SOME NUMERICAL RESULTS USING AN ADDITIVE SCHWARZ METHOD FOR MAXWELL'S EQUATIONS

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Abstract. We present some numerical results for a two-level additive overlapping Schwarz method applied to the 2-D Maxwell's equations. Nédélec finite elements defined on rectangles are employed. Numerical examples show the convergence properties of the method, when varying the mesh size of the coarse and fine problems, the overlap and the time step of the implicit finite difference scheme employed.

1. Introduction. The numerical approximation of Maxwell's equations generally requires suitable finite element (FE) spaces that ensure the correct continuity properties for the physical fields involved across the elements of the triangulation of the domain considered, see [6].

When time-dependent partial differential equations are discretized with implicit finite difference (FD) schemes, a linear system is to be solved at each time step. The condition number of such a system grows when the triangulation is refined. Overlapping Schwarz methods generally ensure a condition number that is independent of the problem size and, when applied in their additive form, are easily parallelizable. We know of only one study closely related to ours; see [1]. In that paper, an overlapping Schwarz method is applied to 2D mixed elements for $H(\mathrm{div}, \Omega)$; in two dimensions their theoretical results are also valid for $H(\mathrm{curl}, \Omega)$.

In this paper, we will present some numerical results concerning low order Nédélec FE spaces built on rectangles, and discuss the convergence properties of two additive Schwarz methods, when varying the diameter of the triangulation, the time step, the overlap, and the dimension of the coarse problem.

In section 2, we will introduce the problem and the FE spaces employed, while in section 3 we will describe the additive Schwarz methods considered. Section 4 is devoted to the discussion of the numerical results.

2. Continuous and discrete problems. Let Ω be an open, bounded, connected set in $\mathbb{R}^2$ and let a bilinear form $a(\cdot,\cdot)$ be defined by

$$a(\mathbf{u},\mathbf{v}) = \lambda\,(\mathbf{u},\mathbf{v}) + (curl\,\mathbf{u}, curl\,\mathbf{v}), \tag{1}$$

for every $\mathbf{u}$, $\mathbf{v}$ belonging to $H(\mathrm{curl}, \Omega)$. Here $(\cdot,\cdot)$ denotes the L^2 inner product and λ is a strictly positive fixed parameter. For the properties of the Hilbert space $H(\mathrm{curl}, \Omega)$ and its trace space, see [3]. In particular, we recall that for every function in $H(\mathrm{curl}, \Omega)$ it is possible to define a tangential trace over $\partial\Omega$ as an element of $H^{-\frac{1}{2}}(\partial\Omega)$ and that the functions of $H(\mathrm{curl}, \Omega)$ with vanishing tangential trace form a proper subspace of $H(\mathrm{curl}, \Omega)$, denoted by $H_0(\mathrm{curl}, \Omega)$.

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Let $\mathbf{f}$ be a function of $L^2(\Omega)$: we are seeking the solution $\mathbf{u} \in H_0(\mathrm{curl}, \Omega)$ of the variational problem

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \, \mathbf{v} \in H_0(\mathrm{curl}, \Omega). \tag{2}$$

When $\mathbf{f}$ and the boundary of Ω are sufficiently regular, $\mathbf{u}$ is also solution of the differential equation

$$\lambda \mathbf{u} + \mathbf{curl}\, curl\, \mathbf{u} = \mathbf{f}, \text{ in } \Omega, \tag{3}$$

with the natural boundary condition

$$\mathbf{u} \times \mathbf{n} = 0, \text{ on } \partial\Omega, \tag{4}$$

where $\mathbf{n}$ is the normal to $\partial\Omega$.

Suppose now that the domain Ω is a rectangle with sides parallel to the axes and consider a quasiuniform triangulation $\mathcal{T}_h$, obtained by subdividing Ω into rectangles. The mesh size h is defined as the maximum diameter of the rectangles in $\mathcal{T}_h$. Let $V_h \subset H_0(\mathrm{curl}, \Omega)$ be the first Nédélec space of order $k = 1$ based on rectangles, as described in [5]. The degrees of freedom of the functions in V_h are defined on the sides of $\mathcal{T}_h$. We are led to the linear system

$$Ax = (\lambda A_1 + A_2)\, x = b, \tag{5}$$

where x is a vector consisting of the N_h degrees of freedom of the approximation of the solution $\mathbf{u}$. The integer N_h is the number of sides of $\mathcal{T}_h$ not belonging to $\partial\Omega$. The matrices A_1 and A_2 are built using the basis functions of V_h and their curls.

If $\{\boldsymbol{\psi}_i\}_{i=1}^{N_h}$ are the basis functions of V_h, then the matrix A_1 is defined by

$$A_1 = \left[a_{ij}^{(1)}\right] = \left[(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i)\right], \tag{6}$$

and the matrix A_2 by

$$A_2 = \left[a_{ij}^{(2)}\right] = \left[(\mathrm{curl}\boldsymbol{\psi}_j, \mathrm{curl}\boldsymbol{\psi}_i)\right]. \tag{7}$$

In particular A_1 is a symmetric, positive definite matrix, while A_2 is symmetric and positive semi-definite.

Equation (3) is obtained from

$$\mathbf{curl}\, curl\, \mathbf{u} + \mu\varepsilon \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\mu \frac{\partial \mathbf{J}}{\partial t}, \tag{8}$$

once an implicit FD time scheme is employed: the parameter λ is thus positive and proportional to $(\Delta t)^{-2}$, where Δt is the time step.

Equation (8) is derived from Maxwell's equations and gives the 2-D electric field $\mathbf{u}$, once the 2-D density of current $\mathbf{J}(\mathbf{x}, t)$ is known. For an explanation of the physical quantities involved and for further details on the FE approximation of time-dependent Maxwell's equations, see [6]; for error estimates for the semi-discrete problem, see [4], and for a discussion of FE approximations of hyperbolic equations, see [7].

3. An Additive Schwarz Method. Once λ is fixed, the condition number of A in equation (5) increases as h decreases, while for fixed h, it increases as λ decreases, or, equivalently, when the time step Δt increases. We have studied an additive Schwarz-type algorithm for the solution of (5).

Suppose now that the triangulation $\mathcal{T}_h$ is obtained by refining a coarser regular triangulation $\mathcal{T}_H = \{\Omega_i\}_{i=1}^{J_H}$, with diameter $H > h$. Consider then a covering $\{\Omega_i'\}$, such that each Ω_i' is the union of rectangles of $\mathcal{T}_h$ and contains Ω_i. Suppose also that there is a constant β such that every pont in Ω is contained in at most β sets in $\{\Omega_i'\}$. Let the overlapping parameter δ be

$$\delta = \min_i \{\mathrm{dist}(\partial\Omega_i', \Omega_i)\}. \tag{9}$$

Let V_H be the the first Nédélec space of order $k = 1$ built on the coarse triangulation $\mathcal{T}_H$ and let V_i be

$$V_i = \left\{\mathbf{u} \in V_h \,|\, \mathbf{u}(\mathbf{x}) = 0,\, \forall \mathbf{x} \in \overline{\Omega} \setminus \Omega_i'\right\},\ \forall i = 1, \cdots, J_H. \tag{10}$$

The subspaces V_H and $\{V_i\}$ are contained in V_h. Let R_i $(i = 1, \cdots, J_H)$ be the restriction matrix that returns the vector of coefficients defined in Ω_i' for every function $\mathbf{u} \in V_h$. The preconditioner B_F for the 1-level algorithm is defined as

$$B_F = \sum_{i=1}^{J_H} R_i^T (R_i A R_i^T)^{-1} R_i, \tag{11}$$

and the *1-Level Additive Algorithm* (1LAA) consists of solving the system

$$B_F A x = B_F b, \tag{12}$$

with the conjugate gradient method, without any farther preconditioning, employing the inner product $(\cdot,\cdot)_A$ defined by

$$(x, y)_A = x^T A y,\ \forall x, y \in \mathbb{R}^{N_h}. \tag{13}$$

For the 2-level algorithm define R_0^T as the matrix representation of the linear interpolation from the coarse space V_H into the fine space V_h, and define the preconditioner B as

$$B = R_0^T (R_0 A R_0^T)^{-1} R_0 + \eta B_F = B_C + \eta B_F, \tag{14}$$

where η is a suitable scaling parameter greater than zero. The *2-Level Additive Algorithm* (2LAA) consists of solving the system

$$BAx = Bb, \tag{15}$$

with the conjugate gradient method, without preconditioning, employing the inner product $(\cdot,\cdot)_A$. For further details on Schwarz Methods, see [2] and [8].

Our work has been inspired by [1], where two- and multi- level Schwarz algorithms are studied for problems defined by the bilinear form $\overline{a}(\mathbf{u}, \mathbf{v}) = \lambda\,(\mathbf{u}, \mathbf{v}) + (div\mathbf{u}, div\mathbf{v})$, with $\lambda \geq 1$, for functions $\mathbf{u}, \mathbf{v}$ in $H(\mathrm{div}, \Omega)$, $\Omega \subset \mathbb{R}^2$. Raviart-Thomas FE spaces, defined on triangles, are employed in that study. In [1], in particular, it is proven that the spectral condition number of problem (15) is bounded independently of h, H, $\lambda \geq 1$, and δ, if δ/H is bounded away from zero. Note that, since *in two dimensions* functions in $H(\mathrm{curl}, \Omega)$ are obtained from functions in $H(\mathrm{div}, \Omega)$ simply by a rotation of 90 degrees and the basis functions of the first Nédélec space (defined on triangles and rectangles) are obtained from those of the Raviart-Thomas space by the same rotation, the matrices and the preconditioners for the bilinear form $a(\cdot, \cdot)$ are the same as those used for $\overline{a}(\cdot, \cdot)$. See [3] for further details, and [1] for some applications of the space $H(\mathrm{div}, \Omega)$.

4. Numerical Results. In our numerical tests, we have considered rectangular domains and uniform triangulations. In particular, meshes are obtained by intersecting straight lines parallel to the axes. For the 2LAA, all the results presented have been obtained without rescaling ($\eta = 1$ in (14)). On each subdomain problem and for the coarse problem an exact band Choleski solver has been employed.

A property of our method is that, once the domain Ω, the diameter of the coarse triangulation H, and the overlap δ are specified, the condition number of the preconditioned problem is independent of the diameter of the fine mesh h, both for the 1LAA and the 2LAA. Detailed results are not shown here, but, as an example, compare the fourth column of Table 1 ($H = 1/2$) with the third column of Table 4 ($\lambda = 1$), for the 1LAA, and the fourth column of Table 2 ($H = 1/2$) with the third column of Table 5 ($\lambda = 1$), for the 2LAA. Tables 1 and 2 refer to a (64×64) triangulation of a square , while Tables 4 and 5 to a (32×32) mesh for the same domain: a detailed description of these tables is given below.

	$H = 2$	$H = 1$	$H = 1/2$	$H = 1/4$
$H/\delta = 4$	5.67	8.49	12.5	20.1
$H/\delta = 2$	4.15	5.41	10.3	21.4
$H/\delta = 1$	1.0	2.87	5.50	11.1
$H/\delta = 1/2$	-	1.81	3.18	4.21
$H/\delta = 1/3$	-	1.0	3.08	4.34

Table 1. The condition number of the preconditioned system as a function of H/δ and H: 1LAA, a square with side 4, mesh (64,64), $\lambda = 1$, problem size $n = 8064$.

	$H = 2$	$H = 1$	$H = 1/2$	$H = 1/4$
$H/\delta = 4$	4.47	4.82	4.94	4.84
$H/\delta = 2$	4.39	4.71	4.85	4.82
$H/\delta = 1$	1.0	2.36	3.81	6.48
$H/\delta = 1/2$	-	1.89	2.84	5.47
$H/\delta = 1/3$	-	1.0	2.94	4.00

Table 2. The condition number of the preconditioned system as a function of H/δ and H: 2LAA, a square with side 4, mesh (64,64), $\lambda = 1$, problem size $n = 8064$.

The first example considered (Example 1) is a square with side 4, with a uniform fine mesh of (64×64) elements. For $\lambda = 1$, Tables 1 and 2 show the condition number of the preconditioned system for different values of the overlap parameter H/δ and the coarse diameter H, for the 1LAA and 2LAA, respectively. The values shown should be compared with the condition number of the unpreconditioned system; $\kappa(A)$ is equal to 2314. When the condition number is equal to 1.0, an expanded subregion actually covers the whole domain: in this case the conjugate gradient method converges in one iteration, but this requires the factorization of the original matrix A, and is thus equivalent to a direct solver.

Table 1 shows that, as expected, for a given accuracy, the condition number grows rapidly with $1/H$ for the 1LAA, while it grows slowly for the 2LAA. In this case, and this seems to be a general feature, the 2LAA presents a considerable improvement over the 1LAA, when the overlap is small ($H/\delta \geq 1$) and when the number of subregions is large ($H \leq 1/4$).

We also remark that in some cases, for the same values of δ and H, the 2LAA gives a higher condition number. This can be avoided by scaling the fine problems, making $\eta \neq 1$, but it is not clear exactly how to select η. In Example 1, for instance, a value $\eta < 1$ slightly reduces the number of iterations, while in Example 2, see below, $\eta > 1$ is required.

In addition, for different values of the diameter of the fine mesh, the overlap, and the number of subregions, our results, in term of the number of iterations required to decrease the error by a fixed factor, are very close to the ones presented in [8] for the Laplace operator and structured grids, both for the 1LAA and the 2LAA; these results are not shown here.

	$m_y = 8$	$m_y = 16$	$m_y = 32$	$m_y = 64$
$H/\delta = 8$	11	12	15	20
$H/\delta = 4$	11	12	14	14
$H/\delta = 2$	11	11	13	16
$H/\delta = 1$	11	11	12	14
$H/\delta = 1/2$	8	10	11	11
$H/\delta = 1/3$	10	10	9	10
$H/\delta = 1/5$	10	8	9	9

Table 3. The number of iterations to decrease the error by a factor 10^{-7}, as a function of H/δ and m_y (m_y is the number of subregions and is proportional to $1/H$): 2LAA with a (1×16) rectangle, mesh $(8, 128)$, $\lambda = 0.1$, problem size $n = 1912$.

The second test (Example 2) is a thin rectangle of dimension (1×16) subdivided into (8×128) elements. For the 1LAA, the condition number of the preconditioned problem increases as H decreases (as in Example 1). Table 3 shows the number of iterations required in order to decrease the error of the residual norm by a factor 10^{-7}, for different values of the overlap and the coarse mesh diameter for the 2LAA.

In this example, once the overlap and the required precision are fixed, the number of iterations for the 2LAA is practically independent of H.

Tables 4 and 5 refer to Example 1: they show the condition number of the 1LAA and the 2LAA, respectively, for (8×8) subdomains, when varying λ and H/δ. The last row shows the condition number of the unpreconditioned system.

In this case, once the overlap and the number of subregions is fixed, the condition number of the preconditioned system is bounded independently of λ, both for the 1LAA and the 2LAA. The same property is observed when choosing different domains and varying the number of subdomains of the Schwarz method. This means that, when discretizing (8) with an implicit time scheme and with Nédélec elements, the condition number of the linear system that is to be solved at each step is practically independent of the time step Δt and the mesh size h of the fine problem.

	$\lambda = 10$	$\lambda = 1$	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$
$H/\delta = 4$	7.10	13.2	17.7	15.96	15.99
$H/\delta = 2$	4.82	10.3	14.9	14.9	15.1
$H/\delta = 1$	2.65	5.51	8.61	9.16	9.19
$H/\delta = 1/2$	2.76	3.18	4.52	4.95	5.01
$H/\delta = 1/3$	3.03	3.08	3.37	3.46	3.44
$\kappa(A)$	77.8	1003	5908	56062	446561

Table 4. The condition number of the preconditioned system, as a function of H/δ and λ; the last row shows the condition number of the unpreconditioned system for different λ: 1LAA, a square with side 4, mesh (32,32), $H = 1/2$ (64 subdomains).

	$\lambda = 10$	$\lambda = 1$	$\lambda = 10^{-1}$	$\lambda = 10^{-2}$	$\lambda = 10^{-3}$
$H/\delta = 4$	4.67	4.85	4.92	5.06	5.10
$H/\delta = 2$	4.67	4.82	4.89	4.97	4.99
$H/\delta = 1$	2.33	3.81	5.34	5.73	5.79
$H/\delta = 1/2$	2.79	2.83	3.89	4.19	4.23
$H/\delta = 1/3$	3.03	2.94	3.16	3.23	3.21
$\kappa(A)$	77.8	1003	5908	56062	446561

Table 5. The condition number of the preconditioned system, as a function of H/δ and λ; the last row shows the condition number of the unpreconditioned system for different λ: 2LAA, a square with side 4, mesh (32,32), $H = 1/4$ (64 subdomains).

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