mathcal{T}_h)\,,
\end{array}$$

commutes, which links nodal projectors and differential operators. We have written $\mathcal{V}_d(0, \mathcal{T}_h)$ to denote the space of piecewise polynomials of degree d over $\mathcal{T}_h$ and $\Pi^0_{d,\mathcal{T}_h}$ for the L^2–orthogonal projection onto this space. A proof of the commuting diagram property can be found in [35].

A trivial, but very important consequence of the commuting diagram property is the fact that nodal interpolations as defined above preserve the kernels of the differential operators:

$$x \in \boldsymbol{H}(\mathbf{D}, \Omega) \cap \mathcal{D}(\Pi^{\mathbf{D}}_{d,\mathcal{T}_h}) \quad \wedge \quad \mathbf{D}\,x = 0 \quad \Longrightarrow \quad \mathbf{D}\left(\Pi^{\mathbf{D}}_{d,\mathcal{T}_h}\,x\right) = 0 \tag{5}$$

Here, $\mathcal{D}(\Pi^{\mathbf{D}}_{d,\mathcal{T}_h})$ denotes the set of vector fields for which the interpolation operator is well–defined.

An inconvenient trait of the nodal projectors has to be stressed: they cannot be extended to continuous mappings on the entire function spaces. A slightly enhanced smoothness of the argument function is required. For instance, in the case of $\mathcal{V}_d(\mathbf{curl}, \mathcal{T}_h)$, the integrals along edges occurring in the definition of the d.o.f. are continuous functional only for functions $\boldsymbol{\xi}$, which locally belong to the space $X_p(T)$ for $p > 2$ and any $T \in \mathcal{T}_h$ (see [2, Lemma 4.7]). $X_p(T)$ is given by

$$X_p(T) := \{\boldsymbol{\eta} \in (L^p(T))^3\,;\, \mathbf{curl}\,\boldsymbol{\eta} \in (L^p(T))^3\,;\, \boldsymbol{\eta} \times \boldsymbol{n} \in (L^p(\partial T))^2\}\,. \tag{6}$$

This leads to considerable technical complications. Nevertheless, we cannot dispense with using nodal interpolation; no other projectors are known that satisfy the commuting diagram property (compare Remark 3.1 in [30]).

Affine equivalence techniques along with the commuting diagram property can be used to establish the approximation properties (see [44, 17, 53])

$$\left\| \mathbf{D}\left(x - \Pi^{\mathbf{D}}_{d,\mathcal{T}_h}\right)\right\|_{L^2(\Omega)} \leq C\,h\,|\mathbf{D}\,x|_{H^1(\Omega)}\ ,\forall x \in \boldsymbol{H}(\mathbf{D},\Omega); \mathbf{D}\,x \in H^1(\Omega), \tag{7}$$

with $C > 0$ independent of h.

A highly desirable property of the finite element spaces $\mathcal{V}_d(\mathbf{D},\mathcal{T}_h)$ is that an essential algebraic property of the continuous function spaces is preserved in the discrete setting (see [9, 36]):

THEOREM 2.1 (DISCRETE POTENTIALS). *Let Ω be simply connected with a connected boundary. Then the following sequences of vector spaces are exact for any $d \geq 0$:*

$$\{\text{const.}\} \xrightarrow{Id} \mathcal{V}_d(\mathbf{grad},\mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{V}_d(\mathbf{curl},\mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{V}_d(\mathrm{div},\mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathcal{V}_d(0,\mathcal{T}_h) \longrightarrow \{0\}$$

$$\{0\} \xrightarrow{Id} \mathcal{V}_{d,0}(\mathbf{grad},\mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{V}_{d,0}(\mathbf{curl},\mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{V}_{d,0}(\mathrm{div},\mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathcal{V}_{d,0}(0,\mathcal{T}_h) \longrightarrow \{0\}\ ,$$

where $\mathcal{V}_{d,0}(0,\mathcal{T}_h)$ contains piecewise polynomials with zero mean value.

The gist of this theorem is that for domains Ω complying with the assumptions we have

$$x \in \mathcal{V}_d(\mathbf{D},\mathcal{T}_h)\ \wedge\ \mathbf{D}\,x = 0 \quad\Longleftrightarrow\quad x = \widetilde{\mathbf{D}}y \text{ for an } y \in \mathcal{V}_d(\widetilde{\mathbf{D}},\mathcal{T}_h), \tag{8}$$

where $\widetilde{\mathbf{D}}$ is the "potential operator" associated with $\mathbf{D}$ fulfilling $\mathbf{D}\,\widetilde{\mathbf{D}} = 0$, i.e. for $\mathbf{D} = \mathbf{curl}$, $\widetilde{\mathbf{D}} = \mathbf{grad}$, and for $\mathbf{D} = \mathrm{div}$, $\widetilde{\mathbf{D}} = \mathbf{curl}$. In the sequel we will take for granted that the topology of Ω makes Theorem 2.1 hold.

Remark. If Ω is topologically more complex, Theorem 2.1 is still valid, except for a low dimensional space of functions in the kernel of $\mathbf{D}$ that lack a representation by means of a potential [2, Section 3]. The dimension of this space solely depends on fundamental topological properties of Ω and not on $\mathcal{T}_h$. Because of this invariance, the Schwarz analysis confronts little difficulties on general domains, since kernel functions without potential representation can all be treated on the coarse grid. For the sake of simplicity, we forgo a general discussion, however.

3. Overlapping Schwarz methods. It is well known how to construct efficient Schwarz methods for standard second order elliptic problems; see [54, 27, 12]. In order to adapt these recipes to the case of $\boldsymbol{H}(\mathbf{curl};\Omega)$– and $\boldsymbol{H}(\mathrm{div};\Omega)$-elliptic problems successfully, we have to make sure that the targeted differential operator displays proper ellipticity, as pointed out in the introduction. The bilinear forms (1) and (2) obviously lack this property; functions in the kernel of the differential operators div and **curl** are acted upon differently from functions in the orthogonal complement, even if they may have the same oscillatory character. In the following, the space $\mathcal{N}(\mathbf{D})$ will generically denote the kernel of the differential operator $\mathbf{D}$: the space on which $\mathbf{D}$ is considered will be clear from the context.

On the orthogonal complement $\mathcal{N}(\mathbf{D})^\perp$ we may expect a proper elliptic character of the bilinear forms, since there the $(\mathbf{D}\,\cdot,\mathbf{D}\,\cdot)_{L^2(\Omega)}$–part prevails. To elaborate further, let us temporarily switch to the entire space $\mathbb{R}^3$. Straightforward computations in the frequency domain bear out that for all $u, v \in \boldsymbol{H}(\mathbf{D},\mathbb{R}^3) \cap \mathcal{N}(\mathbf{D})^\perp$

$$a(u,v) := (u,v)_{L^2(\Omega)} + \eta\cdot(\nabla u,\nabla v)_{L^2(\Omega)} \quad .$$

This means that when restricted to $\mathcal{N}(\mathbf{D})^\perp$ the differential operators $Id + \eta\,\mathbf{grad}\,\mathrm{div}$ or $Id + \eta\,\mathbf{curl}\,\mathbf{curl}$ associated with $a(\cdot,\cdot)$ agree with the vector Laplacian plus a zero order term. Informally, we can write

$$A \approx Id + \eta \cdot \Delta \quad \text{on} \quad \mathcal{N}(\mathbf{D})^\perp \; . \tag{9}$$

The crucial question is how to deal with the kernel $\mathcal{N}(\mathbf{D})$ is given by representation theorems stating that $\mathcal{N}(\mathrm{div}) = \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl};\Omega)$ [31, Theorem I.3.4] and $\mathcal{N}(\mathbf{curl}) = \mathbf{grad}\,H^1(\Omega)$ [31, Theorem I.2.9]. In short, we can write

$$\mathcal{N}(\mathbf{D}) = \widetilde{\mathbf{D}}\boldsymbol{H}(\widetilde{\mathbf{D}},\Omega) \; ,$$

which holds for domains topologically equivalent to a ball. This tells us that a lifting into a suitable potential space can convert the problem on $\mathcal{N}(\mathbf{D})$ into an elliptic problem. Tersely writing, we have

$$\widetilde{\mathbf{D}}^* \circ A \circ \widetilde{\mathbf{D}} \approx \Delta \quad \text{on} \quad \mathcal{N}(\widetilde{\mathbf{D}})^\perp \; , \tag{10}$$

where $\mathbf{D}^*$ stands for the formal L^2–adjoint of the differential operator $\mathbf{D}$.

These considerations for the continuous function spaces, are also valid in the discrete setting, since the particular finite element spaces introduced in Section 2 inherit many crucial properties of the function spaces. Relying on these insights we pursue the following policy:

1. We treat the two components $\mathcal{N}(\mathbf{D})$ and $\mathcal{N}(\mathbf{D})^\perp$ of the L^2–orthogonal Helmholtz decomposition separately.

2. In order to tackle $\mathcal{N}(\mathbf{D})^\perp$ we will use a decomposition of the finite element space which resembles those used in the context of Schwarz methods for second order elliptic problems. We note that the components of the splitting do need not be exactly orthogonal to the kernel; approximate orthogonality is enough, since we only aim at constructing a preconditioner.

3. For the treatment of the null space $\mathcal{N}(\mathbf{D})$ we make use of Theorem 2.1 to switch to discrete potentials. We thus resort to an appropriate decomposition of $\mathcal{N}(\widetilde{\mathbf{D}})^\perp$ which can be obtained from the previous guidelines.

3.1. Overlapping domain decomposition method. Let $\mathcal{T}_H$ be a triangulation of the domain Ω, with meshwidth H, consisting of tetrahedra $\{\Omega_i\}_{i=1}^J$. Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$, with meshwidth $h < H$. We suppose that both $\mathcal{T}_H$ and $\mathcal{T}_h$ are *shape-regular* and *quasiuniform*. Consider now an open covering of Ω, say $\{\Omega_i'\}_{i=1}^J$, such that each subregion Ω_i' is the union of tetrahedra of $\mathcal{T}_h$ and contains an element of $\mathcal{T}_H$. In addition define the overlap δ as

$$\delta = \min_i \{\mathrm{dist}(\partial\Omega_i', \Omega_i)\}.$$

We will assume that the following two properties hold:

ASSUMPTION 3.1.

a) [Generous overlap] *The relative overlap (δ/H) is bounded away from zero.*

b) [Finite covering] *For every point $P \in \Omega$, P belongs to at most N_c subregions in $\{\Omega_i'\}_{i=1}^J$.*

```
MSP(x ∈ V_{d,0}(D, T_h), f ∈ V_{d,0}(D, T_h))
  {
    for i = 0, ···, J
      {
        r ← f − Ax
        x ← x + R_i^T A_i^{-1} R_i r
      }
  }
```

FIG. 1. *Evaluation of the multiplicative Schwarz preconditioner* $MSP(x, f)$.

For the sake of brevity we will write V_h for the finite element space $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_h)$ introduced in Section 2. We define the subspaces $\{V_i \subset V_h\}$, for $i = 1, \cdots, J$, by setting the degrees of freedom outside Ω_i' to zero. The space V_h admits the decomposition $V_h = \sum_{i=1}^{J} V_i$. Following standard practice in overlapping domain decomposition for 2nd order elliptic problems [27, 54], we augment this decomposition by a *coarse grid space* $V_H := \mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_H)$. Since $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, V_H is contained in V_h. Keeping in mind the above considerations, we have now achieved a promising decomposition of the orthogonal complement of $\mathcal{N}(\mathbf{D})$ in the finite element space. The very same approach is applied to the space $\widetilde{V}_h := \mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_h)$ of discrete potentials which yields the preliminary decomposition

$$V_h = V_H + \widetilde{\mathbf{D}}\widetilde{V}_H + \sum_{i=1}^{J} \left(V_i + \widetilde{\mathbf{D}}\widetilde{V}_i \right) .$$

Using it for the construction of an overlapping domain decomposition methods would involve unnecessary computational work, since we observe

$$\widetilde{\mathbf{D}}\widetilde{V}_H \subset V_H \quad \text{and} \quad \widetilde{\mathbf{D}}\widetilde{V}_i \subset V_i .$$

Hence the contributions of the potential space can actually be absorbed by the components of the decomposition of $\mathcal{N}(\mathbf{D})^{\perp}$. Thus, the final splitting the overlapping domain decomposition method for $\boldsymbol{H}_0(\mathbf{D}, \Omega)$ is based upon, is

$$V_h = V_H + \sum_{i=1}^{J} V_i . \tag{11}$$

We remark that we have resorted to the discrete Helmholtz decomposition only for derivation of the splitting (11). The decomposition is never calculated in practice: both multiplicative and additive methods can be implemented in a perfectly standard fashion [54]. Figure 1 shows the evaluation of the multiplicative preconditioner: R_i^T is the linear interpolation operator from the subspace V_i to V_h, and A_i is the operator relative to the bilinear form a, defined on the subspace V_i (we have set $V_0 = V_H$).

3.2. Multilevel method. The setting for the multilevel methods assumes that we have a nested sequence of quasiuniform triangulations $\mathcal{T}_l$, $l = 0, \ldots, J$ of Ω, created by regular refinement of an initial mesh as, for instance, described in [6] for simplicial meshes. Then the meshwidths h_l, $l = 0, \ldots, J$, decrease in a geometric progression, i.e. $h_l \approx 2^{-l}$. Without loss of generality we may set $h_0 \approx 1$.

```
Initial guess: x_L, right hand side f_L
MGVC(int k, x_l ∈ V_{d,0}(D, T_l), f_l ∈ V_{d,0}(D, T_l))
  {
    if (l==0) x_0 := A_0^{-1} f_0
    else
      {
        x_l ← S_l(x_l, f_l)      [Presmoothing]
        e_{l-1} ← 0
        MGVC(l-1, e_{l-1}, R_l^{l-1}(f_l - A_l x_l))
        x_l ← x_l + P_{l-1}^l e_{l-1}
        x_l ← S_l(x_l, f_l)      [Postsmoothing]
      }
  }
```

FIG. 2. *Multigrid V(1,1)–cycle for discrete variational problem related to $a(\cdot,\cdot)$ on finite element space $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_L)$.*

Standard multigrid methods and multilevel preconditioners for second order elliptic problems arise from a *nodal multilevel decomposition* of the finite element space $V_h := \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_J)$ (see [12, 32, 61]). This means that the Schwarz algorithm is based on one–dimensional subspaces spanned by the nodal basis functions on all the meshes $\mathcal{T}_1,\dots,\mathcal{T}_J$ and an additional coarse grid space $V_0 := \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_0)$. Following the very same reasoning as above, i.e. employing a nodal multilevel decomposition of both $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ and $\mathcal{V}_{d,0}(\widetilde{\mathbf{D}},\mathcal{T}_h)$, we immediately get the splitting

$$V_h := V_0 + \sum_{l=1}^{J}\left(\sum_{\kappa_l}\mathrm{Span}\left\{\psi_{\kappa_l}\right\} + \sum_{\widetilde{\kappa}_l}\mathrm{Span}\left\{\widetilde{\mathbf{D}}\widetilde{\psi}_{\widetilde{\kappa}_l}\right\}\right)\;, \tag{12}$$

where κ_l runs through all d.o.f. in $\Xi_d(\mathbf{D},\mathcal{T}_l)$ and $\widetilde{\kappa}_l$ covers all d.o.f. in $\Xi_d(\widetilde{\mathbf{D}},\mathcal{T}_l)$. By ψ_κ and $\widetilde{\psi}_{\widetilde{\kappa}}$ we denoted the canonical basis functions of $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ and $\mathcal{V}_{d,0}(\widetilde{\mathbf{D}},\mathcal{T}_h)$, respectively, belonging to a particular d.o.f. κ and $\widetilde{\kappa}$. Evidently, generous overlap between the subspace on different levels is present in (12).

The straightforward symmetric multiplicative version of the multilevel Schwarz method leads to a multigrid V(1,1)–cycle with Gauß–Seidel smoother. The algorithm is outlined in Figure 2, in order to convey that it can be implemented in a perfect multigrid–like fashion:

The operators $P_{l-1}^{l} : \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_{l-1}) \mapsto \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_l)$ and $R_l^{l-1} : \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_l) \mapsto \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_{l-1})$ designate the canonical intergrid transfers, prolongation and restriction, of the finite element spaces, induced by the natural embedding of these spaces in the case of nested meshes (see [34]). They are transposes of each other and lend themselves to a purely local evaluation.

The only special thing about the method is the design of the smoother $S_l(\cdot,\cdot)$, the steps of which are described in Figure 3. It might be called a "hybrid" Gauß–Seidel smoother, since smoothing sweeps both in the current finite element space $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_l)$ and discrete potential spaces $\mathcal{V}_{d,0}(\widetilde{\mathbf{D}},\mathcal{T}_l)$ are carried out. In Figure 3, C_l stands for the linear operator (i.e. the stiffness matrix) related to the bilinear form $(\widetilde{u}_l,\widetilde{v}_l) \mapsto (\widetilde{\mathbf{D}}\,\widetilde{u}_l,\widetilde{\mathbf{D}}\,\widetilde{v}_l)_{L^2(\Omega)}$ in $\mathcal{V}_{d,0}(\widetilde{\mathbf{D}},\mathcal{T}_l)$. The Gauß–Seidel relaxation of any linear system is invariably supposed to be based on the canonical bases of the finite element

```
S_l(c_l ∈ V_{d,0}(D, T_l), s_l ∈ V_{d,0}(D, T_l))
  {
    Gauß-Seidel sweep on A_l c_l = s_l
    r_l ← s_l − A_l c_l
    r̃_l ← T_l^* r_l
    p̃_l ← 0
    Gauß-Seidel sweep on C_l p̃_l = r̃_l
    return c_l + T_l p̃_l
  }
```

FIG. 3. *Evaluation of the hybrid smoother $S_l(c_l, s_l)$.*

spaces.

The lifting into a potential space is reflected by the transfer operator T_l : $\mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_l) \mapsto \mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$. It is defined by the embedding $\widetilde{\mathbf{D}}\, \mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_l) \subset \mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$ and, due to the local nature of the basis functions of both spaces, a local evaluation is also possible in this case. Therefore, a smoothing step requires a computational effort proportional to the dimension of the finite element space on the current level.

Also in the case of the multigrid method the explicit corrections in the spaces of discrete potentials can be discarded, yet at the expense of larger subspaces in the decomposition of $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$, $l = 1, \ldots, J$: Denote by $K_{\widetilde{\kappa},l}$ the minimal set of $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$–basis functions such that $\widetilde{\mathbf{D}}\, \widetilde{\psi}_{\widetilde{\kappa}_l} \subset \mathrm{Span}\left\{K_{\widetilde{\kappa},l}\right\}$. Then a viable multigrid method can be founded on the *clustered decomposition*

$$V_h := V_0 + \sum_{l=1}^{J} \sum_{\widetilde{\kappa}_l} \mathrm{Span}\left\{K_{\widetilde{\kappa},l}\right\} ,$$

which has been introduced for $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problems in 2D by Arnold, Falk and Winther [3, 4]. We do not need to smooth in the potential space any more, but now several nodal values of the spaces $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$ have to be relaxed simultaneously, which affects the savings in computational costs.

4. Decompositions and projections. The considerations that led us to the Schwarz decompositions centered around the notion of a Helmholtz–decomposition of vector fields. For the theoretical investigation of the methods they turn out to be crucial, but in the finite element setting their usefulness is tainted by awkward properties of the orthogonal complements.

In the following, we will carry on our analysis for the space $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_h) \subset \boldsymbol{H}_0(\boldsymbol{D}, \Omega)$ of vectors satisfying homogeneous essential boundary conditions; the results for $\mathcal{V}_d(\mathbf{D}, \mathcal{T}_h) \subset \boldsymbol{H}(\boldsymbol{D}, \Omega)$ (free boundary values) can be obtained in a similar fashion. Accordingly, in the following $\mathcal{N}(\boldsymbol{D})$ will denote the kernel of the operator $\boldsymbol{D}$, defined in $\boldsymbol{H}_0(\boldsymbol{D}, \Omega)$, and $\mathcal{N}(\mathbf{D})^{\perp}$ its orthogonal complement in $\boldsymbol{H}_0(\boldsymbol{D}, \Omega)$, with respect to the L^2–scalar product, giving the *continuous* Helmholtz–decomposition:

$$\boldsymbol{H}_0(\boldsymbol{D}, \Omega) = \mathcal{N}(\boldsymbol{D}) \oplus \mathcal{N}(\mathbf{D})^{\perp} . \tag{13}$$

Denote by $\mathcal{V}^0_{d,0}(\mathbf{D}, \mathcal{T}_h)$ the kernel of the operator $\boldsymbol{D}$ defined in $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_h)$:

$$\mathcal{V}^0_{d,0}(\mathbf{D}, \mathcal{T}_h) := \{v_h \in \mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_h),\, \mathbf{D}\, v_h = 0\},$$

and by $\mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$ the L^2–orthogonal complement of $\mathcal{V}^{0}_{d,0}(\mathbf{D},\mathcal{T}_h)$ in $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$; we have thus obtained the *discrete* Helmholtz–decomposition

$$\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h) = \mathcal{V}^{0}_{d,0}(\mathbf{D},\mathcal{T}_h) \oplus \mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h). \tag{14}$$

For nested meshes, $\mathcal{T}_H \prec \mathcal{T}_h$, we have the following inclusions:

$$\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_H) \subset \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h) \quad \text{and} \quad \mathcal{V}^{0}_{d,0}(\mathbf{D},\mathcal{T}_H) \subset \mathcal{V}^{0}_{d,0}(\mathbf{D},\mathcal{T}_h);$$

but it is easy to see that, in general, $\mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_H) \not\subset \mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$. In addition, the space $\mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$ lacks a set of neatly localized basis functions, which prevents the use of standard finite element techniques. In short, any analysis of the overlapping Schwarz methods relying on discrete orthogonal complement faces formidable difficulties.

As an alternative to the completely discrete Helmholtz–decomposition (14), we can consider the continuous Helmholtz–decomposition (13). Any finite element function $v_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ is also a member of the continuous function space, and, as such, can be decomposed as $v_h = v_h^0 + v_h^{\perp}$, where, according to (13), $\mathbf{D}\, v_h^0 = 0$ and $v_h^{\perp} \in \mathcal{N}(\mathbf{D})^{\perp}$. Writing $\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$ for the image of $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ under the L^2–orthogonal projection onto the continuous space $\mathcal{N}(\mathbf{D})^{\perp}$, it is immediate that

$$\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_H) \subset \mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h) \quad \text{and} \quad \mathbf{D}\,\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h) = \mathbf{D}\,\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)\ .$$

In other words, these spaces are nested and at least their images under the relevant differential operator are contained in proper finite element spaces. Moreover, under some assumptions on the domain Ω, they exhibit some additional regularity:

LEMMA 4.1. *Let Ω be a convex Lipschitz–domain. Then the seminorm $\|\mathbf{D}\cdot\|_{L^2(\Omega)}$ is an equivalent norm on the orthogonal complement of $\mathcal{N}(\mathbf{D})$ in $\boldsymbol{H}_0(\mathbf{D},\Omega)$, which is equivalent to the norm $|\cdot|_{H^1(\Omega)}$.*

Proof. For the proof, it is enough to recall the following identities (see [31, 24]),

$$\begin{aligned}\mathcal{N}(\mathrm{div})^{\perp} &= \{\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div};\Omega);\ \mathbf{curl}\,\boldsymbol{u} = 0\}\ ,\\ \mathcal{N}(\mathbf{curl})^{\perp} &= \{\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega);\ \mathrm{div}\,\boldsymbol{u} = 0\}\ ;\end{aligned}$$

the result then follows from [2, Theorem 2.17]. □

Remark. Lemma 4.1 is also valid if the differential operator $\mathbf{D}$ is defined in the whole space $\boldsymbol{H}(\mathbf{D},\Omega)$ (natural boundary conditions). In this case (see [31, 24]) the orthogonal complements are given by

$$\begin{aligned}\mathcal{N}(\mathrm{div})^{\perp} &= \{\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div};\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega);\ \mathbf{curl}\,\boldsymbol{u} = 0\}\ ,\\ \mathcal{N}(\mathbf{curl})^{\perp} &= \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}_0(\mathrm{div};\Omega);\ \mathrm{div}\,\boldsymbol{u} = 0\}\ ;\end{aligned}$$

and the result also follows from [2, Theorem 2.17]. This remark allows us to extend the convergence analysis of the following section to the case of boundary value problems with natural boundary conditions. In the following, we will always assume that Ω is a convex polyhedron.

The bottom line is that these *semicontinuous spaces* $\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$ offer a more benign environment for the examination of Schwarz methods. Following this idea, we have to introduce a suitable projection which takes us from the finite element space into the continuous orthogonal complement:

DEFINITION 4.1 (ENERGY PROJECTION). *The* energy projection $P_h : \boldsymbol{H}(\mathbf{D},\Omega) \mapsto \mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$ *onto the semicontinuous space* $\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$ *is a linear mapping defined by*

$$(\mathbf{D}(u - P_h\, u), \mathbf{D}\, v_h)_{L^2(\Omega)} = 0 \quad \forall v_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)\ .$$

Please note, that if $\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h) = \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$, as is the case for $H^1_0(\Omega)$–conforming finite elements, then the projection P_h agrees with the usual $a(\cdot,\cdot)$–orthogonal projection. Clearly, whenever P_h is applied to a vector u in $\mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$, it coincides with the L^2–orthogonal projection onto $\mathcal{N}(\mathbf{D})^{\perp}$; in this case we have $\mathbf{D}(u - P_h\, u) = 0$.

The energy projection features the following crucial approximation property. It is a generalization of the approximation estimates that hold for the $|\cdot|_{H^1(\Omega)}$–orthogonal projection in the case of elliptic regularity [23, Section 3.2].

LEMMA 4.2. *Let Ω be* convex *and let $\mathcal{T}_h$ be a quasiunform, shape–regular triangulation of meshwidth $h > 0$. Then the following error estimate holds for the energy projection $P_h : \boldsymbol{H}(\mathbf{D},\Omega) \mapsto \mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$:*

$$\|v - P_h\, v\|_{L^2(\Omega)} \le Ch\, \|\mathbf{D}\, v\|_{L^2(\Omega)}, \quad \forall v \in \mathcal{N}(\mathbf{D})^{\perp}\ ,$$

with $C > 0$ independent of h and v.

Proof. The proof employs classical duality techniques and hinges on the following regularity result which is valid in the case of convex Ω (see [2, 29, 31]) and can be deduced from Lemma 4.1:

$$\left.\begin{array}{c} \mathbf{D}^*\,\mathbf{D}\, u = f \text{ in } \Omega \\ u \in \mathcal{N}(\mathbf{D})^{\perp},\, f \in \mathcal{N}(\mathbf{D})^{\perp} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} u \in H^1(\Omega) \,\wedge\, \mathbf{D}\, u \in H^1(\Omega) \\ \|u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \\ \|\mathbf{D}\, u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}\ . \end{array}\right. \tag{15}$$

For arbitrary $v \in \mathcal{N}(\mathbf{D})^{\perp}$ let $z \in \mathcal{N}(\mathbf{D})^{\perp}$ be determined by

$$(\mathbf{D}\, z, \mathbf{D}\, q)_{L^2(\Omega)} = (v - P_h\, v, q)_{L^2(\Omega)} \quad \forall q \in \mathcal{N}(\mathbf{D})^{\perp}\ .$$

Due to the definition of the energy projection, we get for all $q_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$

$$\|v - P_h\, v\|^2_{L^2(\Omega)} = (\mathbf{D}(z - q_h), \mathbf{D}(v - P_h\, v))_{L^2(\Omega)}\ ,$$

and end up with

$$\begin{aligned} \|v - P_h\, v\|^2_{L^2(\Omega)} &\le \|\mathbf{D}(v - P_h\, v)\|_{L^2(\Omega)} \cdot \inf_{q_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)} \|\mathbf{D}(z - q_h)\|_{L^2(\Omega)} \\ &\le Ch\, \|\mathbf{D}\, v\|_{L^2(\Omega)} \cdot \|v - P_h\, v\|_{L^2(\Omega)}\ , \end{aligned}$$

where we relied on the approximation estimate (7) and the regularity of the boundary value problem from (15). □

However, ultimately the decompositions are set in the original finite element spaces. Thus, we need another projector to return from the semicontinous spaces:

DEFINITION 4.2 (HELMHOLTZ–PROJECTION). *The* Helmholtz–projection B_h : $\boldsymbol{H}(\mathbf{D},\Omega) \mapsto \mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$, *onto the orthogonal complement from the discrete Helmholtz–decomposition (14) of the finite element space $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ is the linear mapping defined by*

$$(\mathbf{D}(u - B_h\, u), \mathbf{D}\, v_h)_{L^2(\Omega)} = 0 \quad \forall v_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)\ .$$

This definition makes sense, since a function from $\mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$ is uniquely determined by its image under the linear differential operator $\mathbf{D}$. For the same reason, we have $\mathbf{D}(u - B_h\, u) = 0$, whenever $u \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ or $u \in \mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$. Note that both the Helmholtz–projection and the energy projection are orthogonal projections w.r.t. the $\|\mathbf{D}\cdot\|_{L^2(\Omega)}$–seminorm.

In order to ensure that our return to the genuine finite element spaces does not destroy the essential properties of the decompositions, we have to rely on yet another error estimate:

LEMMA 4.3. *Let Ω be convex and let $\mathcal{T}_h$ be a quasiuniform, shape–regular triangulation of meshwidth $h > 0$. Then the Helmholtz–projection B_h onto $\mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$ satisfies*

$$\left\| B_h\, v_h^{\perp} - v_h^{\perp} \right\|_{L^2(\Omega)} \leq C h \left\| \mathbf{D}\, v_h^{\perp} \right\|_{L^2(\Omega)}, \quad \forall v_h^{\perp} \in \mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)\ ,$$

where $C > 0$ depends only on Ω and the shape–regularity of $\mathcal{T}_h$.

Proof. Following [44], we immediately conclude from the definition of the Helmholtz–projection and of the semicontinuous spaces $\mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$ that

$$\mathbf{D}(B_h\, v_h^{\perp} - v_h^{\perp}) = 0\ ,$$

which, due to (5), yields

$$\mathbf{D}(B_h\, v_h^{\perp} - \Pi^{\mathbf{D}}_{d,\mathcal{T}_h} v_h^{\perp}) = 0 \quad \Longleftrightarrow \quad B_h\, v_h^{\perp} - \Pi^{\mathbf{D}}_{d,\mathcal{T}_h} v_h^{\perp} \in \mathcal{V}^{0}_{d,0}(\mathbf{D},\mathcal{T}_h)\ .$$

Both $B_h\, v_h^{\perp}$ and $v_h^{\perp}$ are orthogonal to $\mathcal{V}^{0}_{d,0}(\mathbf{D},\mathcal{T}_h)$, so that

$$\begin{aligned} \left\| B_h\, v_h^{\perp} - v_h^{\perp} \right\|^2_{L^2(\Omega)} &= \left(B_h\, v_h^{\perp} - v_h^{\perp}, (B_h\, v_h^{\perp} - \Pi^{\mathbf{D}}_{d,\mathcal{T}_h} v_h^{\perp}) + (\Pi^{\mathbf{D}}_{d,\mathcal{T}_h} v_h^{\perp} - v_h^{\perp}) \right)_{L^2(\Omega)} \\ &\leq \| B_h\, v_h^{\perp} - v_h^{\perp} \|_{L^2(\Omega)} \cdot \| (\Pi^{\mathbf{D}}_{d,\mathcal{T}_h} - Id) v_h^{\perp} \|_{L^2(\Omega)}\ . \end{aligned}$$

It remains to estimate the interpolation error of the nodal projector in the L^2–norm. Naturally, we aim to exploit $v_h^{\perp} \in H^1(\Omega)$ provided by Lemma 4.1:

Consequently, for $\mathbf{D} = \mathrm{div}$ the assertion of the lemma follows from classical interpolation estimates for Raviart–Thomas finite element functions [19, Prop. 3.6].

In the case $\mathbf{D} = \mathbf{curl}$ the proof is more intricate [48, Section 4], [2, Section 4]. Let us first consider an arbitrary element $T \in \mathcal{T}_h$ and write $\widehat{v}$ for the image of $v^{\perp}_{h\,|T}$ under the suitable canonical transformation onto the reference element $\widehat{T}$. Since $\widehat{v} \in H^1(\widehat{T})$ and $\mathbf{curl}\,\widehat{v}$ is polynomial, it is clear from Sobolev's imbedding theorems [1, Ch. 5] that $\widehat{v} \in X_3(\widehat{T})$ (cf. (6)). Thus [2, Lemma 4.7] teaches us that

$$\|\widehat{\Pi}^{\mathbf{curl}}_{d,\widehat{T}}\ \widehat{v}\|_{L^2(\widehat{T})} \leq C(\|\widehat{v}\|_{L^3(\widehat{T})} + \|\mathbf{curl}\,\widehat{v}\|_{L^3(\widehat{T})} + \|\widehat{v} \times \boldsymbol{n}\|_{L^3(\partial\widehat{T})})\ .$$

As $\mathbf{curl}\,\widehat{v}$ belongs to a finite dimensional space of polynomials on $\widehat{T}$, we can further estimate

$$\|\widehat{\Pi}^{\mathbf{curl}}_{d,\widehat{T}}\ \widehat{v}\|_{L^2(\widehat{T})} \leq C\, \|\widehat{v}\|_{H^1(\widehat{T})}\ .$$

Since the interpolation operator $\widehat{\Pi}^{\mathbf{curl}}_{d,\widehat{T}}$ preserves constants, a Bramble–Hilbert argument [23, Section 3.1] shows

$$\|(Id - \widehat{\Pi}^{\mathbf{curl}}_{d,\widehat{T}})\widehat{v}\|_{L^2(\widehat{T})} \leq C\, |\widehat{v}|_{H^1(\widehat{T})}\ .$$

Using the following transformation estimates [48, Section 3], which can be obtained by straightforward affine equivalence techniques,

$$\begin{aligned} |\widehat{v}|_{H^1(\widehat{T})} &\leq Ch^{1/2}\,|v|_{H^1(T)} \\ \|v\|_{L^2(T)} &\leq Ch^{1/2}\|\widehat{v}\|_{L^2(\widehat{T})} \end{aligned}$$

(constants only depending on the shape regularity of the mesh) we get

$$\left\|(\Pi^{\mathbf{D}}_{d,\mathcal{T}_h} - Id)v_h^{\perp}\right\|_{L^2(T)} \leq Ch\left|v_h^{\perp}\right|_{H^1(T)} ,$$

which can be turned into the desired global estimate by summing over all elements and applying Lemma 4.1.

Similar arguments bear out the assertion for $\mathbf{D} = \mathbf{grad}$, since $v_h^{\perp}$ is a piecewise polynomial, continuous function. We skip the technical details. □

Lemma 4.3 allows us to prove another error estimate for the energy projection.

LEMMA 4.4. *Let Ω be* convex *and let $\mathcal{T}_h$ be a quasiunform, shape–regular triangulation of meshwidth $h > 0$. Then the following error estimate holds for the energy projection $P_h : \boldsymbol{H}(\mathbf{D},\Omega) \mapsto \mathcal{V}^{\perp}_{d,0}(\mathbf{D},\mathcal{T}_h)$:*

$$\left\|v_h^{+} - P_h\, v_h^{+}\right\|_{L^2(\Omega)} \leq Ch\left\|\mathbf{D}\, v_h^{+}\right\|_{L^2(\Omega)}, \quad \forall v_h^{+} \in \mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h) ,$$

with $C > 0$ independent of h and v_h^{+}.

Proof. Let $v_h^{+} \in \mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$. Since $B_h\, P_h\, v_h^{+} = v_h^{+}$, for all $v_h^{+} \in \mathcal{V}^{+}_{d,0}(\mathbf{D},\mathcal{T}_h)$, from Lemma 4.3, we have

$$\left\|v_h^{+} - P_h\, v_h^{+}\right\|_{L^2(\Omega)} = \left\|B_h\, P_h\, v_h^{+} - P_h\, v_h^{+}\right\|_{L^2(\Omega)} \leq Ch\left\|\mathbf{D}\, P_h\, v_h^{+}\right\|_{L^2(\Omega)} .$$

The definition of P_h, proves the assertion. □

In our analysis we will also employ the L^2–orthogonal projection $Q_h : L^2(\Omega) \mapsto \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$, onto the finite element space. The following stability and error estimates hold. They can be proved like in the case of standard finite elements (see [16]).

LEMMA 4.5. *Let the mesh $\mathcal{T}_h$ be shape–regular and quasiuniform with meshwidth h. Then the L^2–orthogonal projection Q_h onto $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ satisfies,*

$$\begin{aligned} \|\mathbf{D}\, Q_h\, v\|_{L^2(\Omega)} &\leq C|v|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \\ \|v - Q_h\, v\|_{L^2(\Omega)} &\leq Ch\,|v|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) , \end{aligned}$$

with constants independent of h.

5. Convergence theory. First, we briefly recall the fundamental results of the algebraic theory of abstract Schwarz methods for a selfadjoint positive definite variational problem, characterized by the bilinear form $a : V_h \times V_h \mapsto \mathbb{R}$. For further details, we refer to [54, 61, 64, 17] and the references therein.

We assume that the Schwarz method is based on a decomposition

$$V_h = \sum_{i=0}^{J} V_i \;, \tag{16}$$

where the V_i, $i = 0, \ldots, J$, are closed subspaces of the Hilbert space V_h. Let us now define the following operators for $i = 0, \cdots, J$:

$$T_i : V_h \longrightarrow V_i,$$
$$a\left(T_i u, v\right) = a\left(u, v\right), \ \forall v \in V_i.$$

The additive and symmetric multiplicative Schwarz operators are defined as

$$T_{as} := \sum_{i=0}^{J} T_i ,$$
$$T_{ms} := I - (I - T_0) \cdots (I - T_J)(I - T_J) \cdots (I - T_0) .$$

Different choices of multiplicative operators and various hybrid methods are also possible [54]. The equation $Tu = g$ is then solved with a conjugate gradient type method, without any further preconditioner, employing $a(\cdot, \cdot)$ as the inner product and using a suitable right hand side g. The choices $T = T_{as}$ and $T = T_{ms}$ correspond to the additive and multiplicative algorithm, respectively.

Two basic assumptions then need to be verified to establish the convergence properties of the multiplicative and additive Schwarz schemes:

The first measures the stability of the decomposition with respect to the energy norm defined by the bilinear form $a(\cdot, \cdot)$, while the second is related to the quasi–orthogonality of the subspaces that make up the splitting:

ASSUMPTION 5.1 (STABILITY OF THE DECOMPOSITION). *There exists a constant $C_0 > 0$ such that for all $v_h \in V_h$*

$$\inf\{\sum_i a(v_i, v_i); \sum_i v_i = v_h, \ v_i \in V_i\} \leq C_0^2 a(v_h, v_h) .$$

ASSUMPTION 5.2 (QUASI–ORTHOGONALITY OF SUBSPACES). *There exist constants $0 \leq \epsilon_{ij} \leq 1$ such that the following strengthened Cauchy–Schwarz inequality holds $\forall u_i \in V_i, \ v_j \in V_j, \ i, j \in \{1, \ldots, J\}$:*

$$|a(u_i, v_j)| \leq \epsilon_{i,j} \ a(u_i, u_i)^{\frac{1}{2}} \cdot a(v_j, v_j)^{\frac{1}{2}}.$$

Let $\rho(E)$ be the spectral radius of the $J \times J$ matrix $E = (\epsilon_{i,j})$. We remark that the "coarse" subspace V_0 is not included in Assumption 5.2.

The following lemma provides upper bounds for the condition number of the additive and multiplicative algorithms; the proof can be found in [54].

LEMMA 5.1. *If Assumptions 5.1 and 5.2 hold, then, $\forall \boldsymbol{u} \in V_h$,*

$$C_0^{-2} a(\boldsymbol{u}, \boldsymbol{u}) \leq a(T_{as}\boldsymbol{u}, \boldsymbol{u}) \leq (\rho(E) + 1) a(\boldsymbol{u}, \boldsymbol{u}), \tag{17}$$

$$\left(1 + 2\rho^2(E)\right)^{-1} C_0^{-2} a(\boldsymbol{u}, \boldsymbol{u}) \leq a(T_{ms}\boldsymbol{u}, \boldsymbol{u}) \leq a(\boldsymbol{u}, \boldsymbol{u}). \tag{18}$$

The bound for the multiplicative scheme can be improved by suitably rescaling the local problems, and inexact solvers can also be employed on each subspace.

We will show that C_0 and $\rho(E)$ are uniformly bounded with respect to the number of subdomains of the overlapping domain decomposition method, and the number of refinement levels of the multilevel method, in analogy to the $H^1(\Omega)$–elliptic case.

5.1. Overlapping domain decomposition method. To examine Assumption 5.1 for the decomposition (11) of $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$, we pick an arbitrary $v_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ and establish the stability of a particular splitting of this finite element function with constants independent of v_h.

Using our main idea, we start with the discrete Helmholtz–decomposition of v_h:

$$v_h = v_h^0 + v_h^+ \quad , \ \mathbf{D}\, v_h^0 = 0,\ v_h^+ \in \mathcal{V}_{d,0}^+(\mathbf{D},\mathcal{T}_h)\ , \tag{19}$$

and we then provide separate decompositions for v_h^0 and v_h^+.

As regards to v_h^+, the first step involves eliminating the low frequency components that might cripple stability. To this end we employ the L^2–orthogonal projection Q_H onto $\mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_H)$, in combination with the energy projection $P_h : \boldsymbol{H}(\mathbf{D},\Omega) \mapsto \mathcal{V}_{d,0}^{\perp}(\mathbf{D},\mathcal{T}_h)$ from Def. 4.1:

$$v_h^+ = Q_H\, P_h\, v_h^+ + w_h$$

The remainder $w_h \in \mathcal{V}_{d,0}(\mathbf{D},\mathcal{T}_h)$ is then treated in a classical way [27, 54]. We introduce a piecewise linear partition of unity $\{\chi_i\}_{i=1}^J$ relative to the covering $\{\Omega_i'\}_{i=1}^J$ [27], defined as a set of continuous functions, satisfying the following properties,

$$\begin{aligned}&\chi_i \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}_h(\Omega),\\ &\text{supp}\ (\chi_i) \subset \Omega_i',\\ &0 \le \chi_i \le 1,\ \sum_i \chi_i = 1\ .\end{aligned}$$

Moreover, we can assume

$$\|\mathbf{grad}\,\chi_i\|_{L^\infty(\Omega)} \le C/\delta.$$

Thus the remainder w_h can be decomposed into parts belonging to the local subspaces:

$$w_h = \sum_{i=1}^J \Pi_{d,\mathcal{T}_h}^{\mathbf{D}}\,(\chi_i \cdot w_h)$$

We have thus found the decomposition

$$v_h^+ = \sum_{i=0}^J v_i \quad \text{with} \quad v_0 = Q_H\, P_h\, v_h^+ \text{ and } v_i = \Pi_{d,\mathcal{T}_h}^{\mathbf{D}}\left(\chi_i \cdot (v_h^+ - v_0)\right)\ .$$

First we examine v_0. Using the stability of the L^2–orthogonal projection and the regularity of the space $\mathcal{V}_{d,0}^{\perp}(\mathbf{D},\mathcal{T}_h)$, Given by Lemmas 4.5 and 4.1 we get

$$\|\mathbf{D}\,v_0\|_{L^2(\Omega)} \le C|P_h\, v_h^+|_{H^1(\Omega)} \le C\left\|\mathbf{D}\,P_h\, v_h^+\right\|_{L^2(\Omega)} = C\|\mathbf{D}\,v_h\|_{L^2(\Omega)}\ . \tag{20}$$

Additionally, since the Helmholtz–decomposition is L^2–orthogonal

$$\|v_0\|_{L^2(\Omega)} \le \left\|P_h\, v_h^+\right\|_{L^2(\Omega)} \le \left\|v_h^+\right\|_{L^2(\Omega)} \le \|v_h\|_{L^2(\Omega)}\ . \tag{21}$$

We now proceed in close analogy to Section 5.3.1 in [54]. In order to estimate the energy of the v_i we continue with local considerations. Fix $i \in \{1,\dots,J\}$ and pick a

$T \in \mathcal{T}_h$. Denote by $\bar{\chi}_{i,T}$ the average of χ_i over T. A straightforward application of the triangle inequality and using $\chi_i \leq 1$ shows

$$\|\mathbf{D}\, w_i\|^2_{L^2(T)} \leq 2\left\|\mathbf{D}\, \Pi^{\mathbf{D}}_{d,\mathcal{T}_h}\left((\chi_i - \bar{\chi}_{i,T})w_h\right)\right\|^2_{L^2(T)} + 2\|\mathbf{D}\, w_h\|^2_{L^2(T)} \,. \tag{22}$$

From the definition of the degrees of freedom it is clear that for $\varphi \in \mathcal{P}_1(T)$ and $v_h \in \mathcal{V}_d(\mathbf{D}, T)$

$$\kappa(\varphi \cdot v_h) \leq C\ \|\varphi\|_{L^\infty(T)} \cdot \kappa(v_h) \tag{23}$$

for any degree of freedom $\kappa \in \Xi_d(\mathbf{D}, T)$. The constant C may be chosen in such a way that it depends on the polynomial degree d only. Now recalling the L^2–stability estimate (4), it is immediate that

$$\left\|\Pi^{\mathbf{D}}_{d,\mathcal{T}_h}\left((\chi_i - \bar{\chi}_{i,T})w_h\right)\right\|^2_{L^2(T)} \leq C\,\|\chi_i - \bar{\chi}_{i,T}\|^2_{L^\infty(T)}\ \|w_h\|^2_{L^2(T)} \,.$$

In addition, by virtue of the special choice of the partition of unity $\{\chi_i\}_i$, we get

$$\|\chi_i - \bar{\chi}_{i,T}\|_{L^\infty(T)} \leq Ch/\delta \,.$$

These estimates, in combination with an inverse inequality, permits us to bound the first term on the right hand side of (22) by

$$\left\|\mathbf{D}\, \Pi^{\mathbf{D}}_{d,\mathcal{T}_h}\left((\chi_i - \bar{\chi}_{i,T})w_h\right)\right\|^2_{L^2(T)} \leq C/\delta^2\ \|w_h\|^2_{L^2(T)} \,.$$

By the finite covering property of the subdomains, we can easily switch back to the global finite element space and get

$$\sum_{i=1}^{J} \|\mathbf{D}\, v_i\|^2_{L^2(\Omega)} \leq C/\delta^2\ \|w_h\|^2_{L^2(\Omega)} + C\|\mathbf{D}\, w_h\|_{L^2(\Omega)} \,. \tag{24}$$

Finally, we resort to the approximation estimates from Lemmas 4.5 and 4.4 in combination with Lemma 4.1:

$$\begin{aligned}\|w_h\|^2_{L^2(\Omega)} &\leq 2\left\|(Q_H - Id)\, P_h\, v_h^+\right\|^2_{L^2(\Omega)} + 2\left\|(P_h - Id)v_h^+\right\|^2_{L^2(\Omega)} \\ &\leq CH^2\left(|P_h\, v_h^+|^2_{H^1(\Omega)} + \left\|\mathbf{D}\, v_h^+\right\|^2_{L^2(\Omega)}\right) \leq CH^2\|\mathbf{D}\, v_h\|^2_{L^2(\Omega)} \,.\end{aligned}$$

Plugging this into (24) shows that

$$\sum_{i=1}^{J} \|\mathbf{D}\, v_i\|^2_{L^2(\Omega)} \leq C\left(\frac{H}{\delta}\right)^2 \|\mathbf{D}\, v_h\|^2_{L^2(\Omega)} \,. \tag{25}$$

The L^2–stability of the decomposition is readily concluded from (23) and (21):

$$\|v_0\|^2_{L^2(\Omega)} + \sum_{i=1}^{J} \|v_i\|^2_{L^2(\Omega)} \leq C\|v_h\|^2_{L^2(\Omega)} \,. \tag{26}$$

Merging (20), (25) and (26), we get

$$a(v_0, v_0) + \sum_{i=1}^{J} a(v_i, v_i) \leq C \left(1 + \left(\frac{H}{\delta}\right)^2\right) a(v_h, v_h) \ , \tag{27}$$

with $C > 0$ *independent* of η.

To deal with v_h^0, we use Theorem 2.1 and choose $\widetilde{v}_h^+ \in \mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_h)$ such that $\widetilde{\mathbf{D}}\, \widetilde{v}_h^+ = v_h^0$. Using the very same approach as above in the potential space, we end up with the splitting

$$\widetilde{v}_h^+ = \widetilde{v}_0 + \sum_{i=1}^{J} \widetilde{v}_i \quad \widetilde{v}_i \in \widetilde{V}_i \ ,$$

with the property

$$\|\widetilde{\mathbf{D}}\, \widetilde{v}_0\|^2_{L^2(\Omega)} + \sum_{i=1}^{J} \|\widetilde{\mathbf{D}}\, \widetilde{v}_i\|^2_{L^2(\Omega)} \leq C \left(1 + \left(\frac{H}{\delta}\right)^2\right) \|\widetilde{\mathbf{D}}\, \widetilde{v}_h^+\|^2_{L^2(\Omega)} \ . \tag{28}$$

Adding (28) and (27), we finally obtain

$$C_0^2 \leq C \left(1 + \left(\frac{H}{\delta}\right)^2\right), \tag{29}$$

where C_0 is defined in Assumption 5.1. This proves the first inequality in (17).

The proof of the second inequality in (17) is standard and can be found in [27, 54, 3]. In particular, we obtain

$$\rho(E) \leq (N_c + 1), \tag{30}$$

where N_c is the finite covering parameter defined in Assumption 3.1.

We have thus proved the following theorem:

THEOREM 5.2. *If the domain Ω is convex, and the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ are shape-regular and quasiuniform, the condition numbers of the additive and multiplicative two-level algorithms are bounded uniformly with respect to h, the number of subregions and η. The bounds grow quadratically as $(1 + (H/\delta)^2)$.*

Remark. In the analysis of overlapping Schwarz methods for the scalar H^1–elliptic case, if the L^2 projection on the coarse space is employed, the coarse triangulation $\mathcal{T}_H$ has to be *quasiuniform*, in order for the estimates in Lemma 4.5 to hold. This assumption can be removed by employing alternative interpolation operators, such as Clément interpolation or other suitable local averages [23]. In the vector case studied in this work, the use of the L^2–projection can be avoided, but quasiuniformity is still required for the proof of Lemma 4.4. This is mainly due to the lack of localized properties of of the energy projection P_h and the auxiliary semicontinuous spaces $\mathcal{V}_{d,0}^{\perp}(\mathbf{D}, \mathcal{T}_h)$ employed.

5.2. Multilevel scheme. As in Section 5.1 we start with the discrete Helmholtz–decomposition (19) of an arbitrary $v_h \in \mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_h)$. Unlike before, we then switch to a sort of *skewed Helmholtz–decomposition*

$$v_h = w_h^+ + w_h^0 \ , \tag{31}$$

where

$$\begin{aligned} w_h^+ &:= B_0\, P_0\, v_h^+ + \sum_{l=1}^{J} B_l (P_l - P_{l-1}) v_h^+ \\ w_h^0 &:= v_h^0 + \sum_{l=1}^{J} (B_l - B_{l-1})\, P_{l-1}\, v_h^+ \,. \end{aligned}$$

The operators P_l and B_l are the energy and Helmholtz projections in $\mathcal{V}_d^0(\mathbf{D}, \mathcal{T}_l)$. This splitting was employed in the study of some multilevel decompositions of $\boldsymbol{H}(\mathrm{div};\Omega)$ in 2D [58, 3, 40]. Owing to the "**D**–preserving" property of the Helmholtz–projections B_l from Def. 4.2, we notice that

$$(B_l - B_{l-1})\, P_{l-1}\, v_h^+ \in \mathcal{V}_{d,0}^0(\mathbf{D}, \mathcal{T}_l) \,,$$

and thus $\mathbf{D}\, w_h^0 = 0$.

We first tackle w_h^+. We start with the trivial estimate

$$\left\| \mathbf{D}\, B_0\, P_0\, v_h^+ \right\|_{L^2(\Omega)} \le \left\| \mathbf{D}\, v_h^+ \right\|_{L^2(\Omega)} .$$

Now, for the sake of brevity, set $r_l := B_l (P_l - P_{l-1}) v_h^+$, for which we get by means of Lemmas 4.3 and 4.2 and the properties of projection operators:

$$\begin{aligned} \|r_l\|_{L^2(\Omega)} &\le \left\| (P_l - P_{l-1}) v_h^+ \right\|_{L^2(\Omega)} + \left\| (Id - B_l)(P_l - P_{l-1}) v_h^+ \right\|_{L^2(\Omega)} \\ &\le \left\| (Id - P_{l-1})(P_l - P_{l-1}) v_h^+ \right\|_{L^2(\Omega)} + C h_l \left\| \mathbf{D}(P_l - P_{l-1}) v_h^+ \right\|_{L^2(\Omega)} \\ &\le C h_l \left\| \mathbf{D}(P_l - P_{l-1}) v_h^+ \right\|_{L^2(\Omega)} . \end{aligned}$$

Exploiting the $\|\mathbf{D}\cdot\|_{L^2(\Omega)}$–orthogonality implied by the properties of the energy projection, we arrive at

$$\sum_{i=1}^{J} h_l^{-2} \|r_l\|_{L^2(\Omega)}^2 \le C \sum_{i=1}^{J} \left\| \mathbf{D}(P_l - P_{l-1}) v_h^+ \right\|_{L^2(\Omega)}^2 \le C \left\| \mathbf{D}\, v_h^+ \right\|_{L^2(\Omega)}^2 . \tag{32}$$

In this we followed the general policy of certain regularity–based proofs of the stability of multilevel decompositions of $H^1(\Omega)$–conforming finite element spaces [65, 63]. The crucial duality techniques are concealed in the proof of Lemma 4.2.

Next, we recall that for a basis function $\psi_{\kappa,l}$ on level l, we have straightforwardly that

$$\|\psi_{\kappa,l}\|_{L^2(\Omega)} \le C h_l \, \|\mathbf{D}\, \psi_{\kappa,l}\|_{L^2(\Omega)} .$$

Plugging this estimate into (32) and using the L^2–stability (4) yields

$$\left\| \mathbf{D}\, B_0\, P_0\, v_h^+ \right\|_{L^2(\Omega)}^2 + \sum_{l=1}^{J} \sum_{\kappa} \|\mathbf{D}\, r_{\kappa,l}\|_{L^2(\Omega)}^2 \le C \left\| \mathbf{D}\, v_h^+ \right\|_{L^2(\Omega)}^2 , \tag{33}$$

where, for a $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$–d.o.f. κ, we set $r_{\kappa,l} := \kappa(r_l)\psi_{\kappa,l}$.

We emphasize that the proof of (33) remains valid in the potential space. For a suitable $\widetilde{v}_h^+ \in \mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_h)$ with $\widetilde{\mathbf{D}}\,\widetilde{v}_h^+ = w_h^0$, which can be found according to Theorem 2.1, we have just shown the existence of a nodal multilevel decomposition

$$\widetilde{v}_0 + \sum_{l=1}^{J}\sum_{\widetilde{\kappa}} \widetilde{v}_{\widetilde{\kappa},l} = \widetilde{v}_h^+ \ , \quad \widetilde{v}_{\widetilde{\kappa},l} \in \mathrm{Span}\left\{\widetilde{\psi}_{\widetilde{\kappa},l}\right\} , \ \widetilde{v}_0 \in \mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_0) \ ,$$

with the property

$$\|\widetilde{\mathbf{D}}\,\widetilde{v}_0\|^2_{L^2(\Omega)} + \sum_{l=1}^{J}\sum_{\widetilde{\kappa}} \|\widetilde{\mathbf{D}}\,\widetilde{v}_{\widetilde{\kappa},l}\|^2_{L^2(\Omega)} \leq C\|\widetilde{\mathbf{D}}\,\widetilde{v}_h^+\|^2_{L^2(\Omega)} \ . \tag{34}$$

In addition, we observe

$$\begin{aligned} \|w_h^0\|^2_{L^2(\Omega)} &\leq 2\|v_h^0\|^2_{L^2(\Omega)} + 2\left\|\sum_{l=1}^{J} r_l\right\|^2_{L^2(\Omega)} \\ &\leq 2\|v_h^0\|^2_{L^2(\Omega)} + 2\left(\sum_{l=1}^{J} h_l^2\right)\left(\sum_{l=1}^{J} h_l^{-2}\|r_l\|^2_{L^2(\Omega)}\right) \\ &\leq 2\|v_h^0\|^2_{L^2(\Omega)} + C\|\mathbf{D}\,v_h\|^2_{L^2(\Omega)} \ , \end{aligned} \tag{35}$$

thanks to the geometric decrease of the meshwidths.

Combining (33), (34), and (35) confirms Assumption 5.1 for the nodal multilevel splitting (12) with a stability constant

$$C_0^2 = C(1 + \frac{1}{\eta}) \ ,$$

where $C > 0$ only depends on the shape–regularity of the initial triangulation $\mathcal{T}_0$ and the domain Ω. Compared to the case of the overlapping Schwarz method, this bound degrades as η decreases. This is the price to pay for using the non–orthogonal skewed Helmholtz–decomposition (31).

To establish the strengthened Cauchy–Schwarz inequality of Assumption 5.2 we can resort to tricks that have been conceived, e.g., in [7, 62, 61] for $H^1(\Omega)$–conforming standard finite elements. For applications of these techniques to $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problems in two dimensions, we refer to [40, 4]. It turns out that the approach carries over to three dimensions and $\boldsymbol{H}(\mathbf{curl};\Omega)$ with scarcely any modifications. Thus we will only briefly sketch the idea. A more detailed discussion can be found in [36, 37].

To begin with, we sort the basis functions of $\mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_l)$ and $\mathcal{V}_{d,0}(\widetilde{\mathbf{D}}, \mathcal{T}_l)$, $l = 1, \ldots, J$ into different classes such that the intersection of the supports of any two basis functions in one class has measure zero. A small number of such classes will do on any level of refinement due to the uniform shape regularity of the meshes. Write N and $\widetilde{N}$, respectively, for these numbers. We introduce the notations Y_l^i, $i = 1, \ldots, N$ and $\widetilde{Y}_l^i$, $i = 1, \ldots, \widetilde{N}$, for the subspaces spanned by the basis functions in class i. Note that the basis functions in one class are mutually $a(\cdot,\cdot)$–orthogonal. For this reason we may well replace the one–dimensional subspaces in (12) by Y_l^I and $\widetilde{Y}_l^i$ without the slightest impact on $\rho(E)$ in Assumption 5.2. For this "lumped decomposition", we can prove the following lemma, by means of Green's formula and purely local investigations on elements of the coarser mesh (see [36, Section 6]).

LEMMA 5.3. *For* $0 \leq m < k \leq J$ *and any* $z_m \in \mathcal{V}_{d,0}(\mathbf{D}, \mathcal{T}_m)$, $q_k \in Y_k^i$ $(1 \leq i \leq N)$, *and* $p_k \in \widetilde{Y}_k^i$ $(1 \leq i \leq \widetilde{N})$ *we can estimate*

$$\begin{aligned} a(q_k, z_m) &\leq C\left(\sqrt{\frac{h_k}{h_m}} + h_k\right) \|\mathbf{D}\, q_k\|_{L^2(\Omega)} \cdot \|\mathbf{D}\, z_m\|_{L^2(\Omega)} \\ a(\widetilde{\mathbf{D}}\, p_k, z_m) &\leq C\sqrt{\frac{h_k}{h_m}}\, \|\widetilde{\mathbf{D}}\, p_k\|_{L^2(\Omega)} \cdot \|z_m\|_{L^2(\Omega)} \,. \end{aligned}$$

Using Cauchy-Schwarz inequalities, the geometric decrease of the meshwidths h_l, and estimates for the spectral radius of the matrix E from Assumption 5.2 [61, Lemma 4.6], we end up with

$$\rho(E) \leq C(\frac{1}{\sqrt{\eta}} + 1) \,.$$

Again, we face a deterioration of the bound for very small values of η.

In sum, as a consequence of Lemma 5.1, we have shown

THEOREM 5.4. *For a convex polyhedron* Ω, *in the case of uniform regular refinement, the multigrid method and the multilevel preconditioner based on the decomposition (12), exhibit a rate of convergence and a condition number, respectively, that are bounded independently of the depth* J *of refinement. The theoretical bound behaves like* $O(\eta^{-3/2})$ *for small values of the scaling parameter* η.

Remark. Numerical experiments from [37, 37, 39] strongly hint that for the multigrid scheme, the convergence is hardly affected by a dominant zero order term, whereas the additive preconditioners is not robust for a small η. Since the above analysis is valid for both approaches, it fails to reflect the superiority of the multiplicative strategy.

6. Conclusion. In this paper, we have presented a uniform framework for the analysis of some overlapping and multigrid schemes for elliptic problems in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$, discretized by means of Raviart–Thomas and Nédélec finite elements. Guided by the idea that the kernels of the differential operators div and **curl** require a special treatment, we derived viable decompositions for overlapping domain decomposition methods and multilevel schemes. Their implementation can be carried out in a standard fashion, as for the case of $H^1(\Omega)$–elliptic discrete variational problems. Moreover, we showed that our methods are optimal if the computational domain is convex.

REFERENCES

[1] R. ADAMS, *Sobolev Sapces*, Academic Press New York, 1975.

[2] C. AMROUCHE, C. BERNARDI, M. DAUGE, AND V. GIRAULT, *Vector potential in three–dimensional nonsmooth domains*, Tech. Rep. 96–04, IRMAR, Rennes, France, 1996.

[3] D. ARNOLD, R. FALK, AND R. WINTHER, *Preconditioning in* $H(\mathrm{div})$ *and applications*, Math. Comp., 66 (1997). To appear.

[4] ———, *Multigrid preconditioning in* $H(\mathrm{div})$ *on non-convex polygons*, tech. rep., Penn State University, 1997, submitted to Computational and Applied Mathematics.

[5] R. BANK, T. DUPONT, AND H. YSERENTANT, *The hierarchical basis multigrid method*, Numer. Math., 52 (1988), pp. 427–458.

[6] J. BEY, *Tetrahedral grid refinement*, Computing, 55 (1995), pp. 355–378.

[7] F. BORNEMANN, *A sharpened condition number estimate for the BPX–precondtioner of elliptic finite element problems on highly nonuniform triangulations*, Tech. Rep. SC 91–9, ZIB–Berlin, 1991.

[8] A. BOSSAVIT, *Two dual formulations of the 3D eddy–currents problem*, COMPEL, 4 (1985), pp. 103–116.

[9] ——, *A rationale for edge elements in 3d field computations*, IEEE Trans. Mag., 24 (1988), pp. 74–79.
[10] ——, *Whitney forms: A class of finite elements for three–dimensional computations in electromagnetism*, IEE Proc. A, 135 (1988), pp. 493–500.
[11] ——, *A new viewpoint on mixed elements*, Meccanica, 27 (1992), pp. 3–11.
[12] J. Bramble, *Multigrid methods*, Pitman Research Notes in Mathematics Series, Longman, London, 1993.
[13] J. Bramble, J. Pasciak, and A. Schatz, *The construction of preconditioners by substructuring I*, Math. Comp., 47 (1986), pp. 100–134.
[14] ——, *The construction of preconditioners for elliptic problems by substructuring. IV*, Math. Comput., 53 (1989), pp. 1–24.
[15] J. Bramble, J. Pasciak, and J. Xu, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.
[16] J. Bramble and J. Xu, *Some estimates for a weighted L^2-projection*, Math. Comp., 56 (1991), pp. 463–476.
[17] J. Brandts, *Superconvergence and a posteriori error estimation for triangular mixed finite elements*, Numer. Math., 68 (1994), pp. 311–324.
[18] F. Brezzi, J. Douglas, and D. Marini, *Two families of mixed finite elements for 2nd order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.
[19] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer–Verlag, New York, 1991.
[20] Z. Cai, C. Goldstein, and J. Pasciak, *Multilevel iteration for mixed finite element systems with penalty*, SIAM J. Sci. Comput., 14 (1993), pp. 1072–1088.
[21] Z. Cai, R. Lazarov, T. Manteuffel, and S. McCormick, *First–order system least–squares for partial differential equations: Part I*, SIAM J. Numer. Anal., 31 (1994), pp. 1785–1799.
[22] Z. Cai, R. Parashkevov, T. Russel, and X. Ye, *Domain decomposition for a mixed finite element method in three dimensions*, SIAM J. Numer. Anal., (1997).
[23] P. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.
[24] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology*, Springer-Verlag, New York, 1988.
[25] J. Douglas and J. Roberts, *Global estimates for mixed methods for 2nd order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.
[26] M. Dryja, B. F. Smith, and O. B. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal., 31 (1994), pp. 1662–1694.
[27] M. Dryja and O. B. Widlund, *Domain decomposition algorithms with small overlap*, SIAM J. Sci. Comput., 15 (1994), pp. 604–620.
[28] R. Ewing and J. Wang, *Analysis of the Schwarz algorithm for mixed finite element methods*, M^2AN Math. Modelling and Numer. Anal., 26 (1992), pp. 739–756.
[29] V. Girault, *Incompressible finite element methods for Navier–Stokes equations with nonstandard boundary conditions in $\mathbb{R}^3$*, Math. Comp., 51 (1988), pp. 55–74.
[30] ——, *Curl–conforming finite element methods for Navier–Stokes equations with non–standard boundary conditions in $\mathbb{R}^3$*, vol. 1431 of Lecture Notes in Mathematics, Springer–Verlag, Berlin, 1989, pp. 201–218.
[31] V. Girault and P. Raviart, *Finite element methods for Navier–Stokes equations*, Springer–Verlag, Berlin, 1986.
[32] M. Griebel, *Multilevel algorithms considered as iterative methods on semidefinite systems*, SIAM J. Sci. Stat. Comp., 15 (1994), pp. 547–565.
[33] W. Hackbusch, *Multi–grid Methods and Applications*, Springer–Verlag, Berlin, 1985.
[34] ——, *Theorie und Numerik elliptischer Differentialgleichungen*, B.G. Teubner–Verlag, Stuttgart, 1986.
[35] R. Hiptmair, *Canonical construction of finite elements*, Tech. Rep. 360, Institut für Mathematik, Universität Augsburg, 1996. Submitted to Math. Comp.
[36] ——, *Multilevel Preconditioning for Mixed Problems in Three Dimensions*, PhD thesis, Mathematisches Institut, Universität Augsburg, 1996.
[37] ——, *Multigrid method for $H(\mathrm{div})$ in three dimensions*, Tech. Rep. 368, Institut für Mathematik, Universität Augsburg, 1997. To appear in ETNA.
[38] ——, *Multigrid method for Maxwell's equations*, Tech. Rep. 374, Institut für Mathematik, Uni-

versität Augsburg, 1997. Submitted to SINUM.

[39] R. Hiptmair and R. Hoppe, *Multilevel preconditioning for mixed problems in three dimensions*, Tech. Rep. 359, Mathematisches Institut, Universität Augsburg, 1996. Submitted to Numer. Math.

[40] R. Hiptmair, T. Schiekofer, and B. Wohlmuth, *Multilevel preconditioned augmented Lagrangian techniques for 2nd order mixed problems*, Computing, 57 (1996), pp. 25–48.

[41] A. Klawonn, *An optimal preconditioner for a class of saddle point problems with a penalty term, Part II: General theory*, Tech. Rep. 14/95, Westfälische Wilhelms-Universität Münster, Germany, April 1995. Also available as Technical Report 683 of the Courant Institute of Mathematical Sciences, New York University.

[42] P. Lin, *A sequential regularization method for time-dependent incompressible Navier–Stokes equations*, SIAM J. Numer. Anal., 34 (1997), pp. 1051–1071.

[43] G. Makridakis and P. Monk, *Time–discrete finite element schemes for Maxwell's equations*, RAIRO M^2AN, 29 (1995), pp. 171–197.

[44] J. Nédélec, *Mixed finite elements in* R^3, Numer. Math., 35 (1980), pp. 315–341.

[45] ———, *A new family of mixed finite elements in* R^3, Numer. Math., 50 (1986), pp. 57–81.

[46] P. Oswald, *Preconditioners for discretizations of the biharmonic equation by rectangular finite elements*, tech. rep., Friedrich Schiller Universität, Jena, Germany, 1991.

[47] ———, *Multilevel Finite Element Approximation, Theory and Applications*, Teubner Skripten zur Numerik, B.G. Teubner, Stuttgart, 1994.

[48] J. P. Ciarlet and J. Zou, *Fully discrete finite element approaches for time–dependent Maxwell equations*, Tech. Rep. TR MATH–96-31 (105), Department of Mathematics, The Chinese University of Hong Kong, 1996.

[49] L. F. Pavarino, *Additive Schwarz methods for the p-version finite element method*, Numer. Math., 66 (1994), pp. 493–515.

[50] ———, *Schwarz methods with local refinement for the p-version finite element method*, Numer. Math., 69 (1994), pp. 185–211.

[51] L. F. Pavarino and O. B. Widlund, *Preconditioned conjugate gradient solvers for spectral elements in 3D*, in Solution Techniques for Large-Scale CFD Problems, W. G. Habashi, ed., John Wiley & Sons, 1995, pp. 249–270. Proceedings of the International Workshop on Solution Techniques for Large-Scale CFD Problems held at CERCA, Montréal, Canada, September 26–28, 1994.

[52] A. Pehlivanov, G. Carey, and R. Lazarov, *Least squares mixed finite elements for second order elliptic problems*, SIAM J. Num. Anal., 31 (1994), pp. 1368–1377.

[53] P. A. Raviart and J. M. Thomas, *A Mixed Finite Element Method for Second Order Elliptic Problems*, vol. 606 of Springer Lecture Notes in Mathematics, Springer–Verlag, New York, 1977, pp. 292–315.

[54] B. Smith, P. Bjørstad, and W. Gropp, *Domain decomposition*, Cambridge University Press, Cambridge, 1996.

[55] A. Toselli, *Some numerical results using an additive Schwarz method for Maxwell's equations*, Tech. Rep. 726, Courant Institute, New York University, November 1996.

[56] ———, *Overlapping Schwarz methods for Maxwell's equations in 3D*, Tech. Rep. TR–736, Courant-Institute, New York, 1997.

[57] P. Vassilevski and R. Lazarov, *Preconditioned mixed finite element saddle–point elliptic problems*, Numer. Lin. Algebra, 2 (1995).

[58] P. Vassilevski and J. Wang, *Multilevel iterative methods for mixed finite element discretizations of elliptic problems*, Numer. Math., 63 (1992), pp. 503–520.

[59] O. Widlund, *Iterative substructuring methods: Algorithms and theory forelliptic problems in the plane*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. Golub, G. Meurant, and J. Périaux, eds., SIAM, Philadelphia, 1986, pp. 113–128.

[60] B. Wohlmuth, *Adaptive Multilevel–Finite–Elemente Methoden zur Lösung elliptischer Randwertprobleme*, PhD thesis, TU München, 1995.

[61] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Review, 34 (1992), pp. 581–613.

[62] H. Yserentant, *On the multi–level splitting of finite element spaces*, Numer. Math., 58 (1986), pp. 379–412.

[63] ———, *Two preconditioners based on the multilevel splitting of finite element spaces*, Numer. Math., 58 (1990), pp. 163–184.

[64] ———, *Old and new convergence proofs for multigrid methods*, Acta Numerica, (1993), pp. 285–326.
[65] X. Zhang, *Multilevel Schwarz methods*, Numer. Math., 63 (1992), pp. 521–539.
[66] ———, *Multilevel Schwarz methods for the biharmonic Dirichlet problem*, SIAM J. Sci. Comput., 15 (1994), pp. 621–644.