OVERLAPPING SCHWARZ METHODS FOR MAXWELL'S EQUATIONS IN THREE DIMENSIONS

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Abstract. Two-level overlapping Schwarz methods are considered for finite element problems of 3D Maxwell's equations. Nédélec elements built on tetrahedra and hexahedra are considered. Once the relative overlap is fixed, the condition number of the additive Schwarz method is bounded, independently of the diameter of the triangulation and the number of subregions. A similar result is obtained for a multiplicative method. These bounds are obtained for quasi-uniform triangulations. In addition, for the Dirichlet problem, the convexity of the domain has to be assumed. Our work generalizes well-known results for conforming finite elements for second order elliptic scalar equations.

1. Introduction. When time-dependent Maxwell's equations are considered, the electric field $\mathbf{u}$ satisfies the following equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + \mu\varepsilon\frac{\partial^2\mathbf{u}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{u}}{\partial t} = -\mu\frac{\partial\mathbf{J}}{\partial t}, \quad \text{in } \Omega, \tag{1}$$

where $\mathbf{J}(\mathbf{x}, t)$ is the current density and ε, μ, σ describe the electromagnetic properties of the medium. For their meaning and for a general discussion of Maxwell's equations, see [4]. Here Ω is a bounded domain, with boundary Γ and outside normal $\mathbf{n}$. A similar equation holds for the magnetic field. For a perfect conducting boundary, the electric field satisfies the natural boundary condition

$$\mathbf{u} \times \mathbf{n}|_\Gamma = 0. \tag{2}$$

For the analysis and solution of Maxwell's equations suitable Sobolev spaces must be introduced. If $\Omega \subset \mathbb{R}^3$ is a bounded, open, connected set with Lipschitz continuous boundary Γ, the space $H(\mathrm{curl}, \Omega)$, of square integrable vectors, with square integrable curls, is a Hilbert space with the scalar product

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{u}, \mathbf{v}). \tag{3}$$

Here, $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$ (or $L^2(\Omega)^3$); we will use $\|\cdot\|$ to denote the corresponding norm. For the properties of $H(\mathrm{curl}, \Omega)$, see [6]. In particular, we recall that if Γ is Lipschitz continuous, then for every function $\mathbf{u} \in H(\mathrm{curl}, \Omega)$ it is possible to define a tangential trace over Γ, $\mathbf{u} \times \mathbf{n}$, as an element of $H^{-\frac{1}{2}}(\Gamma)^3$ and that the functions of $H(\mathrm{curl}, \Omega)$ with vanishing tangential trace form a proper subspace of $H(\mathrm{curl}, \Omega)$, denoted by $H_0(\mathrm{curl}, \Omega)$. Additional properties will be mentioned in the next section. The bilinear form $a(\cdot, \cdot)$ is related to the differential operator $L = I + \mathbf{curl}\,\mathbf{curl}$.

Variational problems involving the bilinear form $a(\cdot, \cdot)$, arise, for instance, when equation (1) is discretized with an implicit finite difference time scheme. For the

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spatial approximation of (1), Nédélec spaces can be employed; see [11], [12]: only the continuity of the tangential component across the faces of the triangulation is ensured. See [17], [15], [16], for the finite element approximation of time-dependent Maxwell's equations; and [18], for a discussion of approximations of hyperbolic equations.

When an implicit FD scheme is employed and a finite element space $V \subset H_0(\mathrm{curl}, \Omega)$ is introduced, equation (1) can be approximated by:

Find $\mathbf{u} \in V$ such that

$$a_\eta(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) + \eta(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \forall\, \mathbf{v} \in V, \tag{4}$$

at each time step; η is a positive quantity that vanishes when the time step Δt tends to zero and f depends on the solution at the previous steps, as well as on the right hand side of (1).

In the last few years, a considerable effort has been devoted to the study of Schwarz methods for the solution of linear systems arising from non-conforming finite element problems; see [2], [9], [8], [10], [14], [20]. Our analysis of overlapping methods has been inspired by [2], where a Schwarz method for non-conforming finite element problem in 2D is studied; their result is valid for 2D Maxwell's equations; see [20]. In addition, we will also use the technical tools and the analysis originally developed in [8], where a multigrid method is studied for a divergence-conforming finite element problem in 3D. In particular, we will prove a regularity property that will enable us to extend the tools in [8] and [10] to a general convex polyhedron.

We will only consider the bilinear form $a(\cdot, \cdot)$ for $\eta = 1$. In addition, we will first consider $a(\cdot, \cdot)$ defined on $H_0(\mathrm{curl}, \Omega)$ (Dirichlet problem); the extension to the whole space $H(\mathrm{curl}, \Omega)$ (Neumann problem) will be then carried out. In the following, the capital letter C, possibly with a subscript, will be used for a positive constant that is bounded away from ∞.

Let us introduce the operator

$$T = \sum_{j=0}^{J} T_j : V \longrightarrow V, \tag{5}$$

where T_0 and $\{T_j\}_{j=1}^{J}$ are operators defined on a coarse finite element space and on spaces related to subdomains $\{\Omega'_j\}$, respectively. When using a two-level Schwarz additive algorithm, one solves the equation

$$T\,\mathbf{u} = \mathbf{g}, \tag{6}$$

with the conjugate gradient method, without any further preconditioner, employing $a(\cdot, \cdot)$ as the inner product and a suitable right hand side $\mathbf{g}$ ([5],[19]). We will prove that

$$C_1^{-1} a(\mathbf{u}, \mathbf{u}) \le a(T\mathbf{u}, \mathbf{u}) \le C_2\, a(\mathbf{u}, \mathbf{u}),\ \forall \mathbf{u} \in V, \tag{7}$$

where the constants C_1 and C_2 are independent of the mesh size h and the number of subregions, and depend only on the overlap. The condition number of the operator T is thus bounded uniformly with respect to h.

Iterative two-level multiplicative schemes can also be designed (see [5],[19]). The error $\mathbf{e}_n$ at the n-th step satisfies the equation

$$\mathbf{e}_{n+1} = E\,\mathbf{e}_n = (I - T_J)\cdots(I - T_0)\,\mathbf{e}_n,\ \forall n \ge 0. \tag{8}$$

An upper bound for the norm of E will follow directly from Schwarz theory. Different choices of multiplicative operators are also possible; see [19] for a more detailed discussion.

We end this section by remarking that the bounds in formula (7) allow us to build some optimal block-diagonal preconditioners for mixed problems: the general theory for such preconditioners is developed in [14]. Magnetostatic or electrostatic problems are generally reformulated in a mixed form ([13],[9]). Equations for vector potentials also give rise to mixed problems ([1],[8]) and for time-dependent Maxwell's equations, it is often convenient to consider a mixed formulation ([17]).

In Section 2, we will state some properties of the space $H(\mathrm{curl}, \Omega)$ and prove an embedding theorem that we will need for the proof of (7), while in Section 3 we will describe the Nédélec finite element space V and introduce some operators. Section 4 is devoted to the description of the Schwarz methods.

2. Sobolev spaces and regularity results. In this section, we will describe some results on the space $H(\mathrm{curl}, \Omega)$; as a general reference for this section, see [6] and [4]. In addition we will need the space $H(\mathrm{div}, \Omega)$, that consists of square integrable vector functions, with square integrable divergence. In $H(\mathrm{div}, \Omega)$, it is possible to define a normal trace on the boundary Γ of Ω, as an element of $H^{-\frac{1}{2}}(\Gamma)$. The subspace of functions of $H(\mathrm{div}, \Omega)$, with vanishing normal trace is denoted by $H_0(\mathrm{div}, \Omega)$.

In the following, the domain $\Omega \subset \mathbb{R}^3$ will be a bounded, convex, open polyhedron. We will assume that its boundary consists of a finite number of plane surfaces. Thus Ω is simply connected, and its boundary is connected and Lipschitz continuous.

An orthogonal decomposition of $L^2(\Omega)^3$, valid in a general Lipschitz domain, holds

$$H_0(\mathrm{curl}, \Omega) = \mathbf{grad}\, H_0^1(\Omega) \oplus H_0^{\perp}(\mathrm{curl}, \Omega), \tag{9}$$

where

$$H_0^{\perp}(\mathrm{curl}, \Omega) = H(\mathrm{div}_0, \Omega) \cap H_0(\mathrm{curl}, \Omega), \tag{10}$$

with

$$H(\mathrm{div}_0, \Omega) = \{\mathbf{u} \in H(\mathrm{div}, \Omega) |\, \mathrm{div}\, \mathbf{u} = 0\}\,; \tag{11}$$

see [4, Proposition 1, p. 215, vol. 3]. As usual, $H^1(\Omega)$ is the Sobolev space of functions that are square integrable, together with their first derivatives, and $H_0^1(\Omega)$ its subspace, consisting of functions that vanish on the boundary. Relation (9) is equivalent to

$$H_0^{\perp}(\mathrm{curl}, \Omega) = \left\{\mathbf{u} \in H_0(\mathrm{curl}, \Omega) |\, (\mathbf{u}, \mathbf{grad}\, q) = 0,\ \forall q \in H_0^1(\Omega)\right\}; \tag{12}$$

this implies that the space $\mathbf{grad}\, H_0^1(\Omega)$ is a closed subspace of $H_0(\mathrm{curl}, \Omega)$ and that its orthogonal complement is the space of functions in $H_0(\mathrm{curl}, \Omega)$ with zero divergence.

Since Ω is simply connected and its boundary is connected, the kernel of the curl operator defined on $H_0(\mathrm{curl}, \Omega)$ is $\mathbf{grad}\, H_0^1(\Omega)$ (see [4, Proposition 2, p. 219, vol. 3]) and the following inequality holds:

$$\|\mathbf{u}\| \le C\, \|\mathbf{curl}\, \mathbf{u}\|,\ \forall \mathbf{u} \in H_0^{\perp}(\mathrm{curl}, \Omega). \tag{13}$$

C is a given constant. In particular, inequality (13) implies that the L^2-norm of the curl is an equivalent norm in $H_0^{\perp}(\mathrm{curl}, \Omega)$. We will use this property extensively.

The main result of this section, Theorem 2.3, is an embedding theorem for $H_0^{\perp}(\mathrm{curl}, \Omega)$ and some subspaces of functions with more regular curls. We start by stating a regularity result for the Dirichlet problem for the Laplace operator that is proved in [3, Corollary 18.18]. In the following, we will denote the largest angle between the faces of the given polyhedron Ω, by ω.

LEMMA 2.1. *Given a bounded, open, convex, polyhedron $\Omega \subset \mathbb{R}^3$ and a real number $s \neq -\frac{1}{2}$, such that*

$$s < \min\left\{\frac{3}{2}, \frac{\pi}{\omega} - 1\right\}, \tag{14}$$

then the Laplace operator Δ defines an isomorphism:

$$\Delta :\ H^{s+2}(\Omega) \cap H_0^1(\Omega) \longleftrightarrow H^s(\Omega). \tag{15}$$

REMARK 2.1. *Since, for every fixed bounded, convex, polyhedron the maximum angle ω is strictly smaller than π, Lemma 2.1 implies that there exists a real number $s_\Omega > 0$, such that the mapping (15) is an isomorphism, for any $s \in [0, s_\Omega)$. The exponent s_Ω can always be chosen to be less than $1/2$.*

Our second lemma ensures the existence of a vector potential for solenoidal functions; for a proof see [6, Theorem 3.4, Corollary 3.3 and Remark 3.12].

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, Lipschitz region, with a connected boundary, and let $s \in [0,1]$. A function $\mathbf{u} \in H^s(\Omega)^3$ satisfies*

$$\mathrm{div}\, \mathbf{u} = 0,$$

if and only if there exists a $\mathbf{v} \in H^{1+s}(\Omega)^3$, such that

$$\mathbf{u} = \mathbf{curl}\, \mathbf{v}, \tag{16}$$

$$\mathrm{div}\, \mathbf{v} = 0. \tag{17}$$

We are now ready to prove an embedding theorem.

THEOREM 2.3. *Given a bounded, open, convex, polyhedron $\Omega \subset \mathbb{R}^3$, then there exists a real number $s_\Omega \in (0, 1/2)$, such that, for every $t \in [0, s_\Omega)$, the space of functions $\mathbf{w} \in H(\mathrm{curl}, \Omega)$, satisfying the conditions*

$$\mathbf{w} \times \mathbf{n}|_\Gamma = 0, \tag{18}$$

$$\mathrm{div}\, \mathbf{w} = 0 \text{ in } \Omega\ , \tag{19}$$

$$\mathbf{curl}\, \mathbf{w} \in H^t(\Omega)^3, \tag{20}$$

is continuously embedded in $H^{1+t}(\Omega)^3$.

Proof. The argument is the same as in the proof of similar embedding theorems: see [1, Proposition 3.7]. It employs the existence and regularity of the vector potential of Lemma 2.2 and the regularity result for the Laplace operator given in Lemma 2.1.

Let $s_\Omega \in (0, 1/2)$ be the exponent of Remark 2.1 and let $t \in [0, s_\Omega)$. Given $\mathbf{w} \in H_0(\mathrm{curl}, \Omega)$, satisfying (18), (19), (20), define

$$\mathbf{u} = \mathbf{curl}\, \mathbf{w} \in H^t(\Omega)^3.$$

Let $\mathcal{O}$ be an open ball that contains $\overline{\Omega}$ and let $\tilde{\mathbf{u}}$ be the extension by zero of $\mathbf{u}$ to $\mathcal{O}\backslash\Omega$: $\tilde{\mathbf{u}}$ belongs to $H^t(\mathcal{O})^3$, as $t < 1/2$. Since, by Stokes' theorem, the normal component of $\mathbf{u}$ on Γ is zero, therefore $\tilde{\mathbf{u}}$ belongs to $H(\mathrm{div}, \mathcal{O})$, and $\mathrm{div}\,\tilde{\mathbf{u}} = 0$ in $\mathcal{O}$.

The vector $\tilde{\mathbf{u}}$ satisfies the hypothesis of Lemma 2.2. There exists a vector $\mathbf{v} \in H^{1+t}(\mathcal{O})^3$, satisfying

$$\begin{aligned} \tilde{\mathbf{u}} &= \mathbf{curl}\,\mathbf{v}, \\ \mathrm{div}\,\mathbf{v} &= 0. \end{aligned}$$

Consider now the vector $\mathbf{v}$ in $\mathcal{O}\backslash\Omega$: since $\Omega \subset \mathbb{R}^3$ is simply connected, $\mathcal{O}\backslash\Omega$ is also simply connected. From

$$\mathbf{curl}\,\mathbf{v} = \tilde{\mathbf{u}} = 0 \text{ in } \mathcal{O}\backslash\Omega;$$

we deduce that there exists a function $q \in H^{2+t}(\mathcal{O}\backslash\Omega)$, such that $\mathbf{v} = \mathbf{grad}\,q$ in $\mathcal{O}\backslash\Omega$.

Define now χ by

$$\Delta\chi = 0 \text{ in } \Omega, \tag{21}$$

$$\chi|_\Gamma = q|_\Gamma. \tag{22}$$

Then the vector $\mathbf{grad}\,\chi$ has zero divergence and curl in Ω, and satisfies the boundary conditions

$$\mathbf{grad}\,\chi \times \mathbf{n} = \mathbf{grad}\,q \times \mathbf{n} = \mathbf{v} \times \mathbf{n}, \text{ on } \Gamma;$$

It is easy to see that the vector $(\mathbf{w} - \mathbf{v} + \mathbf{grad}\,\chi)$ has zero divergence and curl in Ω and has zero tangential trace on Γ. By [6, Remark 3.9] it then follows that

$$\mathbf{w} = \mathbf{v} - \mathbf{grad}\,\chi.$$

Consider now the Laplace problem given by (21), (22). For each face F of Ω, $q|_F \in H^{\frac{3}{2}+t}(F)$ and the traces of q match along the edges of Ω; for the definition of trace spaces in polyhedral domains, see [7]. By Lemma 2.1, the solution of (21), (22) belongs to $H^{2+t}(\Omega)$ and, finally, $\mathbf{w}$ belongs to $H^{1+t}(\Omega)$. $\square$

REMARK 2.2. *For $t = 0$, the result of Theorem 2.3 is well-known; see [6, Theorem 3.7]. The constraint $t < s_\Omega < 1/2$ is necessary for the extension by zero of* $\mathbf{curl}\,\mathbf{w}$ *to be in $H^t(\Omega)$. Theorem 2.3 is part of a more general embedding result that is stated in [1, Remark 3.8], where some embedding results are linked to the regularity of the Laplace problem. Observe that the H^{2+t} regularity of problem (21), (22), for $t \geq 0$, is employed. The conclusion of Theorem 2.3 is false for a general non-convex polyhedron.*

3. Finite element spaces and projections.

Let $\mathcal{T}_H$ be a triangulation of the bounded, open, convex polyhedron Ω, consisting of tetrahedra $\{\Omega_i\}_{i=1}^J$. H is the maximum diameter of the triangulation. Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$, with characteristic diameter $h < H$. We suppose that $\mathcal{T}_H$ and $\mathcal{T}_h$ are *shape-regular* and *quasi-uniform*. The second property is required for the proof of Lemma 3.3.

We will consider the Nédélec spaces of the first kind, built on tetrahedra, which were introduced in [11]; see also [6] and [8]. Other choices of finite element spaces are also possible, see [12], as well as triangulations made of hexahedra and prisms, see [11], [12]. Given a tetrahedron K and an integer $k \geq 1$, we define the following spaces:

$$\mathcal{R}_k(K) = \left\{ \mathbf{u} + \mathbf{v};\ \mathbf{u} \in \mathbb{P}_{k-1}(K)^3,\ \mathbf{v} \in \tilde{\mathbb{P}}_k(K)^3,\ \mathbf{v} \cdot \mathbf{x} = 0 \right\}, \tag{23}$$

where $\tilde{\mathbb{P}}_k(K)$ is the space of homogeneous polynomials of degree k. Functions in $\mathcal{R}_k(K)$ are uniquely defined by the following sets of degrees of freedom, see [6],

$$m_1(\mathbf{u}) = \left\{ \int_a \mathbf{u} \cdot \mathbf{t}_a p\,, \text{ for all } p \in \mathbb{P}_{k-1}(a), \text{ for the six edges } a \text{ of } K \right\}, \tag{24}$$

and, for $k > 1$,

$$\begin{aligned} m_2(\mathbf{u}) \quad = \quad & \left\{ \int_f (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{p}\,, \right. \\ & \left. \text{for all } \mathbf{p} \in \mathbb{P}_{k-2}(F)^2, \text{ for the four faces } F \text{ of } K \right\}, \end{aligned} \tag{25}$$

and, additionally, for $k > 2$,

$$m_3(\mathbf{u}) = \left\{ \int_K \mathbf{u} \cdot \mathbf{p}\,, \text{ for all } \mathbf{p} \in \mathbb{P}_{k-3}(K)^3 \right\}. \tag{26}$$

Here $\mathbf{t}_a$ denotes the unit vector in the direction of the edge a. Let us remark that they involve integrals of the tangential components over the edges and the faces of each tetrahedron. It can be proven that this finite element is conforming in $H(\text{curl}, \Omega)$; see [6]. Thus the following finite element spaces are well-defined:

$$V^k = V = \{\mathbf{u} \in H_0(\text{curl}, \Omega);\ \mathbf{u}|_K \in \mathcal{R}_k(K),\ \forall K \in \mathcal{T}_h\}, \tag{27}$$

$$V_0^k = V_0 = \{\mathbf{u} \in H_0(\text{curl}, \Omega);\ \mathbf{u}|_K \in \mathcal{R}_k(K),\ \forall K \in \mathcal{T}_H\}. \tag{28}$$

A nodal interpolation operator $\Pi_h^k = \Pi$ can now be defined. The functionals in $m_1(\mathbf{u})$ are not defined for all vectors in $H^1(\Omega)$, but it follows from Sobolev's inequality that they are well defined for $\mathbf{u} \in H^s(\Omega)$, for $s > 1$. We will employ the error estimates proved in [8], that we summarize in the following lemma.

Lemma 3.1. *Let $\mathcal{T}_h$ be a shape-regular triangulation. The following estimate holds, for $k \geq 2$,*

$$\left\| \mathbf{u} - \Pi_h^k \mathbf{u} \right\|_{0,\Omega} \leq C\, h^s \left\| \mathbf{u} \right\|_{s,\Omega},\ \forall \mathbf{u} \in H^s(\Omega),\ 1 < s \leq 2, \tag{29}$$

with a constant C independent of $\mathbf{u}$ and h.

We will also need the usual spaces of continuous, piecewise polynomial functions, contained in $H^1(\Omega)$, together with their gradients:

$$S^k = S = \left\{ q \in H_0^1(\Omega);\ q|_K \in \mathbb{P}_k(K),\ \forall K \in \mathcal{T}_h \right\}, \tag{30}$$

$$S_0^k = S_0 = \left\{ q \in H_0^1(\Omega);\ q|_K \in \mathbb{P}_k(K),\ \forall K \in \mathcal{T}_H \right\}, \tag{31}$$

$$Z^k = Z = \mathbf{grad}\, S, \tag{32}$$

$$Z_0^k = Z_0 = \mathbf{grad}\, S_0. \tag{33}$$

The following lemma provides an orthogonal decomposition of V and a characterization of the kernel of the curl operator; for a proof see [6, Lemma 5.10 and Proposition 5.1].

Lemma 3.2. *Suppose that $\mathcal{T}_h$ is shape-regular. Then;*

a) if $\mathbf{u} = \mathbf{grad}\, q$*, with* $q \in H_0^1(\Omega)$*, and if* $\Pi_h^k \mathbf{u}$ *is well defined, then there exists a unique* $p \in S^k$*, i.e. a unique* $\mathbf{v} \in Z^k$*, such that* $\Pi_h^k \mathbf{u} = \mathbf{v} = \mathbf{grad}\, p$*;*

b) let $Z^\perp$ *be the orthogonal complement of* Z *in* V*:*

$$Z^{\perp,k} = Z^\perp = \left\{ \mathbf{u} \in V \,|\, (\mathbf{u}, \mathbf{grad}\, q) = 0, \ \forall q \in S^k \right\}. \tag{34}$$

Then, if the mesh $\mathcal{T}_h$ *is quasi-uniform, the following inequality holds*

$$\|\mathbf{u}\| \le C \|\mathbf{curl}\, \mathbf{u}\|, \ \forall \mathbf{u} \in Z^\perp, \tag{35}$$

with a constant C *independent of* $\mathbf{u}$ *and* h*.*

As a consequence of Lemma 3.2, the L^2-norm of the curl is an equivalent norm in $Z^\perp$. We will also need the orthogonal complement of Z_0 in V_0:

$$Z_0^{\perp,k} = Z_0^\perp = \{ \mathbf{u} \in V_0 |\, (\mathbf{u}, \mathbf{grad}\, q) = 0, \ \forall q \in S_0 \}. \tag{36}$$

The decomposition given by Lemma 3.2 and inequality (35) are the discrete analogs of (9), (12) and (13). But, while the following inclusions hold

$$V_0 \subset V \subset H_0(\mathrm{curl}, \Omega),$$
$$Z_0 \subset Z \subset \mathbf{grad}\, H_0^1(\Omega),$$

the space $Z_0^\perp$ is not contained in $Z^\perp$, and neither of them is contained in $H_0^\perp(\mathrm{curl}, \Omega)$. This fact, together with the regularity required by the interpolation operator, makes the analysis of multilevel methods for $H(\mathrm{curl}, \Omega)$-conforming elements particularly cumbersome. In order to obtain suitable projections onto $Z_0^\perp$ and $Z^\perp$, Hiptmair [8] has introduced auxiliary subspaces, defined in the following way:

Let

$$\Theta : \ H_0(\mathrm{curl}, \Omega) \longrightarrow H_0^\perp(\mathrm{curl}, \Omega),$$

be the orthogonal projection onto $H_0^\perp(\mathrm{curl}, \Omega)$. In particular, $\Theta \mathbf{u}$ is defined by

$$\Theta \mathbf{u} = \mathbf{u} - \mathbf{grad}\, q, \ \forall \mathbf{u} \in H_0(\mathrm{curl}, \Omega), \tag{37}$$

where $q \in H_0^1(\Omega)$ satisfies

$$(\mathbf{grad}\, q, \mathbf{grad}\, p) = (\mathbf{u}, \mathbf{grad}\, p), \ \forall p \in H_0^1(\Omega). \tag{38}$$

It is readily seen that Θ preserves the curl and does not increase the L^2-norm. Define now ϑ_0 and ϑ as the restrictions of Θ to $Z_0^\perp$ and $Z^\perp$, respectively, and the following spaces:

$$Z_0^+ = \vartheta_0(Z_0^\perp) = \Theta(Z_0^\perp), \tag{39}$$
$$Z^+ = \vartheta(Z^\perp) = \Theta(Z^\perp). \tag{40}$$

We note that we use different notations than those in [8]. The spaces Z_0^+ and Z^+ are finite dimensional. They are not finite element spaces, but the curls of these functions are finite element functions. It can be proven that Z_0^+ is contained in Z^+, and that they are both contained in $H_0^\perp(\mathrm{curl}, \Omega)$; moreover the operators

$$\vartheta_0 : \ Z_0^\perp \longrightarrow Z_0^+, \tag{41}$$
$$\vartheta : \ Z^\perp \longrightarrow Z^+, \tag{42}$$

are isomorphisms. Their inverses satisfy the following L^2-bounds.

LEMMA 3.3. *Let $k \geq 2$ and suppose that the triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$ are shape regular and quasi-uniform. Then, there exists a constant C, depending only on k and Ω, such that*

$$\|\mathbf{v}\| \leq C\,\left(\|\vartheta_0(\mathbf{v})\| + H\,\|\mathbf{curl}\,\mathbf{v}\|\right), \ \forall \mathbf{v} \in Z_0^{\perp,k}, \tag{43}$$

$$\|\mathbf{v}\| \leq C\,\left(\|\vartheta(\mathbf{v})\| + h\,\|\mathbf{curl}\,\mathbf{v}\|\right), \ \forall \mathbf{v} \in Z^{\perp,k}. \tag{44}$$

REMARK 3.1. *Lemma 3.3 is the main result of this section. Its proof can be found in [8, Lemma 5.15]. Lemma 2.3, the validity of which has been proven for a general convex polyhedron, is applied to functions in Z_0^+ and Z^+, and the error estimate in Lemma 3.1 is employed. A quasi-uniform mesh is required, since an inverse estimate for the H^s-norm of $\mathbf{curl}\,\mathbf{v}$ is used.*

We end this section, by introducing a projection onto the coarse space Z_0^+. We recall that, it follows from (13), that the L^2-norm of the curl is an equivalent norm in Z_0^+. Define P_0 by

$$P_0 :\ H_0^{\perp}(\mathrm{curl},\Omega) \longrightarrow Z_0^+, \tag{45}$$

$$\left(\mathbf{curl}\,(P_0\mathbf{v}), \mathbf{curl}\,\mathbf{w}\right) = \left(\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{w}\right), \ \forall \mathbf{w} \in Z_0^+. \tag{46}$$

The operator P_0 is well defined, by the Lax-Milgram lemma, and it does not increase the L^2-norm of the curl.

Given a function $\mathbf{v} \in Z^+$, some important properties of the splitting

$$\mathbf{v} = P_0\mathbf{v} + (I - P_0)\mathbf{v}, \tag{47}$$

are given in the following lemma.

LEMMA 3.4. *Let Ω be a convex polyhedron and let $\mathbf{v} \in Z^+$. Then,*

$$\|P_0\mathbf{v}\| \leq C\,H\,\|\mathbf{curl}\,(P_0\mathbf{v})\|, \tag{48}$$

$$\|(I - P_0)\mathbf{v}\| \leq C\,h\,\|\mathbf{curl}\,((I - P_0)\mathbf{v})\|, \tag{49}$$

with C independent of h, H and $\mathbf{v}$.

Proof. The proof can be found in [8, Lemma 5.19]. We remark that it requires a regularity result that is only valid for a convex polyhedron. □

4. Overlapping methods. Given the two triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$ of Ω, defined in Section 3, let us consider a covering of Ω, say $\{\Omega_i'\}_{i=1}^J$, such that each subregion Ω_i' is the union of tetrahedra of $\mathcal{T}_h$ and contains Ω_i. We will assume that the following two properties hold.

ASSUMPTION 4.1.

a) There is a constant $\alpha > 0$, such that $\mathrm{dist}(\partial\Omega_i', \Omega_i) \geq \alpha H$;

b) for every point $P \in \Omega$, P belongs to at most β subregions in $\{\Omega_i'\}_{i=1}^J$.

Given the finite element spaces introduced in (27) and (28), we define for $i = 1, \cdots, J$, the subspaces $V_i \subset V$, by setting the degrees of freedom outside Ω_i' to zero. The space V admits the decomposition $V = \sum_{i=0}^J V_i$.

Let us now define the following operators for $i = 0, \cdots, J$:

$$T_i :\ V \longrightarrow V_i, \tag{50}$$

$$a\,(T_i\mathbf{u}, \mathbf{v}) = a\,(\mathbf{u}, \mathbf{v}), \ \forall \mathbf{v} \in V_i, \tag{51}$$

where $a(\cdot,\cdot)$ is defined in (3). The additive and multiplicative Schwarz operators have been introduced in Section 1, in (5) and (8). The three following fundamental lemmas state the convergence properties of the additive and multiplicative algorithms. We refer to [5], [19], and to the references contained therein for proofs and additional comments.

LEMMA 4.1. *Let T_i and T be defined by (50), (51), and (5). If a representation, $\mathbf{u} = \sum \mathbf{u}_i$, can be found, such that*

$$\sum a(\mathbf{u}_i, \mathbf{u}_i) \leq C_0^2\, a(\mathbf{u}, \mathbf{u}),\ \forall \mathbf{u} \in V, \tag{52}$$

then

$$a(T\mathbf{u}, \mathbf{u}) \geq C_0^{-2}\, a(\mathbf{u}, \mathbf{u}),\ \forall \mathbf{u} \in V. \tag{53}$$

LEMMA 4.2. *Let $\|\cdot\|_a$ be the norm induced by $a(\cdot,\cdot)$ and let $\mathcal{E} = \{\varepsilon_{ij}\}$ be the smallest constants for which*

$$|a(\mathbf{u}_i, \mathbf{u}_j)| \leq \varepsilon_{ij} \|\mathbf{u}_i\|_a\, \|\mathbf{u}_j\|_a,\ \forall \mathbf{u}_i \in V_i,\ \forall \mathbf{u}_j \in V_j,\ i, j \geq 1, \tag{54}$$

holds. Then,

$$a(T\mathbf{u}, \mathbf{u}) \leq C_1\, a(\mathbf{u}, \mathbf{u}),\ \forall \mathbf{u} \in V, \tag{55}$$

where $C_1 = (\varrho(\mathcal{E}) + 1)$, with $\varrho(\mathcal{E})$ the spectral radius of $\mathcal{E}$.

LEMMA 4.3. *Assume that Lemma 4.1 and 4.2 hold. Then,*

a) *the condition number $\kappa(T)$ of the operator T of the additive Schwarz method satisfies*

$$\kappa(T) \leq (\varrho(\mathcal{E}) + 1) C_0^2; \tag{56}$$

b) *for the multiplicative Schwarz method, the error operator, E, satisfies*

$$\|E\|_a \leq \sqrt{1 - \frac{1}{(2\varrho(\mathcal{E})^2 + 1) C_0^2}}. \tag{57}$$

The bound (57) can be improved by suitably rescaling the local problems.

We are now ready to prove our main results.

LEMMA 4.4. *Inequality (55) holds with $C_1 = (\beta + 1)$, where β is defined in Assumption 4.1.*

Proof. The proof can be carried out in the same way as in [2, Theorem 4.1]. The bound (55) is proved directly, without employing Lemma 4.2. ☐

We recall that a partition of unity $\{\chi_i\}_{i=1}^J$, relative to the covering $\{\Omega_i'\}_{i=1}^J$, is a set of functions, satisfying the following properties,

$$\chi_i \in C^\infty(\Omega),\ \text{supp}\ (\chi_i) \subset \Omega_i',$$
$$0 \leq \chi_i \leq 1,\ \sum_i \chi_i = 1.$$

Before proving inequality (53), we need the following lemma.

LEMMA 4.5. *Let* $\mathbf{v} \in V$ *and* $\{\chi_i\}_{i=1}^J$ *be a partition of unity relative to the covering* $\{\Omega_i'\}_{i=1}^J$. *If* Π *is the nodal interpolation operator onto* V, *then the following inequalities hold,*

$$\|\Pi(\chi_i \mathbf{v})\| \le C\ \|\chi_i \mathbf{v}\|,\ \forall i = 1, \cdots, J, \tag{58}$$

$$\|\mathbf{curl}\ (\Pi(\chi_i \mathbf{v}))\| \le C\ \|\mathbf{curl}\,(\chi_i \mathbf{v})\|,\ \forall i = 1, \cdots, J, \tag{59}$$

with constants C *independent of* $\mathbf{v}$, i, h *and* H.

Proof. We recall that the degrees of freedom (24), (25), (26), involve integrals of the tangential components over the edges and the faces, as well as values of the function in the interior of each tetrahedron in $\mathcal{T}_h$.

Let us first consider (58). The vector $\mathbf{v}$ has continuous tangential component across the edges and faces of the tetrahedra; since the scalar function χ_i belongs to $C(\Omega)$, the vector $\chi_i \mathbf{v}$ has continuous tangential component across each element and, thus, the degrees of freedom are well defined.

The interpolation operator Π is local. On each tetrahedra $K \in \mathcal{T}_h$, the degrees of freedom are calculated and the interpolated function is built from the appropriate basis functions. Therefore, we need only consider one tetrahedra. We also note that the vector $\chi_i \mathbf{v}$ is C^∞ over each element. Since on the reference tetrahedra the interpolation operator is bounded in the L^2-norm, it is easily seen, by a scaling argument, that, on the generic tetrahedra K, it is bounded independently of the diameter of K. Inequality (58) is, then, obtained by summing over all the elements of $\mathcal{T}_h$.

Let us next consider inequality (59). The space $\mathbf{curl}\, V$ can be fully characterized. It is contained in $H_0(\mathrm{div}, \Omega)$ and is a proper subspace of the Raviart-Thomas finite element space of degree k, $W = W^k$; see [11], [6], [8]. The interpolation operator, Π_{RT}, onto W involves integrals of the normal component on the faces of the tetrahedra, as well as the value of the function in the interior.

The vector $\mathbf{curl}\,(\chi_i \mathbf{v})$ is a C^∞ function over each element and has continuous normal component: the interpolant $\Pi_{RT}(\mathbf{curl}\,(\chi_i \mathbf{v}))$ is, therefore, well defined. First on each tetrahedron, and then in Ω, by the commuting diagram property, see [8, Theorem 2.30], we obtain

$$\mathbf{curl}\,(\Pi(\chi_i \mathbf{v})) = \Pi_{RT}(\mathbf{curl}\,(\chi_i \mathbf{v})). \tag{60}$$

The inequality

$$\|\Pi_{RT}(\mathbf{curl}\,(\chi_i \mathbf{v}))\| \le C\ \|\mathbf{curl}\,(\chi_i \mathbf{v})\|, \tag{61}$$

can be then obtained in the same way as (58), and this proves (59). □

THEOREM 4.6. *Let* $k \ge 2$. *Then, for every* $\mathbf{u} \in V^k$, *inequality (53) holds.* C_0 *depends on* k, *the domain* Ω, *the overlap constants,* α *and* β, *and the shape-regularity and quasi-uniformity constants for* $\mathcal{T}_H$ *and* $\mathcal{T}_h$, *but is independent of* h, H *and* $\mathbf{u}$.

Proof. Let $\mathbf{u} \in V$. By Lemma 4.1, we have to find a suitable decomposition, such that inequality (52) holds. By Lemma 3.2, $\mathbf{u}$ can be decomposed as

$$\mathbf{u} = \mathbf{grad}\, q + \mathbf{w}, \tag{62}$$

where $\mathbf{grad}\, q \in Z$ and $\mathbf{w} \in Z^\perp$. We will decompose $\mathbf{grad}\, q$ and $\mathbf{w}$ separately.

Let us first consider the gradient term. Using the domain decomposition theory for scalar elliptic operators (see [5] and [19]), for $q \in S \subset H_0^1(\Omega)$, we can obtain a decomposition $q = \sum q_i$, for $q \in S \subset H_0^1(\Omega)$, and the following bound:

$$\sum a(\mathbf{grad}\, q_i, \mathbf{grad}\, q_i) = \sum |q_i|^2_{H^1(\Omega)} \le C\ \ |q|^2_{H^1(\Omega)} = C\ a(\mathbf{grad}\, q, \mathbf{grad}\, q), \tag{63}$$

where C depends on the overlap, linearly; see [5].

Consider now $\mathbf{w} \in Z^{\perp}$. We will first employ the decomposition described in [8], by projecting onto Z^{+}, then onto the coarse space Z_0^{+} and finally go back to $Z^{\perp}$. We will, then, divide the reminder into a sum of functions supported on the individual subdomains $\{\Omega_i'\}$.

The first step is performed in the following way. Define

$$\mathbf{w}^{+} = \Theta(\mathbf{w}) \in Z^{+},$$

and consider the splitting

$$\mathbf{w}^{+} = \mathbf{v}_0^{+} + \mathbf{v}^{+},$$

where

$$\begin{aligned} \mathbf{v}_0^{+} &= P_0 \mathbf{w}^{+} \in Z_0^{+}, \\ \mathbf{v}^{+} &= (I - P_0)\mathbf{w}^{+} \in Z^{+}. \end{aligned}$$

The operator P_0 is defined in (45) and (46).

Since ϑ and ϑ_0 are invertible on $Z^{\perp}$ and $Z_0^{\perp}$, respectively, the following vectors are well defined

$$\begin{aligned} \mathbf{v}_0 &= \vartheta_0^{-1}(\mathbf{v}_0^{+}) \in Z_0^{\perp}, \\ \mathbf{v} &= \vartheta^{-1}(\mathbf{v}^{+}) \in Z^{\perp}. \end{aligned}$$

The sum $\mathbf{v}_0 + \mathbf{v} = \mathbf{w}'$ is not equal to the original vector $\mathbf{w}$, but it can easily be seen that the difference $(\mathbf{w} - \mathbf{w}')$ is curl-free and, thus, by Lemma 3.2, is the gradient of a function $p \in S$. Consequently, we have found the decomposition

$$\mathbf{w} = \mathbf{v}_0 + \mathbf{v} + \mathbf{grad}\, p. \tag{64}$$

Before proceeding, we have to find some bounds for the terms in (64) and their curls. Since the operators Θ, ϑ and ϑ_0 preserve the curl and P_0 does not increase the L^2-norm of the curl, it can be easily seen that

$$\|\mathbf{curl}\,\mathbf{v}_0\| \leq \|\mathbf{curl}\,\mathbf{w}\|, \tag{65}$$

$$\|\mathbf{curl}\,\mathbf{v}\| \leq \|\mathbf{curl}\,\mathbf{w}\|. \tag{66}$$

We employ Lemma 3.3 to bound the L^2-norm of $\mathbf{v}_0$ and $\mathbf{v}$. We remark that Lemma 3.3 is only valid for $k \geq 2$. Consider, first, $\mathbf{v}_0$. By (43), we can write

$$\|\mathbf{v}_0\| \leq C\ \left(\|\mathbf{v}_0^{+}\| + H\ \|\mathbf{curl}\,\mathbf{v}_0\|\right) = C\ \left(\|\mathbf{v}_0^{+}\| + H\ \|\mathbf{curl}\,\mathbf{v}_0^{+}\|\right),$$

and, by Lemma 3.4,

$$\|\mathbf{v}_0\| \leq C\ \left(H\ \|\mathbf{curl}\,\mathbf{v}_0^{+}\|\right).$$

Finally, we obtain

$$\|\mathbf{v}_0\| \leq C\ H\ \|\mathbf{curl}\,\mathbf{w}\|. \tag{67}$$

Through Lemma 3.3 and 3.4, we find, in the same way,

$$\|\mathbf{v}\| \leq C\ h\ \|\mathbf{curl}\,\mathbf{w}\|. \tag{68}$$

Since the L^2-norm of $\mathbf{v}_0$ and $\mathbf{v}$ is bounded, we can bound the L^2-norm of $\mathbf{grad}\, p$ in (64) in terms of the norm of $\mathbf{w}$ in $H(\mathrm{curl}, \Omega)$. The term $\mathbf{grad}\, p$ can therefore be decomposed in the same way as the gradient part of $\mathbf{u}$ in (62).

We now decompose the vector $\mathbf{v}$ as a sum of terms in $\{V_i\}_{i=1}^J$. Let $\{\chi_i\}_{i=1}^J$ be a partition of unity, relative to the covering $\{\Omega_i'\}_{i=1}^J$. We define

$$\mathbf{w}_i = \Pi(\chi_i \mathbf{v}) \in V_i, \ i = 1, \cdots, J, \tag{69}$$

where $\Pi = \Pi_h^k$ is the interpolation operator introduced in Section 3. The function $\mathbf{w}' = \mathbf{v}_0 + \mathbf{v}$ is thus decomposed as $\mathbf{w}' = \sum_{i=0}^J \mathbf{w}_i$, with

$$\mathbf{w}_0 = \mathbf{v}_0. \tag{70}$$

We have to check that the sum of the squares of the a-norm of the $\mathbf{w}_i$ is bounded by the square of the a-norm of $\mathbf{w}$. The bounds for $\mathbf{w}_0$ are given by (65) and (67).

By inequality (58) of Lemma 4.5, we can write

$$\|\mathbf{w}_i\|_{L^2(\Omega)} \le C\, \|\chi_i \mathbf{v}\|_{L^2(\Omega)} \le C\, \|\mathbf{v}\|_{L^2(\Omega)},$$

and, by (68),

$$\|\mathbf{w}_i\|_{L^2(\Omega)} \le C\, h\, \|\mathbf{curl}\, \mathbf{w}\|_{L^2(\Omega)}. \tag{71}$$

Employing (59), we can also write

$$\begin{aligned}
\|\mathbf{curl}\, \mathbf{w}_i\|_{L^2(\Omega)} &\le C\, \|\mathbf{curl}\, (\chi_i \mathbf{v})\|_{L^2(\Omega)} \le \\
&\le C\, \left(\|\mathbf{grad}\, \chi_i \times \mathbf{v}\|_{L^2(\Omega)} + \|\chi_i\, \mathbf{curl}\, \mathbf{v}\|_{L^2(\Omega)}\right) \le \\
&\le C\, \left(\|\mathbf{grad}\, \chi_i\|_{L^\infty(\Omega)}\, \|\mathbf{v}\|_{L^2(\Omega_i')} + \|\chi_i\|_{L^\infty(\Omega)}\, \|\mathbf{curl}\, \mathbf{v}\|_{L^2(\Omega_i')}\right) \le \\
&\le C\, \left(H^{-1} \|\mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \mathbf{v}\|_{L^2(\Omega)},\right)
\end{aligned}$$

where, for the last inequality, we have used Assumption 4.1.a. Finally, by (66) and (68), we obtain

$$\begin{aligned}
\|\mathbf{curl}\, \mathbf{w}_i\|_{L^2(\Omega)} &\le C\, \left((h/H)\|\mathbf{curl}\, \mathbf{w}\| + \|\mathbf{curl}\, \mathbf{w}\|\right) \le \\
&\le C\, \|\mathbf{curl}\, \mathbf{w}\|.
\end{aligned} \tag{72}$$

By summing over i, employing Assumption 4.1.b and (65), (67), (71) and (72), we find

$$\sum_{i=0}^{J} a(\mathbf{w}_i, \mathbf{w}_i) \le C a(\mathbf{w}, \mathbf{w}). \tag{73}$$

Since (62) is an a-orthogonal decomposition, inequality (52) is valid, and Lemma 4.1 proves the theorem. □

THEOREM 4.7. *The conclusion of Theorem 4.6 is valid for $k = 1$.*

Proof. The proof is the same as the one for [10, Corollary 1]. The decomposition for V^2 and the hierarchical decomposition of the degrees of freedom for the Nédélec spaces are employed, in order to obtain a stable splitting for V^1. □

The following theorem gives the final result.

THEOREM 4.8. *There exist two constants C_2 and C_3, such that*

$$\kappa(T) \leq C_2, \tag{74}$$

$$\|E\|_a \leq C_3 < 1. \tag{75}$$

C_2 and C_3 depend on the domain Ω, on the overlap constants, α and β, and on the shape-regularity and quasi-uniformity constants for $\mathcal{T}_H$ and $\mathcal{T}_h$, but are independent of h and H.

Proof. The first inequality is a consequence of Lemma 4.4, and Theorems 4.6 and 4.7. The second bound can be easily found by using Lemma 4.3.b. □

For the Neumann problem, convexity does not have to be assumed.

THEOREM 4.9. *When the whole space $H(\text{curl}, \Omega)$ is considered, the conclusions of Lemma 4.4, and Theorems 4.6 and 4.7 are still valid, for a general polyhedral domain.*

Proof. For inequality (55), the proof is the same as in Lemma 4.4. For the lower bound for the minimum eigenvalue of the additive method, the proof can be carried out as in [10, Theorem 5]. The domain Ω is embedded in a larger convex domain, $\tilde{\Omega}$, and the decomposition for $H_0(\text{curl}, \tilde{\Omega})$, together with an extension theorem, is exploited. The result for the multiplicative method is straightforward. □

We conclude with some remarks on our assumptions. A *convex* polyhedral domain is considered for the Dirichlet problem: this is necessary for the Embedding Theorem 2.3 to hold. As pointed out in Remark 2.2, the theorem is not valid for a general non-convex domain, unless the boundary is sufficiently regular. This assumption is also required for the proof of Lemma 3.4.

Quasi-uniform triangulations are assumed, for the proof of the inequalities in Lemma 3.3. As is pointed out in Remark 3.1, the proof of Lemma 3.3 relies on an inverse estimate.

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