Schwarz Preconditioners for Spectral and Mortar Finite Element Methods with Applications to Incompressible Fluids

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ABSTRACT

The spectral element method has been used extensively for the simulation of fluid flows. The resulting linear systems are often not amenable to direct methods of solution, and are especially ill-conditioned. Domain decomposition preconditioners, well adapted to the solution on parallel computers, are proposed and analyzed; both two and three space dimensions are considered.

Second-order elliptic equations are considered first, and the now well-developed theory of domain decomposition methods for finite elements is fully extended to spectral elements. This includes an analysis of exotic coarse spaces, which have proven necessary for the efficient solution of elliptic problems with large discontinuities in the coefficients, as well as a study of overlapping methods. Estimates of the condition numbers of the Schur complement restricted to an edge (in two dimensions) or to a face (in three dimensions) are also given; in particular, a fast method is designed and studied in full detail for problems with many subregions.

The Stokes problem, when restricted to the space of discrete divergence free velocities, is symmetric positive definite. A number of preconditioners are proposed, which are based on previous results for the scalar elliptic case, and new global models. The construction of a basis for the constrained velocity space is not required, and the resulting condition numbers grow only weakly with the degree N and are independent of the number of subdomains.

We also consider the stationary Navier-Stokes equations, solved with Newton's method. In each iteration, a non-symmetric indefinite problem is solved using a Schwarz preconditioner. A new coarse space is proposed which satisfies the usual properties required by the elliptic theory, and also a specific H^1-approximation property. The rate of convergence of the algorithm grows only weakly with N, and does not depend on the number of subdomains, or the Newton step.

Finally, a hierarchical basis preconditioner for the mortar finite element method in two dimensions is proposed and analyzed. The result shows that the flexibility allowed by the mortar method can be combined with a good preconditioner to produce an attractive and fast method. This is further demonstrated by extensive numerical experiments.

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Contents

Chapter 1

Introduction

1.1 A Brief Overview

In order to find approximate solutions to partial differential equations discretized by finite elements or finite differences, discrete linear systems have to be solved. Domain decomposition methods have been developed for this purpose; see, e.g., [62, 40, 41, 63, 67, 98, 68]. These are usually of iterative type, and involve the solution of many local and one global problem in each iteration. A major goal is to obtain algorithms that are scalable, i.e. the solution of a problem of size $2S$ can be obtained with $2P$ processors in about the same time as a similar problem of size S on P processors. Therefore, the iteration count has to be approximately independent of the number of processors; this has been achieved for a number of domain decomposition preconditioners, and in particular for all the methods studied in this thesis. The global solve corresponding to a coarse model plays a major role in accomplishing this goal.

This thesis can be viewed as part of a research program which now focuses at extending these algorithms to harder and more important problems and discretization methods. The techniques and framework previously developed for domain decomposition preconditioners of finite element discretizations of second-order elliptic problems is used in many ways throughout this thesis.

The spectral element method may be regarded as a domain decomposition method, since the discretization space is based on a partition of the domain into logically square or cubic subdomains. Several advantages result from this property: the use of tensor product bases and quadrature rules, geometrical flexibility, and good potential for parallelization. Some of the main contributors to the theoretical and practical development of these methods are Christine Bernardi, Paul Fischer, Yvon Maday, Anthony Patera, and Einar Rønquist [12, 58, 73, 74, 57]; their work has been central to ours.

In the main part of this thesis, we consider spectral element discretizations of second-order elliptic, and the incompressible stationary Stokes and Navier-Stokes equations. A number of different domain decomposition methods have already been proposed for solving the resulting system of linear equations, many without rigorous theoretical justification. An important goal of this thesis is to develop an appropriate theory that explains the good convergence behavior of some of these algorithms. The next step is then to design improved versions of the existing methods, given the insight provided by this new theory. We believe that we have accomplished these goals, to a reasonable extent, for a number of problems. Finally, in the last chapter, we develop, analyze and test an efficient Schwarz algorithm for the mortar finite element method.

In Chapter 3, we present a theory that supports the very clever and insightful experimental work of Shannon Pahl [88] and Einar Rønquist [100] on domain decomposition methods for elliptic second-order equations. This theory allows us to design other efficient preconditioners, and also provides several tools for further developments in the subsequent chapters. We note that while the principal goal is to develop algorithms for parallel architectures, many of these methods can also be used efficiently on machines with a single processor. Some of our results provide new proofs of results obtained by Pavarino and Widlund, but the new tools are also used to study and develop new algorithms, in particular the overlapping methods for the spectral element discretization; see Section 3.1.

Chapter 4 starts by developing a theory to account for the very good experimental results obtained by Babuška et al. [5, 6, 7] for an iterative substructuring method for the p-version finite element in two dimensions. The most efficient variant of their method, actually implemented, is not covered by the theory in [5]. We have been able to develop such an analysis, and as an outgrowth of our study, we have proposed an algorithm for the spectral element in three dimensions which substantially improves the weakest part of one of the methods previously proposed. In our improved algorithm, the face components of the residual are preconditioned with a diagonal matrix, without significantly decreasing the rate of convergence of the overall method; see subsection 4.6.2. Known alternatives involve relatively expensive solutions of Poisson problems in the union of two subregions. Our new method has been tested by Einar Rønquist, and the results show a significant improvement; the iteration count is within a factor of two of some of the best preconditioners for the finite element method (a low-order discretization), while the work per iteration is substantially reduced. Some of these experiments are described at the end of Chapter 4.

Rønquist has also proposed very ingenious and efficient domain decomposition preconditioners for the incompressible, stationary Stokes and Navier-Stokes equations; see [100]. Motivated by his work and also by the important work of Fischer and Patera [57] and Fischer and Rønquist [58] on the unsteady problem, we have developed a theory that attempts to explain rigorously the success of Rønquist's algorithm. Chapter 5 contains our results for the Stokes problem. As with Rønquist's method, our algorithm is based on iterative substructuring. However, his algorithm uses the GMRES method for the Stokes problem, and iterates over the velocity and the pressure, while we propose an iterative method that constrains the velocities to the subspace of discretely divergence free velocities. The Stokes equation then becomes a symmetric positive definite problem, for which the preconditioned conjugate gradient method can be used. We note that our method does not require the construction of an explicit basis for this divergence free subspace.

In Chapter 6, we introduce a domain decomposition method for the incompressible, stationary Navier-Stokes equation. Although we again work in the space of discretely divergence free velocities, the GMRES method is used, since the linearized problem resulting from the Newton iteration is not symmetric positive definite. We remark that our work does not fully explain the success of Rønquist's algorithm, but we believe that this attempt brings a much greater understanding of the mechanisms involved, as well as some promising new methods that will be implemented and compared with Rønquist's in the near future.

The Schwarz framework of Dryja and Widlund [52] accommodates, in an elegant fashion, the use of overlapping spaces in the design of preconditioners. The preconditioner is viewed as consisting of different modules, each of which is often associated with a geometrical object, or with a simplified global model. We try to stress, throughout the thesis, that the judicious use of this flexibility, by adding some simple local spaces and reducing both the dimension of the global problem and the overall condition number, is beneficial and very cost effective. This theme is present in Sections 3.5, 4.6, and Chapters 5 and 6.

We remark that the modularity of the methods is very important for two different reasons:

- the codes generated by this technology are modular in nature, and therefore inherently easy to implement and modify, especially on a parallel machine;

- the analysis for a variant of a previously known method often requires only new insight into one of the components of the complete algorithm; new methods can also be assembled using components from other methods.

The first point has been very well illustrated for finite elements by the release of PETSc, a software package designed to take advantage of this flexibility. Many new methods can be implemented with ease, by modifying the components one at a time, e.g., using different inexact solvers or iterative methods. The second aspect is used to great advantage especially in Section 4.6 and Chapter 5, and has been stressed in [66, 109].

Mortar methods attempt to increase the geometrical flexibility of both the spectral and the finite element method; see [14]. We only consider the geometrically conforming version of the h-version of the method, for which the intersection between two subdomains is either empty, a vertex, or a whole edge of a subdomain. The meshes within the substructures into which the original domain has been partitioned do not necessarily match on the interfaces; an integral constraint is imposed, which still preserves the convergence properties of the method. In Chapter 7, we propose hierarchical basis preconditioners for these methods in the plane, and obtain condition numbers that grow only polylogarithmically with the maximum number of points on an edge of the substructures. The flexibility of the Schwarz framework is again used to substantially decrease the dimension of the global space. In Section 7.4, we report on relatively extensive experiments to support our theory.

Chapter 3 has appeared in a slightly different form as a technical report [37]. We note that Chapter 4, except for Section 4.6 was published as [36]. Chapter 7 represents joint work with Olof Widlund [39]. These three reports have already been submitted for publication. Section 4.6 represents joint work with Einar Rønquist and Olof Widlund.

The remainder of this introductory chapter introduces some background material on Sobolev spaces and the spectral element discretization of the various problems we will consider. For details on the mortar finite element method for second-order elliptic problems, we refer to Section 7.2. Chapter 2 contains a brief discussion of some iterative methods and the abstract framework of Schwarz methods. For a more detailed overview of the material of Chapters 3-7, we point to the introductory sections of each chapter. We have also included an index of symbols as an appendix, in order to help the reader.

1.2 Sobolev and Trace Spaces

Sobolev and trace spaces provide essential tools for the study of partial differential equations, for the analysis of numerical algorithms, and for the analysis of domain decomposition algorithms for the resulting linear systems of equations. In this section we collect some well-known results about these spaces that are used throughout this thesis. A more thorough introduction to the most important tools, specifically those used in domain decomposition theory, may be found in [102, Section 1.2]. For a more complete development of the theory, we refer to [85] and [64].

Let Ω be a bounded Lipschitz region in $\mathbb{R}^d$. Let x represent a point in Ω or on its boundary $\partial\Omega$, let u, v, f, g be scalar valued functions, let $\mathbf{u}$, $\mathbf{v}$ be vector valued functions, and let u_i be the i-th component of $\mathbf{u}$, a vector with d components.

The space $L^2(\Omega)$ is defined as the closure of $C^\infty(\Omega)$ in the norm

$$\|u\|_{L^2(\Omega)}^2 = \int_\Omega |u|^2 \, dx < \infty.$$

The H^1-semi-norm is defined by

$$|u|^2_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla u \, dx,$$

and the scaled norm is given by

$$\|u\|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega)} + \frac{1}{H_\Omega^2} \|u\|^2_{L^2(\Omega)},$$

where H_Ω is the diameter of Ω; this scale factor is generated by dilation starting from a region of unit diameter. The spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ are the closures of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$, respectively, under the H^1-norm.

Lemma 1.2.1 *(***Poincaré's inequality)** *Let*

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u \, dx,$$

be the average value of u, where $|\Omega|$ denotes the volume of Ω. There exists a constant $C(\Omega)$, which depends only on the Lipschitz constants of $\partial\Omega$, such that

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C\,(\Omega)\, H_\Omega |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega).$$

We also need the Poincaré-Friedrichs inequality. The main idea of its proof can be found in [43, Theorem 6.1] and in [85, Chapter 2.7.2].

Lemma 1.2.2 (Poincaré-Friedrichs' inequality) *Let Λ be a an open subset of $\partial\Omega$ with positive measure. There exists a constant $C(\Omega, \Lambda)$ such that $\forall u \in H^1(\Omega)$,*

$$\|u\|^2_{L^2(\Omega)} \leq C(\Omega, \Lambda)\, H_\Omega^2 \,(|u|^2_{H^1(\Omega)} + \frac{1}{H_\Omega} (\int_\Lambda u \, dx)^2). \tag{1.1}$$

The constant $C(\Omega, \Lambda)$ depends only on the Lipschitz constant of $\partial\Omega$ and on the measure of Λ relative to $\partial\Omega$.

Let $H^{1/2}(\Lambda)$ be the trace space of $H^1(\Omega)$ on Λ. The K-method of interpolation gives the equivalent definition $H^{1/2}(\Lambda) = [L^2(\Lambda), H^1(\Lambda)]_{1/2}$; see [71, Section I.15].

This space can also be characterized by several equivalent semi-norms. One of them is:

$$|u|^2_{H^{1/2}(\Lambda)} = \min_{\tilde{u}|_\Lambda = u} |\tilde{u}|^2_{H^1(\Omega)}.$$

Another equivalent semi-norm that has the nice property of being intrinsic, i.e. it only refers to the values of the function on Λ, is:

$$|u|^2_{H^{1/2}(\Lambda)} = \int_\Lambda \int_\Lambda \frac{|u(x) - u(y)|^2}{|x-y|^d} \, dS(x) \, dS(y).$$

The norm of the space $H^{1/2}(\Lambda)$ is therefore given by the norm produced by the K-method of interpolation, or by

$$||u||^2_{H^{1/2}(\Lambda)} = |u|^2_{H^{1/2}(\Lambda)} + ||u||^2_{L^2(\Lambda)},$$

where $|\cdot|_{H^{1/2}(\Lambda)}$ is given by any of the two expressions above.

The closures of $C_0^\infty(\Lambda)$ and $C^\infty(\Lambda)$ under the $H^{1/2}(\Lambda)$-norm are both the space $H^{1/2}(\Lambda)$. However, the extension of $u \in H^{1/2}(\Lambda)$ by zero to the whole of $\partial\Omega$ is *not* a bounded operator from $H^{1/2}(\Lambda)$ to $H^{1/2}(\partial\Omega)$. To obtain a bounded extension, we have to restrict ourselves to a strictly included subspace $H_{00}^{1/2}(\Lambda) \subset H^{1/2}(\Lambda)$, and use a stronger norm. The space $H_{00}^{1/2}(\Lambda)$ is formed by the functions $v \in H^{1/2}(\partial\Omega)$ that vanish outside Λ, endowed with the norm $||v||_{H^{1/2}(\partial\Omega)}$. This space is isomorphic to the interpolation space $[L^2(\Lambda), H_0^1(\Lambda)]_{1/2}$; see [71, Chapter 1]. An equivalent norm for $H_{00}^{1/2}(\Lambda)$ is given by:

$$||v||^2_{H_{00}^{1/2}(\Lambda)} = |v|^2_{H^{1/2}(\Lambda)} + \int_\Lambda \frac{v^2(x)}{d(x, \partial\Lambda)} \, dS(x); \tag{1.2}$$

see [85].

1.3 Ellipticity and the Babuška-Brezzi Theory

We start with the following well-known lemma, which establishes existence, uniqueness and well-posedness for the elliptic problems to be considered.

Lemma 1.3.1 (Lax-Milgram Lemma) *Let $\mathcal{B}$ be a bilinear form on a Hilbert space $\mathcal{H}$. Assume that $\mathcal{B}$ is bounded,*

$$|\mathcal{B}(w, v)| \le K \, \|w\|_\mathcal{H} \cdot \|v\|_\mathcal{H} \quad \forall w, v \in \mathcal{H},$$

and coercive, i.e. there exists an $\alpha > 0$ *such that*

$$\mathcal{B}(v,v) \geq \alpha \, \|v\|^2_{\mathcal{H}} \qquad \forall v \in \mathcal{H}.$$

Then, for every bounded functional $f \in \mathcal{H}^*$*, there exists a unique element* $u_f \in \mathcal{H}$ *such that*

$$\mathcal{B}(u_f, v) = f(v) \qquad \forall v \in \mathcal{H},$$

and

$$\|u_f\|_{\mathcal{H}} \leq \frac{\|f\|_{\mathcal{H}^*}}{\alpha}.$$

The counterpart of the Lax–Milgram Lemma for a certain class of saddle point problems is given by the following result; see Brezzi and Fortin [26].

Lemma 1.3.2 (Babuška-Brezzi Lemma) *Let* V *and* Q *be Hilbert spaces with norms* $\|\cdot\|_V$ *and* $\|\cdot\|_Q$*, respectively. Let* $a(\cdot,\cdot)$ *be a continuous bilinear form on* $V \times V$*, let* $b(\cdot,\cdot)$ *be a continuous bilinear form on* $V \times Q$*, and assume that the range of the operator* $B : V \to Q'$*, defined by* $(B\mathbf{v}, q) = b(\mathbf{v}, q)$*, is closed in* Q'*, i.e. there exists* $k_0 > 0$ *such that*

$$\sup_{\mathbf{v}\in V} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_V} \geq k_0 \, \|q\|_{Q/Ker B^T} = k_0 \, (\inf_{q_0 \in Ker B^T} \|q + q_0\|_Q) \;\; \forall q \in Q. \tag{1.3}$$

Assume also that the operator given by the bilinear form $a(\cdot,\cdot)$ *is elliptic on Ker* B*, i.e. there exists* $\alpha_0 > 0$*, such that*

$$\begin{cases} \inf_{\mathbf{u}_0 \in Ker B} \sup_{\mathbf{v}_0 \in Ker B} \frac{a(\mathbf{u}_0,\mathbf{v}_0)}{\|\mathbf{v}_0\|_V \|\mathbf{u}_0\|_V} \geq \alpha_0, \\ \\ \inf_{\mathbf{v}_0 \in Ker B} \sup_{\mathbf{u}_0 \in Ker B} \frac{a(\mathbf{v}_0,\mathbf{u}_0)}{\|\mathbf{u}_0\|_V \|\mathbf{v}_0\|_V} \geq \alpha_0. \end{cases} \tag{1.4}$$

Then, the problem:
Find $\mathbf{u} \in V$ *and* $p \in Q$ *such that*

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = l_1(\mathbf{v}) \qquad \forall \mathbf{v} \in V \\ \\ \qquad b(\mathbf{u},q) = l_2(q) \qquad \forall q \in Q, \end{cases}$$

has a solution $(\mathbf{u}, p)$ $\forall l_1 \in V'$ *and* $\forall l_2 \in Im\, B$*. The first component,* $\mathbf{u}$*, is unique, while* p *is defined up to an element of Ker* B^T*. Furthermore*

$$\|\mathbf{u}\|_V \leq \frac{1}{\alpha_0}\|l_1\|_{V'} + (1 + \frac{\|a\|}{\alpha_0})\,\frac{1}{k_0}\|l_2\|_{Q'}, \tag{1.5}$$

and

$$\|p\|_{Q/KerB^T} \leq \frac{1}{k_0}(1+\frac{\|a\|}{\alpha_0})\|l_1\|_{V'} + \frac{\|a\|}{k_0^2}(1+\frac{\|a\|}{\alpha_0})\|l_2\|_{Q'}. \tag{1.6}$$

We note that (1.3) and (1.4) are not only sufficient but also necessary for the existence of a solution; cf. Brezzi [25].

1.4 Variational Formulation of the Equations

1.4.1 Second Order Elliptic Problems

Let Ω be a bounded polyhedral region in $\mathbb{R}^d$ with diameter of order 1. We consider the following elliptic self-adjoint problem:
Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \quad \forall\ v \in H_0^1(\Omega), \tag{1.7}$$

where

$$a(u,v) = \int_\Omega k(x)\,\nabla u \cdot \nabla v\,dx \quad \text{and} \quad f(v) = \int_\Omega fv\,dx \quad \text{for} \quad f \in L^2(\Omega),$$

where $C \geq k(x) \geq c > 0$; C and c are constants. The results of Chapter 3 and of Section 4.6 also hold for mixed Neumann–Dirichlet boundary conditions, but here we restrict ourselves to homogeneous Dirichlet conditions, to simplify our discussion. In Chapter 7, we present, for a different discretization, the modifications needed to extend our algorithms and analysis to problems with these mixed boundary conditions.

It is elementary to verify, using Lemma 1.2.2, that this problem satisfies the hypothesis of the Lax-Milgram lemma, and hence the solution u exists, is unique, and satisfies:

$$||u||_{H^1(\Omega)} \leq C(\Omega)||f||_{H^{-1}(\Omega)},$$

where $C(\Omega)$ depends on Ω and on the supremum and infimum of the coefficient $k(x)$.

1.4.2 Stokes Problem

We consider the Stokes equation in the velocity-pressure formulation:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{g}, \end{cases} \tag{1.8}$$

where the system has been properly non-dimensionalized and $\nu > 0$ is the viscosity.

We need a few standard definitions in order to state the variational formulation of (1.8). Let $V = (H_0^1(\Omega))^d$, $Q = L_0^2(\Omega) := \{q \in L^2(\Omega), \int_\Omega q\,dx = 0\}$,

$$a(\mathbf{u},\mathbf{v}) = \nu \sum_{i,j=1}^{d} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx,$$

$$b(\mathbf{v},q) = -\textstyle\int_\Omega q\nabla\cdot\mathbf{v}\,dx,$$

and

$$\langle \mathbf{l},\mathbf{v}\rangle = \textstyle\int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx - a(\mathbf{u}_0,\mathbf{v}),$$

for some $\mathbf{u}_0 \in (H^1(\Omega))^d$ with $\mathbf{u}_0|_{\partial\Omega} = \mathbf{g}$ in $H^{1/2}(\Omega)$, and $\nabla\cdot\mathbf{u}_0 = 0$.

Then, for $\mathbf{f} \in (H^{-1}(\Omega))^d$, $\mathbf{g} \in (H^{1/2}(\partial\Omega))^d$, a weak solution of (1.8) can be obtained by first finding $(\mathbf{w},p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that:

$$\begin{cases} a(\mathbf{w},\mathbf{v}) + b(\mathbf{v},p) &= \langle \mathbf{l},\mathbf{v}\rangle \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d, \\ b(\mathbf{w},q) &= 0 \quad \forall q \in L_0^2(\Omega), \end{cases}$$

and then letting $\mathbf{u} = \mathbf{w} + \mathbf{u}_0$. An equivalent formulation is given by:
Find $(\mathbf{u},p) \in (H^1(\Omega))^d \times L_0^2(\Omega)$ such that:

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d, \\ b(\mathbf{u},q) &= 0 \quad \forall q \in L_0^2(\Omega), \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{g}, \end{cases} \tag{1.9}$$

where $\langle \mathbf{f},\mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx$. A straightforward application of Lemma 1.3.2 together with an extension lemma [60, Lemma I.2.2] gives the following result; see [60, Theorem I.5.1].

Theorem 1.4.1 *Let Ω be a bounded and connected Lipschitz region of $\mathbb{R}^d$. Then there exists a unique pair $(\mathbf{u}, p) \in (H^1(\Omega))^d \times L_0^2(\Omega)$ which solves (1.9).*

By restricting the test functions $\mathbf{v}$ to be solenoidal, i.e.

$$\mathbf{v} \in V_\nabla = \{\mathbf{w} \in (H_0^1(\Omega))^d \,|\nabla \cdot \mathbf{w} = 0\},$$

we can determine the velocity $\mathbf{u}$ by solving the following self-adjoint elliptic problem:

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_\nabla. \tag{1.10}$$

1.5 Spectral Element Method

We triangulate Ω into non-overlapping substructures $\{\Omega_i\}_{i=1}^M$ of diameter H_i. In the standard spectral element literature, the substructures Ω_i are usually called elements. However, in what follows, we denote the Ω_i by *substructures* or *subdomains.* Later on, we will further divide the substructures into hexahedrals, which will be called *elements.*

Each Ω_i is the image of the reference substructure $\hat{\Omega} = [-1, +1]^3$ under a mapping $F_i = D_i \circ G_i$ where D_i is an isotropic dilation and G_i a C^∞ mapping such that its Jacobian and the inverse thereof are uniformly bounded by a constant. In Section 3.2, we show that our bounds depend on this constant, and are better the closer this constant is to one, i.e. the closer the substructures are to cubes. We also assume that the partition into substructures is geometrically conforming, i.e. in three dimensions, the intersection between the closures of two distinct substructures is either empty, a vertex, a whole edge or a whole face; this is a standard assumption in finite element theory. Some additional properties of the mappings F_i are required to guarantee an optimal convergence rate. We refer to [12, problem 2], and references therein for further details on this issue, but remark that affine mappings are covered by the available convergence theory for these methods; see also [24, Section 8.4] for an analysis involving isoparametric mappings in a related context. We assume for simplicity that $k(x)$ has the constant value $k_i > 0$ in the substructure Ω_i, with possibly large jumps occurring only across substructure boundaries. The bounds for the iterative substructuring methods will be independent of these jumps. For the overlapping methods, we need to introduce more stringent restrictions on $k(x)$ to obtain bounds that are independent of the jumps; see the discussion after Lemma 3.3.4. We remark also that if $k(x)$ varies moderately in each substructure, all our results are easily seen to hold.

We define the space $P^N(\hat{\Omega})$ as the space of polynomials of degree at most N in each of the variables separately. The space $P^N(\Omega_i)$ is the space of functions v_N such that $v_N \circ F_i$ belongs to $P^N(\hat{\Omega})$. The conforming discretization space $P_0^N(\Omega) \subset H_0^1(\Omega)$ is the space of continuous functions the restrictions of which to Ω_i belong to $P^N(\Omega_i)$.

Let $\Lambda = [-1, 1]$. For each N, the Gauss-Lobatto-Legendre quadrature of order N is denoted by GLL(N) and satisfies:

$$\forall p \in P^{2N-1}(\Lambda), \quad \int_{-1}^{1} p(x)\, dx \ = \sum_{j=0}^{N} p(\xi_j)\rho_j. \tag{1.11}$$

Here, the quadrature points ξ_j are numbered in increasing order, and are the zeros of $(1-x^2)L_n'(x)$, and $L_n(x)$ is the Legendre polynomial of degree n; see (4.2). The weights ρ_j are given by:

$$\rho_j = \frac{2}{N(N+1)L_N^2(\xi_j)} \quad (0 \le j \le N). \tag{1.12}$$

The GLL(N) quadrature has the following important property:

$$\forall p_N \in P^N(\Lambda), \quad ||p_N||_{L^2(\Lambda)}^2 \le \sum_{j=0}^{N} p_N^2(\xi_j)\rho_j \le 3||p_N||_{L^2(\Lambda)}^2; \tag{1.13}$$

see [12, Corollary III.1.3].

In three dimensions, the discrete $L^2(\Omega)$-inner product is defined by

$$(u, v)_N = \sum_{i=1}^{M} \sum_{j,k,l=0}^{N} k_i \cdot (u \circ F_i) \cdot (v \circ F_i) \cdot |J_i|(\xi_j, \xi_k, \xi_l) \cdot \rho_j \rho_k \rho_l, \tag{1.14}$$

where $|J_i|$ is the Jacobian determinant of F_i.

1.5.1 Second Order Elliptic Problems

We define a discrete bilinear form for $u, v \in H^1(\Omega)$ by:

$$a_Q(u,v) = (\nabla u, \nabla v)_N, \tag{1.15}$$

where $(\cdot,\cdot)_N$ is computed component-wise. The discrete problem with homogeneous boundary conditions is then:
Find $u_N \in P_0^N(\Omega)$, such that

$$a_Q(u_N, v_N) = (f, v_N)_N \quad \forall\, v_N \in \; P_0^N(\Omega). \tag{1.16}$$

We number the GLL nodes of all the substructures, and choose nodal basis functions $\phi_j^N \in P_0^N(\Omega)$, which are one at the GLL node j and zero at all the others. This basis gives rise, in the standard way, to the linear system $K_N x = b$. Note that the mass matrix of this nodal basis, generated by the discrete L^2 inner product (1.14), is diagonal. The analysis and experimental evidence show that the spectral element method just described achieves very good accuracy for reasonably small N for a wide range of problems; see [12, 73, 74] and references therein. The practical application of this method for large scale problems, however, depends on fast and reliable solution methods for the system $K_N x = b$. A direct method is often not an economical choice, because of long range interactions between the basis elements, and because this is a discretization of a three dimensional problem, which demands large computer resources even for the much simpler seven-point finite difference stencil; see [53].

1.5.2 The Stokes Problem

Let $(P_0^N(\Omega))^d$ be the space of vector-valued velocities with each component in $P_0^N(\Omega)$. While the velocities are taken to be continuous functions, the pressures can be discontinuous across subdomain boundaries. The restriction of the pressure space $\bar{P}^{N-2}(\Omega)$ to each Ω_i is $P^{N-2}(\Omega_i)$. We note that $\bar{P}^{N-2}(\Omega) \subset L^2(\Omega)$, but $\bar{P}^{N-2}(\Omega) \not\subset H^1(\Omega)$.

To simplify the presentation, we assume that $\mathbf{g} \equiv \mathbf{0}$ in (1.9). In the general case, the discrete formulation is completely analogous, while the analysis of the discretization error is somewhat more involved. Here we are concerned with solution methods for the resulting linear system, and the case $\mathbf{g} \not\equiv \mathbf{0}$ presents no essential additional difficulty; the same is true for mixed Dirichlet and Neumann boundary conditions, and hence we consider only the homogeneous Dirichlet boundary conditions.

The spectral element discretization of (1.9) is given by:
Find $(\mathbf{u}, p) \in (P_0^N(\Omega))^d \times \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega)$ such that:

$$\left\{ \begin{array}{rcl} a_Q(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &=& \langle \mathbf{l}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in (P_0^N(\Omega))^d, \\ \\ b(\mathbf{u}, q) &=& 0 \quad \forall q \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega). \end{array} \right. \tag{1.17}$$

Here, $a_Q(\cdot,\cdot)$ is the same as in (1.15), calculated as the sum over the components of the vector arguments $\mathbf{u}$ and $\mathbf{b}$, and with $k_i = \nu$, for all i. The form $b(\cdot,\cdot)$ in 1.17 is defined in subsection 1.4.2, and can be computed exactly by the GLL(N) quadrature rule only if each substructure is the image of a cube under an affine mapping. However, we assume that the discrete problem being solved is actually (1.17); such an assumption is also made in the analysis of the Stokes approximation given in [75], and we refer to that paper for further details on this issue.

We choose the same basis $\{\phi_j^N\}$ as before for each component of the space $(P_0^N(\Omega))^d$, and use the same notation $\{\phi_j^N\}$, with j now varying over a d-times larger set of values, since there are d degrees of freedom per point. We number the GLL(N) nodes ξ within the subregions Ω_i by an index r, and define a basis for $\bar{P}^{N-2}(\Omega)$ by $\beta_{r_1}(\xi_{r_2}) = \delta_{r_1 r_2}$, for all r_1, r_2, where δ is the Kronecker symbol. We note that any function of $\bar{P}^{N-2}(\Omega)$ is uniquely represented by its values at the interior GLL(N) nodes ξ_r. The choice of the internal GLL(N) nodes is an implementation choice; a basis for $\bar{P}^{N-2}(\Omega)$ could also be generated by using Gauss quadrature points, which are all internal to the substructures.

By writing the solution $(\mathbf{u}, p)$ in terms of these two bases, we arrive, in a standard way, to a linear system equivalent to (1.17):

$$\left\{ \begin{array}{l} A\underline{\mathbf{u}} + B^t \underline{p} = \underline{\mathbf{f}} \\ \\ \quad B\underline{\mathbf{u}} = \underline{0}. \end{array} \right. \tag{1.18}$$

To each component of the velocity, there corresponds a diagonal block of A which is equal to the scalar spectral element stiffness matrix K_N from subsection 1.5.1.

The entries of B are given by $B_{jr} = b(\phi_j^N, \beta_r)$, and $\underline{\mathbf{f}}$ is a vector with components $\underline{\mathbf{f}}_j = \langle \mathbf{f}, \phi_j^N \rangle$. Here and in what follows, an underline is used to denote vectors in an Euclidean space; they are often very large vectors. We note that, in the full stress formulation of the Stokes problem, the matrix A is no longer block-diagonal, and our study is not directly applicable to that case; see [59].

The analysis of the discretization error of this approximation is given in [75], and proceeds along the lines of the standard approximation result for mixed finite elements [60, Theorem I.1.8]. The key point in the analysis is an estimate of the inf-sup constant as a function of the degree N; see [75, Lemmas 5.3 and 5.4].

Lemma 1.5.1 *For each N, there exists a $\beta_N > 0$ such that*

$$\inf_{q \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega)} \quad \sup_{\mathbf{v} \in (P_0^N(\Omega))^d} \frac{b(\mathbf{v}, q)}{||\mathbf{v}||_{H^1(\Omega)} ||q||_{L^2(\Omega)}} \geq \beta_N .$$

If the geometry is rectilinear, i.e. the F_i are affine mappings, then there exists a constant β, independent of N, and such that $\beta_N \geq \beta N^{\frac{1-d}{2}}$, for $d = 1$, 2, or 3.

As far as we know, the only rigorous approximation result for the case of many substructures applies only when the mappings F_i are affine. We refer to [75] for a detailed study of this issue and the full analysis of the discretization error, and remark that very good convergence properties are predicted by the theory and have been extensively verified in practice; see [58, 73].

Chapter 2

Domain Decomposition Methods

2.1 Iterative Methods

For problems in two dimensions, direct methods are still the solution method of choice for most industrial or academic problems discretized with the standard finite element or finite differences methods. These methods take advantage of a well-developed set of mesh ordering techniques and software for Gaussian and Cholesky factorizations. For three-dimensional problems, however, the situation changes dramatically. Even for the seven-point finite difference stencil, the computer resources needed grow very rapidly; see [53].

For spectral element methods, memory and work requirements are even larger for the same number of degrees of freedom, because of the long range interactions between the basis elements. Therefore, iterative methods are almost a necessity in order to use these discretizations. For symmetric positive definite problems, the method of choice is the preconditioned conjugate gradient method; we give a brief review of its most important properties in the next subsection. For non-symmetric or indefinite problems, it is harder to point to one iterative method as the best in all circumstances. Indeed, many different iterative schemes may be used with success for any single problem, although the goal of the preconditioning may vary for each individual problem and iterative method. Depending on the iterative method being used, keeping the eigenvalues of the preconditioned problem in the right half of the complex plane may insure convergence, as is the case for

GMRES. For other iterative solvers like preconditioned conjugate residuals, it is most important to obtain eigenvalues that are bounded away from zero, although the spectrum may have positive and negative parts; see e.g. the recent progress on block preconditioners for the Stokes problem [55, 69, 106]. We consider here only the GMRES method, and present some of its properties in subsection 2.1.2

The important observation is that for all acceleration methods, the preconditioning plays a crucial role in the rate of convergence, and ultimately in the applicability of the methods; if a low quality preconditioner is used, GMRES may even fail to converge.

2.1.1 Preconditioned Conjugate Gradients

We consider a linear system

$$\mathcal{A}x = b, \tag{2.1}$$

where $\mathcal{A}$ is symmetric and positive definite. Let $\tilde{\mathcal{A}}$ be another symmetric positive definite matrix of the same size, and let v be a vector. The preconditioned conjugate gradient (PCG) is an iterative method for solving (2.1), which requires the operations $\mathcal{A}v$ and $\tilde{\mathcal{A}}^{-1}v$ once per iteration. It also requires the storage of a few vectors and the computation of a few inner products per iteration.

Let x_0 be an initial guess, x be the exact solution, and x_n be the nth iterate of the PCG method. Then, the following convergence estimate holds:

$$(x_n - x)^t \mathcal{A}(x_n - x) \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^{2n}(x_0 - x)^t \mathcal{A}(x_0 - x), \tag{2.2}$$

where

$$\kappa = \frac{\lambda_{\max}(\tilde{\mathcal{A}}^{-1}\mathcal{A})}{\lambda_{\min}(\tilde{\mathcal{A}}^{-1}\mathcal{A})}.$$

Therefore, PCG is an effective method when

- $\mathcal{A}v$ is relatively inexpensive to compute,
- $\tilde{\mathcal{A}}^{-1}v$ is also relatively easy to evaluate, and
- κ is small.

Most of the domain decomposition methods of this thesis are designed having this method in mind, and we will have the explicit goal of substantially reducing κ while being able to compute $\tilde{A}^{-1}v$ efficiently, especially in a parallel environment. We note that if $\mathcal{A}$ is the spectral element stiffness matrix K_N, efficient ways of computing $\mathcal{A}v$ efficiently are available; see [12, 74].

2.1.2 GMRES

The GMRES method [101] is an iterative method that in each step minimizes the residual over a certain Krylov subspace. For a general non-symmetric matrix $\mathcal{A}$, there is no three-term recurrence relation available, and the work and storage grow linearly with the number of iterations. Therefore, the restarted or truncated versions of the algorithm are also used; see [101]. We note that conditions for a short-term recurrence relation to exist are well known; see [56].

Since this is a well-known scheme, we do not present a full algorithmic description, and state only one convergence result that will be used in Chapter 6 to analyze a preconditioned method for the stationary Navier-Stokes equation. We note that there are several different estimates of the convergence of the GMRES algorithm; see [54, 101, 111].

Let $[\cdot,\cdot]$ be an inner product, which is usually generated by an auxiliary symmetric positive definite matrix A, i.e. $[\cdot,\cdot] = (A\cdot,\cdot)$. Let $\tilde{\mathcal{A}}$ be a preconditioner of $\mathcal{A}$, and define

$$c_{\tilde{\mathcal{A}}^{-1}\mathcal{A}} = \inf_{x\neq 0} \frac{[x, \tilde{\mathcal{A}}^{-1}\mathcal{A}x]}{[x,x]} \quad \text{and} \quad C_{\tilde{\mathcal{A}}^{-1}\mathcal{A}} = \sup_{x\neq 0} \frac{||\tilde{\mathcal{A}}^{-1}\mathcal{A}x||}{||x||}, \tag{2.3}$$

where the norm $||\cdot||$ is generated by $[\cdot,\cdot]$. The following theorem can be established, by considering the decrease of the norm of the residual $r_m = \tilde{\mathcal{A}}^{-1}\mathcal{A}x_m - b$ in a single step of the algorithm; see [54].

Theorem 2.1.1 *(Eisenstat, Elman, and Schultz) If $c_{\tilde{\mathcal{A}}^{-1}\mathcal{A}} > 0$, then the GMRES method converges and after m steps, the norm of the residual is bounded by*

$$||r_m|| \leq \left(1 - \frac{c^2_{\tilde{\mathcal{A}}^{-1}\mathcal{A}}}{C^2_{\tilde{\mathcal{A}}^{-1}\mathcal{A}}}\right)^{m/2} ||r_0||.$$

Compared to the estimate given above for the PCG method, this estimate seems rather poor; it emphasizes the need for a good preconditioner $\tilde{\mathcal{A}}$, so that the parameters in (2.3) fall within reasonable limits to make the method of practical value.

2.2 Multiplicative and Additive Schwarz Methods

As early as 1869, H. A. Schwarz [105] used a domain decomposition idea to construct solutions to the Laplace problem for easily decomposable domains. Pierre-Louis Lions restated the algorithm in variational form, in a paper that provided a starting point for the revival of the interest in the method [72]. We concern ourselves with extensions of the method for the solution of finite element or other Galerkin methods for problems in a region Ω decomposed into the union of many subdomains.

We consider for simplicity (1.7), discretized by choosing a subspace $V^h = V^h(\Omega)$ (e.g., a spectral or finite element space), and look for an approximate solution $u_h \in V^h$ that satisfies

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in V^h. \tag{2.4}$$

Assume that we are given subspaces $\{V_s^h\}_{s=0}^M \subset V^h(\Omega)$, such that their sum spans the whole of $V^h(\Omega)$:

$$V^h = V_0^h + V_1^h + \cdots + V_M^h,$$

and inner products $b_s(\cdot,\cdot)\colon\ V_s^h \times V_s^h \to \mathbb{R}$. We define approximate projections $T_s : V^h \to V_s^h$ by:

$$b_s(T_s w_h, v_h) = a(w_h, v_h) \quad \forall v_h \in V_s^h\ \forall w_h \in V^h. \tag{2.5}$$

We remark that if $b_s(\cdot,\cdot) = a(\cdot,\cdot)$, T_s is an orthogonal projection in the $a(\cdot,\cdot)$-inner product.

Let u_h be the unique solution of (2.4), assumed to exist for any reasonable choice of a space V^h. The multiplicative Schwarz method is given by the following iterative scheme, where u_h^0 is an initial guess, and u_h^k the kth iterate:

$$(u_h^{k+1} - u_h) = (I - T_0)(I - T_1)\cdots(I - T_M)(u_h^k - u_h). \tag{2.6}$$

This may be written as

$$u_h^{k+1} = u_h^k - T_{\mathrm{ms}}(u_h^k - u_h),$$

where $T_{\mathrm{ms}} = I - (I - T_0)(I - T_1) \cdots (I - T_M)$. This iteration can be interpreted as a Richardson iterative scheme for solving the equation

$$T_{\mathrm{ms}} u_h = g, \tag{2.7}$$

where g may be computed without knowing the solution u_h, since for each s, and for $w_h = u_h$, the right hand side of (2.5) is, $\forall v_h \in V_s^h$,

$$a(u_h, v_h) = f(v_h).$$

If there are only two subspaces $V_1 = H_0^1(\Omega_1')$ and $V_2 = H_0^1(\Omega_2')$, where Ω_1' and Ω_2' are overlapping subdomains satisfying $\Omega_1' \cup \Omega_2' = \Omega$, and $b_s(\cdot,\cdot) = a(\cdot,\cdot)$, this is exactly the reformulation by Lions of the method proposed by Schwarz.

The rate of convergence of u_h^k to u_h will of course depend on the choice of the spaces V_s^h and the bilinear forms $b_s(\cdot,\cdot)$. Even if these elements of the algorithm are fixed, there are many variants of the method. For the operator T_{ms} itself, we may use iterative methods more powerful than the Richardson iteration scheme; for efficiency reasons equation (2.7) may be accelerated by the GMRES method. We also mention briefly a few alternatives to T_{ms}, and refer to [27, 48, 80] for more detailed studies; all the methods are acceleration schemes applied to an equation $Tu_h = g$, where T is a polynomial of the operators T_s, and g can be calculated without knowing u_h, as in (2.7).

For the operator T_{ms}, the problems (2.5) have to be solved in a sequential manner. This may be remedied in part by observing that if the intersection of $V_{s_1}^h$ with $V_{s_2}^h$ is trivial, then T_{s_1} and T_{s_2} may be computed at the same time, which is extremely important to enhance the efficiency of computations on parallel computers. This is called the coloring strategy [29, 30]. If, however, there is one space, for example V_0^h, which has a non-trivial intersection with all the other subspaces, then this is 'a possible bottleneck' for parallel computations, 'with many processors idly waiting for the solution' of this single problem; see [102, p. 26]. It turns out that for any scalable method for an elliptic problem, the presence of such a space is absolutely necessary, as shown in [115], and hence we need to look for an alternative.

The additive Schwarz operator T_{as} proposed by Dryja and Widlund [49] and Matsokin and Nepomnyaschikh [84] is the simplest polynomial of the arguments $T_0, T_1, \ldots, T_M$, for which one can still compute $g = T_{\text{as}} u_h$, given only the right hand side $f(v_h)$:

$$T_{\text{as}} = T_0 + T_1 + \cdots + T_M.$$

The algorithm then amounts to solving the equation $T_{\text{as}} u_h = g$ using a Richardson iteration or a conjugate gradient method, since all the T_s (and hence T_{as}) are symmetric with respect to the inner product given by $a(\cdot,\cdot)$. For an additive algorithm, all the solves $T_s v_h$ may be performed concurrently, which has obvious advantages for the implementation on parallel computers. In the next subsection, we will state some theoretical results which show that both algorithms are convergent at approximately the same rate, under certain assumptions.

In practice, the multiplicative methods usually converge faster, and one would like to preserve this fast convergence and the good parallelization properties of T_{as}. In this direction, Cai [27] has proposed a hybrid method, given by

$$T_{\text{Cai}} = \gamma T_0 + (I - (I - T_M) \cdots (I - T_1)),$$

combining the multiplicative and additive schemes. For these and other reasons related to the solvability of some local problems, Mandel and Brezina [80] proposed an alternative hybrid method:

$$T_{\text{MB}} = T_0 + (I - T_0)(T_1 + \cdots + T_M)(I - T_0).$$

We conclude this discussion of the many different combinations of these operators by pointing out that the best choice of polynomial of $T_0, \ldots, T_M$ depends on the particular problem and computer system where the algorithm will be implemented. For example, the particular properties of convection-dominated flows have been used to great advantage in practical computations, by selecting a particular polynomial of the T_s; see [31].

2.3 Abstract Condition Number Estimates

We state two abstract results on the condition number of the operators T_{as} and T_{ms}. The framework given by these theorems is sufficiently general to analyze several domain decomposition and multigrid algorithms that have been proposed for elliptic problems; see [47, 48] for several applications of these methods.

The basic assumptions of these theorems are the following three hypothesis; here and in what follows we will drop the subscript h in the notation for discrete functions.

H1: There exists a constant C_0^2 such that every $u \in V^h(\Omega)$ can be decomposed into a sum $u = \sum_s u_s$, with $u_s \in V_s^h$, which satisfies

$$\sum_s b_s(u_s, u_s) \leq C_0^2 a(u, u);$$

in other words, the sum of the approximate energies of the components can be estimated in terms of the energy of the original function.

H2: There exists a constant $\omega > 0$ such that

$$a(u, u) \leq \omega b_s(u, u) \quad \forall u \in V_s^h, \; s = 0, 1, \ldots, M.$$

H3: There exist constants $\mathcal{E}_{ij}$, $i, j = 1, \ldots, M$, such that

$$a(u_i, u_j) \leq \mathcal{E}_{ij} a(u_i, u_i)^{1/2} a(u_j, u_j)^{1/2} \quad \forall u_i \in V_i^h, \; \forall u_j \in V_j^h.$$

Let $\rho(\mathcal{E})$ be the spectral radius of the matrix $\mathcal{E}$, with entries $\mathcal{E}_{ij}$.

Theorem 2.3.1 *Assume that* **H1, H2,** *and* **H3** *hold. Then,*

$$C_0^2 a(u, u) \leq a(T_{as} u, u) \leq (\rho(\mathcal{E}) + 1)\omega a(u, u) \quad \forall u \in V^h(\Omega).$$

The proof of this lemma can be found, e.g., in [52].

We state now a result for the operator T_{ms}. Let $e^k = u^k - u$ be the error at step k of the iterative algorithm (2.6). It is clear that an upper bound for the relative decrease of the a-norm of the residual at each iteration is given by an upper bound on the a-norm of the error propagation operator $E_s = (I - T_0) \cdots (I - T_S)$.

Theorem 2.3.2 *Assume that* **H1, H2**, *and* **H3** *hold. Assume further that* $\omega < 2$. *Then,*

$$||E_S||_a^2 \leq 1 - \frac{2 - \hat{\omega}}{(1 + 2\hat{\omega}^2 \rho(\mathcal{E})^2) C_0^2},$$

where $\hat{\omega} = \max(1, \omega)$

We note that $\omega < 2$ is a natural assumption; otherwise for at least one of the factors we would have

$$||(I - T_s)||_a > 1.$$

A quite similar theorem was first proved in [20]; see also [52].

Chapter 3

Quasi-Optimal Schwarz Methods for the Conforming Spectral Element Discretization

3.1 Introduction

The spectral element method has been used extensively to discretize a variety of partial differential equations, and its efficiency has been demonstrated both analytically and numerically; see [73, 74], and references therein. The method uses polynomials of high degree in each element, and a particular choice of basis and numerical quadrature rules. In large scale problems, long range interactions between the basis elements within each substructure produce quite dense and expensive factorizations of the stiffness matrix, and the use of direct methods is often not economical because of the large memory requirements [58]. In the past decade, many preconditioners have been developed for *finite element* discretizations of these equations; see, e.g., [67, 68, 98]. For both families of discretizations, the design of preconditioners for three dimensional problems is especially challenging.

Early work on preconditioners for spectral methods was carried out by Canuto and Funaro [35] and Pavarino [91, 92, 93]. Some of the algorithms studied by Pavarino are numerically scalable (i.e., the number of iterations is independent of the number of substructures) and optimal (the number of iterations does not grow or grows only slowly with the degree of the polynomials). However, each application of his preconditioners can be very expensive.

Several iterative substructuring methods, which preserve quasi-optimality and scalability, were later introduced by Pavarino and Widlund [94, 95]. These preconditioners can be viewed as block-Jacobi methods after the stiffness matrix has been transformed by using a certain basis. The subspaces used are analogues of those proposed by Smith [108] for piecewise linear finite element discretizations. The bound for the condition number of the preconditioned operator grows only slowly with the polynomial degree, and is independent of the number of substructures.

The tensorial character of the spectral element matrix can be exploited when evaluating its action on a vector [74], but does not help when evaluating the action of the inverse of certain blocks of this matrix, as required when using these preconditioners. Orzag [86] and Deville and Mund [45] have proposed the use of a finite difference and a Q_1 or P_1 finite element model, respectively, as preconditioners for the spectral element matrix. The triangulation for this finite element method is based on the hexahedrals defined by the Gauss-Lobatto-Legendre (GLL) mesh of one substructure. This preconditioner has been demonstrated both numerically, in [45], and theoretically, by Canuto [34], to have a condition number independent of the degree of the polynomials. We note that ideas similar to those in [34] and [45] also appear in Quarteroni and Zampieri [99] and references therein. The spectral equivalence results of Canuto [34] and generalizations for other boundary conditions were also obtained independently by Parter and Rothman [89].

Based on these ideas, extended to the case of several substructures, Pahl [88] proposed efficient, easily parallelizable preconditioners for the spectral element method using iterative substructuring and overlapping Schwarz methods applied to the GLL finite element model. Pahl also performed experiments for a model problem in two dimensions, demonstrating that these preconditioners can be very efficient. In other words, high order accuracy is combined with efficient and inexpensive low-order preconditioning. The work of Pahl, however, did not contain any rigorous theoretical justification for the experimental results obtained. Einar Rønquist has also proposed some iterative substructuring-based methods for three-dimensional problems, and conducted relatively large experiments [100]. Again, no rigorous theory was proposed to support these results. In Section 4.6, we present a variant of the original algorithm by Rønquist together with some numerical results.

The previous analysis of Schwarz preconditioners for the h-method has relied upon the shape-regularity of the mesh, see [19, 48, 51], which does not hold at all for the GLL mesh. In this chapter, we analyze some Schwarz finite element preconditioners defined on this mesh, and derive polylogarithmic bounds on the condition number of the preconditioned operators for iterative substructuring methods, and a result analogous to the standard finite element bound for overlapping Schwarz algorithms. Then, by applying Canuto's result, [34], we propose and analyze a new overlapping preconditioner that uses only blocks of the spectral element matrix to define the local contributions of the preconditioner. We also give a new proof of one of the estimates in [94]. In summary, the equivalence between the spectral and finite element matrices, and the tools we develop here, allow us to extend the analysis available for the domain decomposition preconditioners of the standard finite element case to the spectral element case. We remark that our techniques may also be used to estimate the convergence of a large class of domain decomposition preconditioners on some non-regular meshes.

The remainder of the chapter is organized as follows. The motivation and strategy of our analysis are presented in detail in Section 3.2. In Section 3.3, we state and prove our core technical results. In sections 3.4 and 3.5, we formulate and analyze several representative iterative substructuring and overlapping algorithms. Section 3.6 briefly describes some numerical experiments performed by Pahl [88].

3.2 Finite Element Preconditioning and Some Simplifications

The condition number of K_N is very large even for moderate values of N; see [12]. Our approach is to solve this system by a preconditioned conjugate gradient algorithm. The following low-order discretization is used to define several preconditioners in the next sections.

The GLL points of degree N, denoted GLL(N), define, in a natural way, a triangulation $\mathcal{T}^{\hat{h}}$ of $\hat{\Omega}$ into N^3 parallelepipeds, and on this triangulation we define the space $P^{\hat{h}}(\hat{\Omega})$ of continuous functions that are trilinear (Q_1) in each parallelepiped of $\mathcal{T}^{\hat{h}}$. The spaces $P^h(\Omega_i)$ and $P_0^h(\Omega)$ are defined by mapping in the same way as for $P^N(\Omega_i)$ and $P_0^N(\Omega)$. The finite element discrete problem associated with (1.7) is:

Find $u_h \in P_0^h(\Omega)$, such that

$$a(u_h, v_h) = f(v_h) \quad \forall\, v_h \in \; P_0^h(\Omega). \tag{3.1}$$

The standard nodal basis $\{\hat{\phi}_j^{\hat{h}}\}$ of $P^{\hat{h}}(\hat{\Omega})$ is mapped by the F_i into a basis for $P^h(\Omega_i)$, for $1 \leq i \leq M$. These bases and the bilinear form $a(\cdot,\cdot)$ give rise to a system $K_h x = b$.

We could also define a finite element system generated by dividing each hexahedral of $\mathcal{T}^{\hat{h}}$ into tetrahedrals, and using P_1 finite elements on this new triangulation. As will become clear in the following, the analysis for P_1 elements carries over immediately from the analysis for Q_1 elements, and from the following equivalence of norms. Let K be an element of $\mathcal{T}^{\hat{h}}$, and let $u_1 \in Q_1(K)$. If u_2 is the continuous function with restrictions to each tetrahedral into which K has been decomposed belonging to P_1, and coinciding with u_1 at the vertices of K, then it is easily seen that the H^1-norms of u_1 and u_2 are respectively equivalent to each other, with constants independent of N; the same is true for the L^2-norm. We remark that the P_1 elements have been shown to produce smaller condition numbers when used as a preconditioner, and should be prefered in a practical implementation. For the sake of simplicity, we restrict our analysis to the case of Q_1 elements.

Let $\hat{h}$ be the distance between the two leftmost GLL(N) points ξ_0 and ξ_1 in the interval $[-1,+1]$; $\hat{h}$ is on the order of $1/N^2$, while the distance between two consecutive GLL points increases to a maximum, close to the origin, which is on the order of $1/N$; see [12]. Hence, the aspect ratios of some of the elements of the triangulation $\mathcal{T}^{\hat{h}}$ grow in proportion to N.

We use the following notations: $x \preceq y$, $z \succeq u$, and $v \asymp w$ to express that there are strictly positive constants C and c such that

$$x \leq C\,y, \quad z \geq c\,u, \quad \text{and} \quad c\,w \leq v \leq C\,w, \quad \text{respectively.}$$

Here and elsewhere c and C are moderate constants independent of H, N, and $k(x)$.

Let $\hat{u}_N$ belong to $P^N(\hat{\Omega})$, and let $\hat{u}_h = \hat{I}_N^h(\hat{u}_N)$ be the unique function of $P^{\hat{h}}(\hat{\Omega})$ for which

$$\hat{u}_h(x_G) = \hat{u}_N(x_G),$$

for every GLL(N) point $x_G \in \overline{\hat{\Omega}}$. Then, by corollary 1.13, page 75 of [12] and the results in [34], we have:

$$||\hat{u}_h||^2_{L^2(\hat{\Omega})} \asymp ||\hat{u}_N||^2_{L^2(\hat{\Omega})} \asymp (\hat{u}_N, \hat{u}_N)_N, \tag{3.2}$$

and

$$|\hat{u}_h|^2_{H^1(\hat{\Omega})} \asymp |\hat{u}_N|^2_{H^1(\hat{\Omega})} \asymp a_{\hat{Q}}(\hat{u}_N, \hat{u}_N), \tag{3.3}$$

where $a_{\hat{Q}}$ is given by (1.14) and (1.15) with $J_i \equiv 1$ and $k_i = 1$. The basis of the proof of this last result is the H^1-stability of the polynomial interpolation operator at the GLL nodes for functions in $H^1([-1, +1])$, proved by Bernardi and Maday [11, 12]. The L^2-stability of the GLL quadrature of order N for polynomials of degree N, and properties of the GLL nodes and weights are also important in the argument. We remark that the first equivalence of (3.3) and generalizations to other boundary conditions were obtained independently by Parter and Rothman [89].

Consider now a finite element function u defined in a substructure Ω_i with diameter of order H. Changing variables to the reference substructure by $\hat{v}(\hat{x}) = v(F_i(\hat{x}))$, and using the bounds on the Jacobian of F_i, we obtain

$$\|u\|^2_{L^2(\Omega_i)} \asymp H^d \|\hat{u}\|^2_{L^2(\hat{\Omega})}, \tag{3.4}$$

and

$$|u|^2_{H^1(\Omega_i)} \asymp H^{d-2} |\hat{u}|^2_{H^1(\hat{\Omega})}, \tag{3.5}$$

where d is the dimension and is equal to 1, 2, or 3.

These estimates can be viewed as spectral equivalences of the stiffness and mass matrices generated by the norms and the basis introduced above. Indeed, the nodal basis $\{\hat{\phi}_j^{\hat{h}}\}$ is mapped, by interpolation at the GLL nodes, to the nodal basis of $P^N(\hat{\Omega})$. Then, (3.3) can be written as

$$\underline{\hat{u}}^T \hat{K}_h \underline{\hat{u}} \asymp \underline{\hat{u}}^T \hat{K}_N \underline{\hat{u}}, \tag{3.6}$$

where $\underline{\hat{u}}$ is the vector of nodal values of both $\hat{u}_N$ or $\hat{u}_h$, and $\hat{K}_h$ and $\hat{K}_N$ are the stiffness matrices associated with $|\cdot|^2_{H^1(\hat{\Omega})}$ and $a_{\hat{Q}}(.,.)$.

Let $K_h^{(i)}$ and $K_N^{(i)}$ be the stiffness matrices generated by the bases $\{\phi_j^h\}$ and $\{\phi_j^N\}$, respectively, for all nodes j in the closure of Ω_i, and by using $|\cdot|^2_{H^1(\Omega_i)}$ and $a_{Q,\Omega_i}(\cdot,\cdot)$, respectively. Here, $a_{Q,\Omega_i}(\cdot,\cdot)$ is the restriction of $a_Q(\cdot,\cdot)$ to the subdomain Ω_i. If $\underline{u}$ is the vector of nodal values, and $\underline{u}^{(i)}$ is its restriction to $\overline{\Omega}_i$, then

$$\underline{u}^{(i)^T} K_h^{(i)} \underline{u}^{(i)} \asymp \underline{u}^{(i)^T} K_N^{(i)} \underline{u}^{(i)},$$

by (3.3) and (3.5). The stiffness matrices K_N and K_h are formed by subassembly [48]:

$$\underline{u}^T K_h \underline{u} = \Sigma_i \quad \underline{u}^{(i)T} K_h^{(i)} \underline{u}^{(i)}; \tag{3.7}$$

an analogous formula holds for K_N. These last two relations imply that

$$\underline{u}^T K_h \underline{u} \asymp \underline{u}^T K_N \underline{u}. \tag{3.8}$$

This shows that K_h is an optimal preconditioner for K_N in terms of number of iterations. All these matrix equivalences, and their analogues in terms of norms, are hereafter called the FEM-SEM equivalence.

We next show that the same results also hold for the Schur complements S_h and S_N. The *interface* of the decomposition is defined as $\Gamma = \cup_{i=1}^M \partial\Omega_i \setminus \partial\Omega$. The Schur complement matrices S_h and S_N are obtained by the elimination of the interior nodes of each Ω_i by Cholesky's algorithm; see [48]. A function u_N is said to be (piecewise) Q-discrete harmonic in Ω if, $\forall i$,

$$a_{Q,\Omega_i}(u_N, v_N) = 0, \quad \forall v_N \in P^N(\Omega_i) \cap H_0^1(\Omega_i).$$

The definition of (piecewise) h-discrete harmonic functions is analogous. It is clear that $\underline{u}_\Gamma^T S_N \underline{u}_\Gamma = a_Q(u_N, u_N)$ and that $\underline{u}_\Gamma^T S_h \underline{u}_\Gamma = a(u_h, u_h)$, where u_h and u_N are respectively $h-$ and $Q-$discrete harmonic and $\underline{u}_\Gamma$ is the vector of nodal values on Γ of u_h and u_N.

The matrices S_h and S_N are spectrally equivalent. Indeed, by subassembly (3.7), it is enough to verify the spectral equivalence for each substructure separately. For the substructure Ω_i, we find:

$$\underline{u}_\Gamma^{(i)^T} S_N^{(i)} \underline{u}_\Gamma^{(i)} = a_{Q,\Omega_i}(u_N, u_N) \succeq a_{\Omega_i}(I_N^h(u_N), I_N^h(u_N)) \geq \qquad (3.9)$$
$$a_{\Omega_i}(\mathcal{H}_h(I_N^h u_N), \mathcal{H}_h(I_N^h u_N)) = a_{\Omega_i}(u_h, u_h) = \underline{u}_\Gamma^{(i)^T} S_h^i \underline{u}_\Gamma^{(i)},$$

where $\mathcal{H}_h$ is the h-discrete harmonic extension of the interface values, and I_N^h is the composition of $\hat{I}_N^h$ with F_i. Here, we have used the FEM-SEM equivalence and the well-known minimizing property of the discrete harmonic extension. The reverse inequality is obtained in the same way.

This equivalence implies that S_h is an optimal preconditioner for S_N, in terms of number of iterations. However, as before, the action of the inverse of S_h is too expensive to produce an efficient preconditioner for large problems.

In his Master's thesis [88], Pahl proposed the replacement of K_h and S_h by preconditioners $\tilde{K}_h$ and $\tilde{S}_h$, respectively. If the condition number satisfies

$$\kappa(\tilde{K}_h^{-1} K_h) \leq C(N), \qquad (3.10)$$

with a moderately increasing function $C(N)$, then a simple Rayleigh quotient argument shows that $\kappa(\tilde{K}_h^{-1} K_N) \preceq C(N)$; an analogous bound can be derived for $\tilde{S}_h^{-1} S_N$. $\tilde{K}_h$ and $\tilde{S}_h$ are domain decomposition preconditioners based on $\mathcal{T}^h$, and are designed so that the action of their inverses on vectors are inexpensive to evaluate.

In the next three sections, we define our preconditioners and then establish (3.10) and its analogue for S_h and $\tilde{S}_h^{-1}$. We note that the triangulation $\mathcal{T}_h$ is not shape-regular, and that all the bounds of this form for Schwarz preconditioners previously established in the literature require some kind of inverse condition, or regularity of the triangulation, which, as pointed out in Section 3.2, does not hold for the GLL mesh.

3.3 Technical Results

In this section, we present the technical lemmas needed to prove our results. As is clear from the start, we draw heavily upon the results and techniques of Dryja, Smith, and Widlund [48].

3.3.1 Some estimates for non-regular triangulations

In this section, we develop all the estimates necessary to extend the technical tools developed in [48] to the case of non-regular hexahedral triangulations. We recall that $\hat{\Omega} = [-1, +1]^3$ is the reference *substructure*, and $\mathcal{T}^{\hat{h}}$ its triangulation generated by the GLL mesh. Let $\breve{K} = [-1, +1]^3$ be the reference *element*, and let $\hat{K} \subset \hat{\Omega}$ be a parallelepiped of $\mathcal{T}^{\hat{h}}$ with sides h_1, h_2 and h_3; these mesh parameters are not necessarily comparable in size. The function $\hat{u}$ is a trilinear (Q_1) function defined in $\hat{K}$. In this subsection, we use hats to represent functions defined in $\hat{K}$, and no superscript for points of $\hat{K}$.

Our first result provides expressions of the $L^2(\hat{K})$- and $H^1(\hat{K})$-norms of a trilinear function $\hat{u}$ in terms of its nodal values. Let e_i be one of the coordinate directions of $\hat{K}$, and let a, b, c and d be the vertices of one of the faces that are perpendicular to e_i. Let a', b', c', and d' be the corresponding points on the parallel face. x_α denotes a generic vertex of $\hat{K}$.

Lemma 3.3.1 *Let $\hat{u}$ be trilinear in $\hat{K}$. Then,*

$$\|\hat{u}\|^2_{L^2(\hat{K})} \asymp h_1 h_2 h_3 \sum_{x_\alpha \in \hat{K}} (\hat{u}(x_\alpha))^2, \tag{3.11}$$

and

$$\|\partial_{x_i} \hat{u}\|^2_{L^2(\hat{K})} \asymp \frac{h_1 h_2 h_3}{h_i^2} \sum_{x_\alpha = a,b,c,d} (\hat{u}(x_\alpha) - \hat{u}(x'_\alpha))^2. \tag{3.12}$$

Proof. These formulas follow by changing variables, and by using the equivalence of any pair of norms in the finite dimensional space $Q_1(\breve{K})$. □

In the next lemma, we give a bound on the gradient of a trilinear function in terms of bounds on the differences of the nodal values. Its proof is routine.

Lemma 3.3.2 *Let $\hat{u}$ be trilinear in $\hat{K}$ such that $|\hat{u}(a) - \hat{u}(b)| \leq C|a - b|/r$ for some constant C and parameter r, and for any two vertices a and b belonging to one face of $\hat{K}$. Then*

$$|\nabla \hat{u}| \leq \frac{C}{r},$$

where C is independent of the parameter r.

Lemma 3.3.3 *Let $\hat{u}$ be a trilinear function defined in $\hat{K}$, and let $\hat{\vartheta}$ be a C^1 function such that $|\nabla\hat{\vartheta}| \leq C/r$, and $|\hat{\vartheta}| \leq C$ for some constant C and parameter r. Then*

$$|\partial_{x_i} I^{\hat{h}}(\hat{\vartheta}\hat{u})|^2_{L^2(\hat{K})} \leq C(|\hat{u}|^2_{H^1(\hat{K})} + r^{-2}||\hat{u}||^2_{L^2(\hat{K})}). \tag{3.13}$$

Here C is independent of N and r, and $I^{\hat{h}}$ is the Q_1-interpolant using the values at the vertices of $\hat{K}$.

Proof. By equation (3.12), and letting h_1, h_2, and h_3 be the sides of the element $\hat{K}$:

$$||\partial_{x_i} I^{\hat{h}}(\hat{\vartheta}\hat{u})||^2_{L^2(\hat{K})} \preceq \frac{h_1h_2h_3}{h_i^2} \sum_{x=a,b,c,d} (\hat{u}(x)\hat{\vartheta}(x) - \hat{u}(x')\hat{\vartheta}(x'))^2$$

Each term in the sum above can be bounded by

$$\begin{aligned}&\left(\hat{u}(x)\hat{\vartheta}(x) - \hat{u}(x)\hat{\vartheta}(x') + \hat{u}(x)\hat{\vartheta}(x') - \hat{u}(x')\hat{\vartheta}(x')\right)^2 \leq \\ &\qquad 2\left((\hat{u}(x))^2(\hat{\vartheta}(x) - \hat{\vartheta}(x'))^2 + (\hat{u}(x) - \hat{u}(x'))^2(\hat{\vartheta}(x'))^2\right)\end{aligned}$$

The bound on $\nabla\hat{\vartheta}$ implies that $|\hat{\vartheta}(x) - \hat{\vartheta}(x')| \preceq h_i/r$, and therefore

$$\begin{aligned}||\partial_{x_i} I^{\hat{h}}(\hat{\vartheta}\hat{u})||^2_{L^2(\hat{K})} &\preceq \frac{h_1h_2h_3}{h_i^2}\Big(\sum_{x=a,b,c,d} (\hat{u}(x) - \hat{u}(x'))^2 + \sum_{x=a,b,c,d} (\hat{u}(x))^2 \frac{h_i^2}{r^2}\Big) \\ &\preceq |\hat{u}|^2_{H^1(\hat{K})} + r^{-2}||\hat{u}||^2_{L^2(\hat{K})},\end{aligned} \tag{3.14}$$

since $\hat{\vartheta}$ is bounded. $\square$

3.3.2 Further technical tools

The iterative substructuring algorithms are based on subspaces directly related to the interiors of the substructures, and the faces, edges and vertices. Let Ω_{ij} be the union of two substructures Ω_i, Ω_j, and their common face $\mathcal{F}_k$. Let $\mathcal{W}_j$ represent the wirebasket of the subdomain Ω_j, i.e. the union of all its edges and vertices. We note that a face in the interior of the region Ω is common to exactly two substructures, an interior edge is shared by more than two, and an interior vertex is common to still more substructures. All the substructures, faces, and edges are regarded as open sets.

The following observations greatly simplify our analysis in the next sections. The preconditioner $\tilde{S}_h$ that we propose is defined by subassembly of the matrices $\tilde{S}_h^{(i)}$; see Section 3.4. We then restrict our analysis to one substructure. The results for the whole region follow by a standard Rayleigh quotient argument. The assumption that the $\{F_i\}_{i=1}^M$ are arbitrary smooth mappings improves the flexibility of the triangulation, but does not make the analysis essentially different from the case of affine mappings. This is seen from the estimates in Section 3.2, where we have used only bounds on the Jacobian and inverse of the Jacobian of F_i. Therefore, without loss of generality, we assume, from now on, that the F_i are affine mappings. Throughout this subsection, u is a finite element function belonging to P^h.

For a proof of Lemma 3.3.4 and a general discussion, see Bramble and Xu [21].

Lemma 3.3.4 *Let $Q^H u$ be the L^2 projection of $u \in P^h(\Omega)$ onto the coarse space V^H. Then,*

$$||u - Q^H u||^2_{L^2(\Omega)} \preceq H^2 |u|^2_{H^1(\Omega)},$$

and

$$|Q^H u|^2_{H^1(\Omega)} \preceq |u|^2_{H^1(\Omega)}.$$

We remark that these bounds are not necessarily independent of the values k_i of the coefficient. A sufficient condition to guarantee this independence is that the coefficients k_i satisfy a quasi-monotone condition; see [47].

In what follows, some of the results are stated for substructures of diameter proportional to H, but the arguments are given only for a reference substructure. The introduction of the scaling factors into the final formulas is, by the results of Section 3.2, routine.

Lemma 3.3.5 *Let $\bar{u}_{\mathcal{W}_j}$ be the average value of u on $\mathcal{W}_j$, the wirebasket of subdomain Ω_j. Then*

$$||u||^2_{L^2(\mathcal{W}_j)} \preceq (1 + \log(N))||u||^2_{H^1(\Omega_j)},$$

and

$$||u - \bar{u}_{\mathcal{W}_j}||^2_{L^2(\mathcal{W}_j)} \preceq (1 + \log(N))|u|^2_{H^1(\Omega_j)}.$$

Similar bounds also hold for an individual substructure edge.

Proof. In the reference substructure, we know that $P^{\hat{h}} \subset V^{\hat{h}}$, where $P^{\hat{h}}$ was defined in Section 3.2, and $V^{\hat{h}}$ is a Q_1 finite element space defined on a shape-regular triangulation that is a refinement of $\mathcal{T}^{\hat{h}}$; we can refine all the elements of $\mathcal{T}^{\hat{h}}$ with sides bigger than, say, $3\hat{h}/2$, where $\hat{h} \asymp 1/N^2$. Now we apply Lemma 4.3 in [48], a well-known result for shape-regular triangulations, to get both estimates. $\square$

In the abstract Schwarz convergence theory described in Section 2.3, the crucial point in the estimate of the rate of convergence of a two-level algorithm is the proof that all functions in the finite element space can be decomposed into components belonging to the subspaces, in such a way that the sum of the resulting energies are uniformly, or almost uniformly, bounded with respect to the parameters H and N. The main technique for deriving such a decomposition is the use of a suitable partition of unity. In the next two lemmas, we construct functions that are used to define such partitions of unity.

Lemma 3.3.6 *Let $\mathcal{F}_k$ be the face common to Ω_i and Ω_j, and let $\theta_{\mathcal{F}_k}$ be the function in $P^h(\Omega)$ that is equal to one at the interior nodes of $\mathcal{F}_k$, zero at the remaining nodes of $\partial\Omega_i \cup \partial\Omega_j$, and discrete harmonic in Ω_i and Ω_j. Then*

$$|\theta_{\mathcal{F}_k}|^2_{H^1(\Omega_i)} \preceq (1 + \log(N))H.$$

The same bound also holds for the other subregion Ω_j.

Proof. We define functions $\hat{\theta}_{\mathcal{F}_k}$ and $\hat{\vartheta}_{\mathcal{F}_k}$ in the reference cube; $\theta_{\mathcal{F}_k}$ and $\vartheta_{\mathcal{F}_k}$ are obtained, as usual, by mapping; see Section 3.2. We construct the function $\hat{\vartheta}_{\mathcal{F}_k}$ with the same boundary values as $\hat{\theta}_{\mathcal{F}_k}$, and then prove that the bound given in the Lemma holds for $\hat{\vartheta}_{\mathcal{F}_k}$. The standard energy minimizing property of discrete harmonic functions then gives the estimate for $\hat{\theta}_{\mathcal{F}_k}$. The six functions $\hat{\vartheta}_k$ which correspond to the six faces of the cube also form a partition of unity at all nodes belonging to the closure of the substructure except those on the wirebasket; this property is used in the next lemma.

We divide the substructure into twenty-four subtetrahedra by connecting its center C to all the vertices and to all the six centers C_k of the faces, and by drawing the diagonals of the faces of $\hat{\Omega}$; see Fig. 3.1.

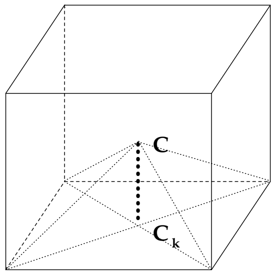

Figure 3.1: One of the segments CC_k

The function $\hat{\vartheta}_{\mathcal{F}_k}$ associated with the face $\mathcal{F}_k$ is defined to be $1/6$ at the point C. The values at the centers of the faces are $\hat{\vartheta}_{\mathcal{F}_k}(C_j) = \delta_{jk}$, for $j = 1, ..., 6$, where δ_{jk} is the Kronecker symbol. $\hat{\vartheta}_{\mathcal{F}_k}$ is linear on the segments CC_j. The values inside each subtetrahedron formed by a segment CC_j and one edge of $\mathcal{F}_j$ are defined to be constant on the intersection of any plane through that edge, and are given by the value, already known, on the segment CC_j. Next, the whole function $\hat{\vartheta}_{\mathcal{F}_k}$ is modified to be a piecewise Q_1 function on $\mathcal{T}^{\hat{h}}$ by interpolating at the GLL nodes; the values of this finite element function at the nodes on the wirebasket are defined to be equal to zero.

We claim that $|\nabla \hat{\vartheta}_{\mathcal{F}_k}(x)| \leq C/r$, where x is a point belonging to any element $\hat{K}$ that does not touch any edge of the cube, and r is the distance from the center of $\hat{K}$ to the closest edge of the cube. Let $\overline{ab}$ be a side of $\hat{K}$. We analyze in detail the situation depicted in Fig. 3.2, where $\overline{ab}$ is parallel to CC_k. Let e be the intersection of the plane containing these two segments with the edge of the cube that is closest to $\overline{ab}$. Then $|\hat{\vartheta}_{\mathcal{F}_k}(b) - \hat{\vartheta}_{\mathcal{F}_k}(a)| \preceq D$, by the construction of $\hat{\vartheta}_{\mathcal{F}_k}$, where D is the size of the radial projection with center e of $\overline{ab}$ onto CC_k. By similarity of triangles, we may write:

$$|\hat{\vartheta}_{\mathcal{F}_k}(b) - \hat{\vartheta}_{\mathcal{F}_k}(a)| \preceq \frac{\mathrm{dist}(a, b)}{r'}, \tag{3.15}$$

where r' is the distance between e and the midpoint of $\overline{ab}$. Here we have used that the distance between e and CC_k is of order 1. If the segment $\overline{ab}$ is not parallel to CC_k, the difference $|\hat{\vartheta}_{\mathcal{F}_k}(b) - \hat{\vartheta}_{\mathcal{F}_k}(a)|$ is even smaller, and (3.15) is still valid. Notice that r' is within a multiple of 2 of r. Therefore Lemma 3.3.2 implies that $|\nabla \hat{\vartheta}_{\mathcal{F}_k}(x)| \leq C/r$.

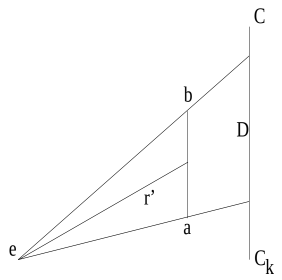

Figure 3.2: Geometry underlying equation (18)

In order to estimate the energy of $\hat{\vartheta}_{\mathcal{F}_k}$, we start with the elements $\hat{K}$ that touch an edge $\mathcal{E}$ of the cube. Let h_3 be the side of $\hat{K}$ which is parallel to $\mathcal{E}$. Then h_3 is greater than or equal to the other sides of $\hat{K}$, by the properties of the GLL nodes, as explained in Section 3.2. Since the nodal values of $\hat{\vartheta}_{\mathcal{F}_k}$ in $\hat{K}$ are bounded by 1, by the construction of $\hat{\vartheta}_{\mathcal{F}_k}$, we have:

$$|\hat{\vartheta}_{\mathcal{F}_k}|_{H^1(\hat{K})} \preceq h_3,$$

by using equation (3.12). Summing over $\hat{K}$, we conclude that the energy of $\hat{\vartheta}_{\mathcal{F}_k}$ is bounded independently of N for the union of all elements that touch the edges of the cube.

To estimate the contribution to the energy from the other elements of the substructure, we consider one subtetrahedron at a time and introduce cylindrical coordinates using the substructure edge, that belongs to the subtetrahedron, as the z-axis. The bound now follows from the bound on the gradient of $\hat{\vartheta}_{\mathcal{F}_k}$ given above and elementary considerations. We refer to [48] for more details, and also to the proof of the next lemma, where a similar computation is performed. □

The following lemma corresponds to Lemma 4.5 in [48]. This lemma and the previous one are the keys to avoiding the use of $H_{00}^{1/2}$ estimates and extension theorems in the analysis of our algorithms.

Lemma 3.3.7 *Let $\vartheta_{\mathcal{F}_k}(x)$ be the function introduced in the proof of Lemma 3.3.6, let $\mathcal{F}_k$ be a face of the substructure Ω_j, and let I^h denote the interpolation operator associated with the finite element space P^h and the image of the GLL points under* *the mapping F_j. Then,*

$$\sum_k I^h(\vartheta_{\mathcal{F}_k} u)(x) = u(x),$$

for all nodal points $x \in \overline{\Omega}_j$ that do not belong to the wirebasket $\mathcal{W}_j$, and

$$|I^h(\vartheta_{\mathcal{F}_k} u)|^2_{H^1(\Omega_j)} \preceq (1 + \log(N))^2 ||u||^2_{H^1(\Omega_j)}.$$

Proof. The first part is trivial from the construction of $\hat{\vartheta}_{\mathcal{F}_k}$ made in the previous lemma. For the second part, we work in the reference substructure, and first estimate the sum of the energy of all the elements $\hat{K}$ that touch an edge $\mathcal{E}$ of the wirebasket. We provide a detailed argument only for $\hat{K}$ touching $\mathcal{F}_k$; the other elements that touch an edge are treated similarly. The nodal values of $I^{\hat{h}}(\hat{\vartheta}_{\mathcal{F}_k} \hat{u})$ in such an element are 0,0,0,0, $\hat{u}(a)$, $\hat{u}(b)$, $\hat{\vartheta}_{\mathcal{F}_k}(c)\hat{u}(c)$ and $\hat{\vartheta}_{\mathcal{F}_k}(d)\hat{u}(d)$; $\hat{\vartheta}_{\mathcal{F}_k}$ lies between 0 and 1. Moreover, let h_3 be the side of $\hat{K}$ that is parallel to $\mathcal{E}$. Then $h_3 \succeq h_1$ and $h_3 \succeq h_2$, by the geometrical properties of the GLL mesh. Now, equation (3.12) implies:

$$|I^{\hat{h}}(\hat{\vartheta}_{\mathcal{F}_k} \hat{u})|^2_{H^1(\hat{K})} \preceq h_3(\hat{u}^2(a) + \hat{u}^2(b) + (\hat{\vartheta}_{\mathcal{F}_k}(c)\hat{u}(c))^2 + (\hat{\vartheta}_{\mathcal{F}_k}(d)\hat{u}(d))^2).$$

Then, applying (3.11) for the segments that are parallel to $\mathcal{E}$, and Lemma 3.3.5, we have:

$$\sum_{\hat{K}} |I^{\hat{h}}(\hat{\vartheta}_{\mathcal{F}_k} \hat{u})|^2_{H^1(\hat{K})} \preceq (1 + \log(N))||\hat{u}||^2_{H^1(\Omega_j)},$$

where this sum is taken over all elements $\hat{K}$ that touch the wirebasket of $\hat{\Omega}$.

We next bound the energy of the interpolant for the other elements. By the proof of the previous lemma, $|\nabla \hat{\vartheta}_{\mathcal{F}_k}| \leq C/r$, where r is the distance between the element $\hat{K}$ and the nearest edge of $\hat{\Omega}$. Then, Lemma 3.3.3 implies that

$$\sum_{\hat{K} \subset \hat{\Omega}} |I^{\hat{h}}(\hat{\vartheta}_{\mathcal{F}_k} \hat{u})|^2_{H^1(\hat{K})} \preceq \sum_{\hat{K} \subset \hat{\Omega}} (|\hat{u}|_{H^1(\hat{K})} + r^{-2}||\hat{u}||^2_{L^2(\hat{K})}),$$

where the sum is taken over all elements $\hat{K}$ that do not touch the wirebasket of $\hat{\Omega}$.

The bound of the first term in the sum is trivial. To bound the second term, we partition the elements of $\hat{\Omega}$ into groups, in accordance to the closest edge of $\hat{\Omega}$; the exact rule for the assignment of the elements that are halfway between is of no importance. For each edge of the wirebasket, we use a local cylindrical coordinate system with the z axis coinciding with the edge, and the radial direction, r, normal to it. The sum restricted to each of these groups of elements can be estimated by an integral

$$\sum_{\hat{K}\subset\hat{\Omega}} r^{-2}||\hat{u}||^2_{L^2(\hat{K})} \preceq \int_{r=\hat{h}}^{C}\int_{\theta}\int_{z}(\hat{u})^2\frac{r}{r^2}\,dz\,d\theta\,dr.$$

The integral with respect to z can be bounded by using Lemma 3.3.5. We obtain

$$\sum_{\hat{K}\subset\hat{\Omega}} r^{-2}||\hat{u}||^2_{L^2(\hat{K})} \preceq (1+\log(C/\hat{h}))||\hat{u}||^2_{H^1(\hat{\Omega})}\int_{r=\hat{h}}^{C} r^{-1}dr$$

and thus

$$\sum_{\hat{K}\subset\hat{\Omega}} |I^{\hat{h}}(\hat{\vartheta}_{\mathcal{F}_k}\hat{u})|^2_{H^1(\hat{K})} \preceq (1+\log(C/\hat{h}))^2||\hat{u}||^2_{H^1(\hat{\Omega})}.$$

□

We note that this proof is an extension of an argument given in [48] for shape-regular meshes, and that equation (3.13) replaces the use of the inverse inequality, which if used here would introduce the bad aspect ratios of the elements into the estimates.

Lemma 3.3.8 *Let $\bar{u}_{\partial\mathcal{F}_k}$, and $\bar{u}_{\mathcal{W}_j}$ be the averages of u on $\partial\mathcal{F}_k$, and $\mathcal{W}_j$, respectively. Then,*

$$(\bar{u}_{\partial\mathcal{F}_k})^2 \preceq \frac{1}{H}||u||^2_{L^2(\partial\mathcal{F}_k)},$$
$$(\bar{u}_{\mathcal{W}_j})^2 \preceq \frac{1}{H}||u||^2_{L^2(\mathcal{W}_j)}.$$

The proof is a direct consequence of the Cauchy–Schwarz inequality.

Lemma 3.3.9 *Let $u \in P^h(\Omega_i)$ be zero on the mesh points of the faces of Ω_j and discrete harmonic in Ω_j. Then,*

$$|u|^2_{H^1(\Omega_j)} \preceq ||u||^2_{L^2(\mathcal{W}_j)}.$$

This result follows by estimating the energy norm of the zero extension of the boundary values using equation (3.12) and by noting that the harmonic extension has a smaller energy.

3.4 Iterative Substructuring Algorithms

At this point, we can propose and analyze several iterative substructuring methods previously developed for finite elements. We choose the wirebasket algorithm proposed by Smith [108] because it is efficient, and its analysis raises all the important technical issues. In a practical problem, the choice between the many alternatives now known should be made on the basis of the theoretical results that can be derived from our theory, as well as numerical experimentation.

Smith's algorithm is a wirebasket based method, and it is also described as Algorithm 6.4 in [48] in the context of standard finite elements. It can be viewed as a block-diagonal preconditioner after transforming S_h into a convenient basis, and the same is true for our algorithm.

By the abstract framework of Schwarz methods of Section 2.3, we know that in order to describe the algorithm we only need to prescribe subspaces, the sum of which spans the whole space of h-discrete harmonic functions of $P_0^h(\Omega)$, and one bilinear form for each subspace.

For each internal face $\mathcal{F}_k$, we let $V_{\mathcal{F}_k}$ be the space of h-discrete harmonic functions that vanish at all the interface nodes that do not belong to this face. The functions in $V_{\mathcal{F}_k}$ have support in $\overline{\Omega}_{ij}$, the union of the two substructures $\overline{\Omega}_i$ and $\overline{\Omega}_j$ that share the face $\mathcal{F}_k$. The bilinear form used for these spaces is $a(\cdot,\cdot)$.

The wirebasket subspace is the range of the following interpolation operator:

$$I_W^h u = \sum_{x_k \in \mathcal{W}_h} u(x_k)\varphi_k + \sum_k \bar{u}_{\partial F^k}\theta_{F^k}.$$

Here, φ_k is the discrete harmonic extension of the standard nodal basis functions ϕ_k, $\mathcal{W}_h$ is the set of nodes in the union of all the wirebaskets, and $\bar{u}_{\partial F^k}$ is the average of u on ∂F^k. The bilinear form for this coarse subspace is given by

$$b_0(u,u) = (1+\log(N))\sum_i k_i \inf_{c_i} ||u-c_i||^2_{L^2(\mathcal{W}_i)}.$$

These subspaces and bilinear forms define, via the Schwarz framework, a preconditioner of S_h that we call $\tilde{S}_{h,WB}$.

Theorem 3.4.1 *For the preconditioner $\tilde{S}_{h,WB}$, we have*

$$\kappa(\tilde{S}_{h,WB}^{-1} S_N) \leq C(1 + log(N))^2,$$

where the constant C is independent of N, H, and the values k_i of the coefficient.

Proof. We can apply, word by word, the proof of theorem 6.4 in [48] to the matrix S_h, using now the tools developed in Section 3.3. This gives

$$\kappa(\tilde{S}_{h,WB}^{-1} S_h) \preceq (1 + log(N))^2.$$

The harmonic FEM-SEM equivalence (3.9) and a Rayleigh quotient argument complete the proof; see Section 3.2. $\square$

The next algorithm is obtained from the previous one by the discrete harmonic FEM-SEM equivalence, by which we find a preconditioner $\tilde{S}_{N,WB}$ from $\tilde{S}_{h,WB}$. The subspaces that define the preconditioner are now contained in the space of Q-discrete harmonic functions of $P_0^N(\Omega)$.

Each face subspace, related to a face $\mathcal{F}_k$, consists of the set of all Q-discrete harmonic functions that are zero at all the interface nodes that do not belong to the interior of the face $\mathcal{F}_k$. The bilinear form for these spaces is $a_Q(\cdot, \cdot)$.

The wirebasket subspaces are defined as before, by prescribing the values at the GLL(N) nodes on a face as the average of the function on the boundary of the face. The bilinear form used for the wirebasket subspace is $b_0^Q(\cdot, \cdot)$, obtained from $b_0(\cdot, \cdot)$ by applying the GLL(N) quadrature to compute the L^2-norm on each edge of the wirebasket. This is exactly the wirebasket method based on GLL quadrature described in [94].

The following lemma shows the equivalence of the two functions u_h and u_N with respect to the bilinear forms $b_0(\cdot, \cdot)$ and $b_0^Q(\cdot, \cdot)$, respectively.

Lemma 3.4.1 *Let u_h be a Q_1 finite element function on the GLL(N) mesh on the interval $I = [-1, +1]$, and let u_N be its polynomial interpolant using the nodes of this mesh. Then,*

$$\inf_c ||u_h - c||_{L^2(I)}^2 \asymp \inf_c \sum_{j=0}^{N} (u_N(\xi_j) - c)^2 \rho_j$$

Proof. The GLL(N) quadrature has the following important property: For any polynomial u_N of degree N defined on I,

$$||u_N||^2_{L^2(\Lambda)} \le \sum_{j=0}^{N} u_N^2(\xi_j)\rho_j \le 3||u_N||^2_{L^2(\Lambda)};$$

see [12, p. 75]. Therefore, it is enough to prove that:

$$\inf_c ||u_h - c||^2_{L^2(I)} \asymp \inf_c ||u_N - c||^2_{L^2(I)}.$$

We prove only the $\preceq$ part of this last estimate, since the opposite inequality is analogous. The inequality without the infimum is valid for the constant c_r that realizes the inf in the right hand side by the FEM-SEM equivalence. By taking the inf of the left hand side the inequality is preserved. $\square$

Theorem 3.4.2 *For the preconditioner $\tilde{S}_{N,WB}$, we have*

$$\kappa(\tilde{S}^{-1}_{N,WB} S_N) \le C(1 + \log(N))^2$$

where the constant is independent of the parameters H, N, and the values k_i of the coefficient.

Proof. In this proof, the functions with indices h and N are h- and Q- discrete harmonic functions respectively, and they agree at the GLL nodes that belong to the interface Γ. As observed in Section 3.2, it is enough to analyze one substructure Ω_i at a time, and prove the following equivalence:

$$b^Q_{0,\mathcal{W}_i}(u_N, u_N) + \sum_{\mathcal{F}_k \subset \Omega_i} k_i |u_N - \overline{u}_{N,\partial\mathcal{F}_k} \theta_{N,\mathcal{F}_k}|^2_{H^1(\Omega_i)} \asymp \tag{3.16}$$
$$b_{0,\mathcal{W}_i}(u_h, u_h) + \sum_{\mathcal{F}_k \subset \Omega_i} k_i |u_h - \overline{u}_{h,\partial\mathcal{F}_k} \theta_{h,\mathcal{F}_k}|^2_{H^1(\Omega_i)},$$

where the subscript $\mathcal{W}_i$ means that only the contribution from the wirebasket of Ω_i is used to define the bilinear form. We prove only the $\preceq$ part; the proof of the reverse inequality is analogous. We first note that Lemma 3.4.1 bounds the first term on the left hand side by the first term on the right hand side.

Each term in the sum on the left hand side can be bounded from above by

$$2k_i |u_N - \overline{u}_{h,\partial\mathcal{F}_k} \theta_{N,\mathcal{F}_k}|^2_{H^1(\Omega_i)} + 2k_i |(\overline{u}_{h,\partial\mathcal{F}_k} - \overline{u}_{N,\partial\mathcal{F}_k}) \theta_{N,\mathcal{F}_k}|^2_{H^1(\Omega_i)}.$$

The first term of this expression can be bounded from above by the corresponding term on the right hand side by using the harmonic FEM-SEM equivalence. The second term is bounded by

$$Ck_iH(1+\log(N))|\overline{u}_{h,\partial\mathcal{F}_k} - \overline{u}_{N,\partial\mathcal{F}_k}|^2 =$$
$$Ck_iH(1+\log(N))|\overline{(u_h - c_{h,\mathcal{W}_i})}_{\partial\mathcal{F}_k} - \overline{(u_N - c_{h,\mathcal{W}_i})}_{\partial\mathcal{F}_k}|^2,$$

where $c_{h,\mathcal{W}_i}$ is the average of u_h over $\mathcal{W}_i$. Here we have used that the estimate on the energy norm of $\theta_{h,\mathcal{F}_k}$, given in Lemma 3.3.6, implies a similar estimate for $\theta_{N,\mathcal{F}_k}$, by (3.9). Applying the Cauchy-Schwarz inequality, as in Lemma 3.3.8, and the FEM-SEM equivalence, we can bound this last expression in terms of the first term of the right hand side of equation (3.16). $\square$

3.5 Overlapping Schwarz Algorithms

We now consider the additive overlapping Schwarz method, which is discussed, for standard finite element discretizations, e.g., in [50, 51]. We recall that an abstract framework, Theorem 2.3.1, is available for the analysis of this type of algorithm. Here we only discuss the additive version, but the analysis can also be extended in a standard way to the multiplicative variant by using Theorem 2.3.2. This variant has proven more effective in many practical problems.

As in the previous section, a preconditioner $\tilde{K}_h$ for K_h is specified by a set of local spaces together with a coarse space. We also have to provide bilinear forms (approximate solvers) for the elliptic problems restricted to each of these subspaces. Here we work with exact solvers, i.e. the bilinear form is $a(\cdot,\cdot)$. The extension to approximate solvers is straightforward.

In the context of spectral elements, the following construction was first proposed by Pahl [88]. The domain Ω is covered by substructures Ω_i, which are the original spectral elements. We enlarge each of them, to produce overlapping subregions Ω_i', in such a way that the boundary of Ω_i' does not cut through any element of the triangulation $\mathcal{T}^h$ generated by the GLL nodes. The overlap δ denotes the minimum distance between the boundaries of Ω_i and Ω_i'. When δ is proportional to H the overlap is called generous, and when δ is comparable to the size of the elements of

$\mathcal{T}^h$, we speak of a small overlap. For the sake of simplicity, we again restrict our analysis to the case when all the mappings F_i are affine. The general situation can be treated similarly.

The local spaces are given by $P_0^h(\Omega_i')$, the set of functions in $P_0^h(\Omega)$ that vanish at all the nodes on and outside $\partial\Omega_i'$. The coarse space is the Q_1 finite element space defined on the mesh generated by the subregions Ω_i, the elements of the coarse triangulation, which are shape-regular by assumption; see Section 1.5. This setting incorporates both small and generous overlap.

Theorem 3.5.1 *Pahl's additive Schwarz algorithm satisfies:*

$$\kappa(\tilde{K}_{h,AS}^{-1} K_N) \preceq (1 + H/\delta)$$

The constant C is independent of the parameters H, N, and δ.

Proof. As before, we follow the proof of the analogous theorem for shape-regular finite elements; see Theorem 3 in [51]. The proof applies, word by word, except for the estimate of $a_K(I_h(\theta_i w_h), I_h(\theta_i w_h))$ where I_h is the interpolation operator, $\{\theta_i\}$ a partition of unity (different from the one described in Lemma 3.3.6), w_h a finite element function, and $a_K(\cdot,\cdot)$ the restriction of $a(\cdot,\cdot)$ to the single element $K \in \mathcal{T}^h$. It is known that θ_i can be found such that $|\theta_i| \leq 1$, and $|\nabla\theta_i| \leq C/\delta$. Lemma 3.3.3 gives:

$$a_K(I_h(\theta_i w_h), I_h(\theta_i w_h)) \leq C(|w_h|^2_{H^1(K)} + \frac{1}{\delta^2}\|w_h\|^2_{L^2(K)}),$$

where the constant C depends on the coefficients k_i. The rest of the proof follows without any change, and we obtain

$$\kappa(\tilde{K}_{h,AS}^{-1} K_h) \leq C(1 + H/\delta).$$

The FEM-SEM equivalence and a Rayleigh quotient argument complete the proof. $\square$

Remark 3.5.1 *Even though the theory does not rule out the possibility of growth of the constant C of Theorem 3.5.1 when the coefficient $k(x)$ has large jumps, only* *a very moderate increase has been observed in numerical experiments; see e.g. [65]. We also note that when the overlap is generous, the method is optimal in the sense that the condition number is uniformly bounded with respect to N and H.*

Remark 3.5.2 *In the present algorithm, the local spaces are allowed to be more general than those considered by Pavarino [91, 92, 93]. For each crosspoint x_ℓ, Pavarino defines an extended subdomain Ω'_ℓ as the union of all the subdomains that contain x_ℓ as a vertex. Therefore, δ is always on the order of H.*

We now apply the FEM-SEM equivalence to the subspaces that define $\tilde{K}_{h,AS}$, to propose yet another preconditioner; this is the same technique used to derive the preconditioner $\tilde{S}_{N,WB}$ from $\tilde{S}_{h,WB}$. The coarse space is the same as the one for $\tilde{K}_{h,AS}$, while the local spaces are given by

$$V^N_{\Omega'_i} = \{v_N \in P^N_0(\Omega) \;\text{ such that }\; I^h_N(v_N) \in P^h_0(\Omega'_i)\}.$$

Notice that the polynomials of $V^N_{\Omega'_i}$ are generally not equal to zero outside Ω'_i, and therefore $V^N_{\Omega'_i} \not\subset P^N_0(\Omega'_i)$.

These subspaces and the use of the bilinear forms $a_Q(\cdot,\cdot)$ and $a(\cdot,\cdot)$ for the local and coarse spaces, respectively, define our new preconditioner $\tilde{K}_{N,AS}$. Theorem 3.5.1 and a simple application of the FEM-SEM equivalence for each of the local spaces immediately give:

Theorem 3.5.2

$$\kappa(\tilde{K}^{-1}_{N,AS} K_N) \leq C(1 + H/\delta).$$

Remark 3.5.3 *To the best of our knowledge, this preconditioner $\tilde{K}_{N,AS}$ is new. Even though $\tilde{K}_{h,AS}$ is superior to $\tilde{K}_{N,AS}$ for the model problem considered here, because the local problems are much easier to solve, the comparative efficiency in more complicated problems can only be determined by experiments.*

3.6 Comments on the Numerical Experiments by Pahl

We describe here some of the experiments performed by Pahl [88]. These experiments have motivated our analysis, and are used to illustrate the efficiency of some of the algorithms considered in the last two sections. For more details on this very thorough study, we refer to [88].

The region Ω was taken to be the unit square in the plane, and the tests were based on the two dimensional analogues of some of the methods described in the previous two sections. Ω was subdivided into a uniform $M \times M$ mesh of squares, with sides $H = 1/M$. In each square, polynomials of degree N were used. The coefficient $k(x)$ was equal to one, and the right hand side of (1.7) was chosen so that the exact solution was $u = xy(1-x)(1-y)$. The stopping criterion for the PCG iteration was a reduction of 10^{-5} in the Euclidean norm of the residual, and only iteration counts were reported. The experiments were performed for the finite element based preconditioners of the spectral element stiffness matrix; see our Theorems 3.4.1 and 3.5.1.

In a first set of experiments, the iterative substructuring methods were considered. Several preconditioners were studied. We focus our attention on the wirebasket preconditioner analyzed in Section 3.4, and an analogue of the balancing preconditioner of Mandel and Brezina [80]. For $N = 4$ and M between 2 and 12, the number of iterations was bounded by 10, and grew hardly at all, with increasing M. For $M = 7$ and N between 4 and 12, the iteration count increased very slowly with N, and was bounded by 11 for the wirebasket and balancing preconditioners. We remark that the analysis of the finite element balancing preconditioner for the spectral element method is a straightforward application of the results of Section 3.4 and the existing theory for shape-regular finite elements; cf. [90]. Pahl's results corroborate, in a clear cut way, the results of the theory.

The overlapping Schwarz preconditioner was studied in a second set of experiments. The domain was divided as before into $M \times M$ subdomains, polynomials of degree N were used within each subdomain, and δ was taken to be on the order of one or two mesh intervals. From the geometrical properties of the GLL mesh, it is easy to see that the bound on the condition number given by Theorem 3.5.1 grows like $(1 + H/\delta) \asymp N^2$. Hence, our theory predicts an iteration count which is linear in N and independent of M. The experiments performed by Pahl showed that for $N = 4$, the number of iterations grows very slowly with M between 2 and 12, and presents a sublinear growth when N increases from 4 to 12, for $M = 7$. The maximum iteration count was 22, achieved for $M = 7$ and $N = 12$. Our estimate seems pessimistic in its dependence on N, at least for this range of values, while it describes the dependence on the number of subdomains quite well.

An important practical question, also addressed by Pahl's experiments, is the choice of the most efficient overlap δ for a particular problem and decomposition of the domain. The greater the overlap, the smaller the iteration count, but since a more generous overlap also increases the work to solve the local problems, it is hard to decide in advance what the best δ would be. The results obtained by Pahl indicate that, for $M = 7$ and N between 4 and 9, one mesh size overlap appears to be a good choice in terms of total work on a serial machine, while a more generous overlap seems to be appropriate for larger N.

Chapter 4

Diagonal Edge Preconditioners in p-version and Spectral Element Methods

4.1 Introduction

Polynomials of high degree have been used extensively to approximate second order elliptic partial differential equations in the plane. Two well-known discretization schemes are the p-version finite element method [112], and the spectral element method [73, 74].

For each substructure Ω_i, the basis of the polynomial space $P^N(\Omega_i)$ is usually chosen so that it can be partitioned into sets of functions associated with the interior of the element, the individual edges, or the vertices.

Let the stiffness matrices corresponding to the p-version and spectral element methods for the homogeneous Dirichlet problem defined in one element be denoted by K_p and K_N, respectively. Let the usual bases for these methods, which will be described in Section 4.2, be used to generate these matrices. Then, the condition numbers satisfy:

$$\kappa(K_p) \asymp N^4 \quad \text{and} \quad \kappa(K_N) \asymp N^3; \tag{4.1}$$

see [12] and [97]. Here, and in what follows, $\asymp$ means that the ratio of the quantities being compared is bounded from above and below by constants independent of the degree N. These conditioning results are even worse for a domain partitioned into many elements, and they suggest that an unpreconditioned conjugate gradient method is likely to require many iterations; this is actually seen in numerical tests. Diagonal preconditioning of these full matrices has also been used, but the condition number still increases quadratically with N; see [12, 97].

Many domain decomposition preconditioners can be viewed as block-Jacobi preconditioners after an appropriate change of basis has been made. Each block is determined by a subspace of the discrete space, and by an exact or inexact solver; see [48]. The decomposition into subspaces corresponds to the elimination of the coupling between different sets of basis functions. We note that it has been determined experimentally that there is a very strong coupling between the interior and the standard interface basis functions [7]. A block-Jacobi preconditioner that eliminates the problem associated with this strong coupling has been proposed by Babuška, Craig, Mandel, and Pitkäranta [5] for the p-version finite element method in two dimensions. A change of basis is performed by computing the Schur complement with respect to the interior degrees of freedom; the new interface basis functions are orthogonal to the interior ones. In this new basis, the preconditioner is built from one block of relatively small dimension associated with a global problem, one block for each edge of the triangulation into elements, and one block for the interior of each element; exact solvers are used for all blocks. The condition number of this algorithm is bounded from above by $C(1 + \log(N))^2$; see [5]. This result can be extended straightforwardly to the spectral element method.

However, for all the implementations that we know of, the Schur complement blocks associated with the edges are preconditioned by their diagonals; in other words, inexact solvers are used to totally decouple the edge degrees of freedom. This substantially reduces the amount of work in constructing and evaluating the action of the preconditioner, because it eliminates the need to assemble and factor the edge Schur complement blocks, or, alternatively, the need to solve, in each iteration, Dirichlet problems in the unions of pairs of subregions; see [5, 48]. The use of this diagonal preconditioner has been found not to increase the condition number of the overall iterative process appreciably, if at all; see [5, 6, 7, 77]. No theoretical result is derived in [5] to support this particular variant of the algorithm.

In this chapter we prove that the blocks of the Schur complement associated with each edge, preconditioned by their diagonal, have condition numbers that grow approximately linearly with N, both for the p-version and for the spectral element method; see Theorems 4.3.1 and 4.4.1.

There are at least two applications of these results: the first immediate consequence is that, for the algorithm as actually implemented in [5] and [6], the condition number κ grows faster than polylogarithmically in N. In fact, κ satisfies

$$CN \leq \kappa \leq CN(1 + \log(N))^3.$$

A very similar estimate holds for the spectral element case. These results can also be easily extended to the p-version finite element applied to triangular elements; see Remark 4.3.1. The numerical results presented here demonstrate that the linear growth predicted by this estimate is present for large N, but also that the actual condition numbers are relatively small, even for N on the order of 50; see Figs. 4.1 and 4.2.

Many domain decomposition algorithms have also been developed for problems in three dimensions; see e.g. [38, 78, 79, 88, 94, 95, 100]. Again, the Schur complement blocks associated with the faces play a major role. We propose diagonal preconditioners for these blocks that produce very reasonable condition numbers for the overall process. In Section 4.6 we analyze one such diagonal preconditioner, applied to a three-dimensional elliptic equation. As above, an essentially linear upper bound on the condition number is derived, and some results of numerical experiments performed by Einar Rønquist are discussed, showing a substantial improvement over his original preconditioner, which used a different scaling of the variables on the interfaces.

4.2 On Polynomials and Trace Norms

Let $\Omega = [-1, +1]^2$, with the side $[-1, +1] \times \{-1\}$ identified with $\Lambda = [-1, +1]$. Let $P^N(\Lambda)$ be the space of polynomials of degree less than or equal to N, and let $P_0^N(\Lambda)$ be the set of polynomials in $P^N(\Lambda)$ that vanish at -1 and 1.

The space $P^N(\Omega)$ is given by tensorization of $P^N(\Lambda)$; analogously, $P_0^N(\Omega)$ is the tensor product of $P_0^N(\Lambda)$ with itself.

The Legendre polynomial basis $\{L_n\}_{n\geq 0}$ results from applying the Gram-Schmidt procedure to the set $1, x, x^2, \ldots,$ and normalizing so that $L_n(1) = 1$. The following properties are classical, and can be found in [12]:

$$((1-x^2)L_n'(x))' + n(n+1)L_n(x) = 0 \quad (n \geq 0), \tag{4.2}$$

$$\int_{-1}^{1} L_n^2(t)\, dt \;=\; \frac{1}{n+1/2} \quad (n \geq 0), \tag{4.3}$$

$$\int_{-1}^{x} L_n(t)\, dt \;=\; \frac{1}{2n+1}(L_{n+1}(x) - L_{n-1}(x)) \quad (n \geq 1). \tag{4.4}$$

We recall that for each N, the Gauss-Lobatto-Legendre quadrature of order N is denoted by GLL(N) and satisfies:

$$\forall p \in P^{2N-1}(\Lambda), \qquad \int_{-1}^{1} p(x)\, dx \;=\; \sum_{j=0}^{N} p(\xi_j)\rho_j.$$

Here, the quadrature points ξ_j are numbered in increasing order, and are the zeros of $(1-x^2)L_n'(x)$.

We next describe the basis functions used in the two methods. Following Babuška and Szabó [112], a polynomial basis for the p-version finite element method on $P^N(\Lambda)$ is defined by $\eta_0(x) = (1-x)/2$, $\eta_1(x) = (1+x)/2$, and

$$\eta_i(x) = \frac{1}{||L_{i-1}||_{L^2(\Lambda)}} \int_{-1}^{x} L_{i-1}(t)\, dt \quad (i \geq 2). \tag{4.5}$$

A p-version polynomial basis for $P^N(\Omega)$ is given by tensorization of this one dimensional basis.

The basis for the spectral element method on $P^N(\Lambda)$ is given by $\{\ell_j\}_{j=0}^{N}$, the Lagrange interpolation basis at the GLL points, i.e. $\ell_j(\xi_i) = \delta_{ij}$. The spectral element basis in two and three dimensions are also given by tensorization of the one dimensional basis.

The remainder of this section describes some Schur complement and trace norm properties. They are valid for both the p-version and the spectral element method. In each case, the basis can be partitioned into two sets of functions. The first is formed by the basis functions vanishing on $\partial\Omega$; these are the interior (i) basis functions. The others are the boundary (b) basis functions. The Schur complement is defined by $S = K_{bb} - K_{ib}^t K_{ii}^{-1} K_{ib}$, where the subscripts refer to blocks of the stiffness matrix K, ordered appropriately.

Let w be the restriction of a function of $P^N(\Omega)$ to $\partial\Omega$, let $\underline{w}_b$ be the vector of its boundary degrees of freedom, and let $||\cdot||_{H^1(\Omega)}$ and $|\cdot|_{H^1(\Omega)}$ be the standard Sobolev norm and semi-norm, respectively. We easily find that

$$\underline{w}_b^t S \underline{w}_b = \min_u |u|^2_{H^1(\Omega)} = |\mathcal{H}w|^2_{H^1(\Omega)}, \tag{4.6}$$

where the minimum is taken over all functions $u \in P^N(\Omega)$ such that $u|_{\partial\Omega} = w$, and $\mathcal{H}w$ is the function achieving the minimum. It is also easy to see that $(w_i, w_b)^t = \underline{\mathcal{H}w}$ satisfies:

$$K_{ii} w_i + K_{ib} w_b = 0.$$

The first expression of (4.6) defines a Schur complement symmetric bilinear form that only depends on the boundary values of the function, and can be estimated in terms of a trace norm. By Theorem 7.4 of [5], for any $w \in P^N(\Omega)$, there is a $u \in P^N(\Omega)$ with $u = w$ on $\partial\Omega$, such that

$$||u||_{H^1(\Omega)} \leq C||w||_{H^{1/2}(\partial\Omega)}. \tag{4.7}$$

We recall that the space $H_{00}^{1/2}(\Lambda)$ is the space of functions $v \in H^{1/2}(\partial\Omega)$ that vanish outside Λ, endowed with the norm $||v||_{H^{1/2}(\partial\Omega)}$. This space is isomorphic to the interpolation space $[L^2(\Lambda), H_0^1(\Lambda)]_{1/2}$; see [71]. An equivalent norm for $H_{00}^{1/2}(\Lambda)$ is given, in this context, by:

$$||v||^2_{H_{00}^{1/2}(\Lambda)} = \int_{-1}^{1}\int_{-1}^{1} (\frac{v(x) - v(y)}{x - y})^2 \, dx \, dy \; + \int_{-1}^{1} \frac{v^2(x)}{1 - x^2} \, dx; \tag{4.8}$$

see [85].

Let v_Λ be the trace on $\Lambda \sim [-1, 1] \times \{-1\}$ of a function of $P^N(\Omega)$ that vanishes on $\partial\Omega \setminus \Lambda$. Let $\underline{v}_\Lambda$ be the vector of degrees of freedom associated with the interior of Λ, and let S_Λ be the Schur complement restricted to these degrees of freedom. Then, by using (4.6) and (4.7), we obtain, $\forall v_\Lambda \in P_0^N(\Lambda)$:

$$c||v_\Lambda||^2_{H_{00}^{1/2}(\Lambda)} \leq \underline{v}_\Lambda^t S_\Lambda \underline{v}_\Lambda \leq C||v_\Lambda||^2_{H_{00}^{1/2}(\Lambda)}. \tag{4.9}$$

4.3 Diagonal Edge Preconditioning for the p-version in 2d

In what follows, we only work with S_Λ, the Schur complement related to Λ, as in (4.9), and we therefore drop the subscript Λ. Accordingly, the vectors consist of the degrees of freedom associated with the interior of Λ. The p-version and spectral element Schur complements are denoted by S_p and S_N, respectively.

Theorem 4.3.1 *Let D_p be the diagonal of S_p. Then, $\forall u \in P_0^N(\Lambda)$,*

$$\lambda_{min}^N(\underline{u}^t D_p \underline{u}) \leq \underline{u}^t S_p \underline{u} \leq \lambda_{max}^N(\underline{u}^t D_p \underline{u}), \tag{4.10}$$

with

$$c \leq \lambda_{max}^N \leq C, \tag{4.11}$$

and

$$\frac{c}{N\log(N)} \leq \lambda_{min}^N \leq \frac{C\log(N)}{N}. \tag{4.12}$$

Proof. Let $u(x) = \sum_{i=2}^N a_i \eta_i(x)$. By using (4.9) and the Courant-Fischer characterization of the extreme eigenvalues in terms of a Rayleigh quotient, we only need to estimate $||u||^2_{H_{00}^{1/2}(\Lambda)}$ in terms of $\sum_{i=2}^N a_i^2 ||\eta_i||^2_{H_{00}^{1/2}(\Lambda)}$. We start by showing that $||\eta_i||^2_{H_{00}^{1/2}(\Lambda)} \asymp 1/i$. Indeed, from (4.2), we have

$$\eta_i = -\frac{\sqrt{i-1/2}}{i(i-1)}(1-x^2)L'_{i-1}. \tag{4.13}$$

Then, by integrating by parts and using (4.2) again, the second term of (4.8) is easily seen to be of order $1/i^2$. To compute the first term of (4.8), we note that it is the square of the L^2-norm of a polynomial of degree less than or equal to $i-1$. We use the GLL$(i-1)$ quadrature rule which, by (1.13), gives the value of the integral, to within a multiplicative constant. The use of this quadrature rule results in a double sum that can be reduced to

$$\sum_{j=0}^{i-1}(\eta'_i(\xi_j))^2\rho_j^2,$$

since the ξ_j are zeros of the η_i, by (4.13). This last sum can be computed exactly by using (1.12) and (4.5) for $N = i-1$, and we find that $||\eta_i||^2_{H^{1/2}_{00}(\Lambda)} \asymp 1/i$.

We prove only the right inequality of (4.11), since the left inequality is clear by taking $u = \eta_2$. Given $u \in P_0^N(\Lambda)$, we define an extension of u, $E(u) \in P^N(\Omega)$ such that $E(u) = u$ on Λ, and $E(u)$ vanishes on $\partial\Omega \setminus \Lambda$. By (4.6) and (4.9), it suffices to show that $|E(u)|^2_{H^1(\Omega)} \leq C\sum_{i=2}^N (a_i^2/i)$. We choose $E(u)(x,y) = \sum_{i=2}^N a_i\eta_i(x)\psi_i(y)$, for some $\psi_i \in P^N(\Lambda)$, $\psi_i(-1) = 1$, $\psi_i(1) = 0$, that will be chosen momentarily. A simple computation shows that

$$|E(u)|^2_{H^1(\Lambda)} \;=\; \sum_{i,j=2}^N a_i a_j((\eta_i', \eta_j')(\psi_i, \psi_j) + (\eta_i, \eta_j)(\psi_i', \psi_j')).$$

Here, $(\cdot,\cdot)$ is the $L^2(-1,1)$ inner product. By using (4.3), (4.4), and (4.5), we find:

$$(\eta_i, \eta_i) \asymp \frac{1}{i^2}, \quad (\eta_i', \eta_i') \asymp 1. \tag{4.14}$$

Moreover, we also have $(\eta_i', \eta_j') = 0$ if $i \neq j$, and $(\eta_i, \eta_j) = 0$ if $|i-j| > 2$. These estimates together with the Cauchy-Schwarz inequality imply

$$|E(u)|^2_{H^1(\Lambda)} \;\leq\; C\sum_{i=1}^N a_i^2(||\psi_i||^2_{L^2(\Lambda)} + (1/i^2)||\psi_i'||^2_{L^2(\Lambda)}).$$

The piecewise linear interpolant using the GLL(N) points I_N^h is defined for any $v_N \in P^N(\Lambda)$ and is given by $v_h = I_N^h(v_N)$, $v_N(\xi_j) = v_h(\xi_j)$, for $j = 0, 1, \ldots, N$. The inverse of I_N^h is denoted by I_h^N. By using results of Canuto [34], we have:

$$||v_N||_{L^2(\Lambda)} \asymp ||v_h||_{L^2(\Lambda)} \;\text{ and }\; ||v_N'||_{L^2(\Lambda)} \asymp ||v_h'||_{L^2(\Lambda)}. \tag{4.15}$$

For $i = 2, \ldots, N$, let $\xi_{j(i)}$ be one of the GLL(N) points, with a distance to -1 proportional to $1/i$. Let $\psi_{i,h}(x)$ be the piecewise linear function that goes from 1 to 0 linearly between -1 and $\xi_{j(i)}$, and is zero for $x \geq \xi_{j(i)}$, and choose $\psi_i = I_h^N(\psi_{i,h})$. By (4.15), we have $||\psi_i||^2_{L^2(\Lambda)} \asymp 1/i$, and $||\psi_i'||^2_{L^2(\Lambda)} \asymp i$, since this is true for $\psi_{i,h}$. Then,

$$|E(u)|^2_{H^1(\Lambda)} \;\leq\; C\sum_{i=1}^N a_i^2((1/i) + (1/i^2)i),$$

which implies the right inequality of (4.11).

We next prove the left inequality of (4.12). We recall that $u(x) = \sum_{i=2}^{N} a_i \eta_i(x)$. Since $\{\eta_i'\}$ is an orthonormal set in $L^2(\Lambda)$, we have

$$a_i \;=\; \int_{-1}^{1} u'(x)\eta_i'(x)\,dx\;. \tag{4.16}$$

By integration by parts and a duality argument, we get

$$\begin{aligned}
a_i &\leq \left| \int_{-1}^{1} u(x)\eta_i''(x)\,dx \right| \\
&\leq ||u||_{H_{00}^{1/2}(\Lambda)} ||\eta_i''||_{(H_{00}^{1/2})'} \\
&\leq ||u||_{H_{00}^{1/2}(\Lambda)} ||\eta_i'||_{H^{1/2}} \\
&\leq ||u||_{H_{00}^{1/2}(\Lambda)} \sqrt{i-1/2}\,||L_{i-1}||_{H^{1/2}}.
\end{aligned}$$

Here, the penultimate inequality follows from [71, Proposition 12.1]. The $H^{1/2}$-norm of L_i has been computed in [3], using a Gaussian quadrature rule, and is known to be bounded from above by $C(\log(i))^{1/2}$. Therefore,

$$\sum_{i=2}^{N} a_i^2 ||\eta_i||^2_{H_{00}^{1/2}(\Lambda)} \;\leq\; C\Big(\sum_{i=2}^{N} \log(i-1)\Big)||u||^2_{H_{00}^{1/2}},$$

which implies the left inequality of (4.12).

We prove the right inequality of (4.12) only for the case of N even. For N odd, the same proof applies, with N replaced by $N-1$. Let $u(x) = (L_N(x) - 1) \in P_0^N(\Lambda)$. By (4.16) and integration by parts, we obtain, for $2 \leq i < N$:

$$\begin{aligned}
a_i &= \sqrt{i-1/2}\,\Big(\int_{-1}^{1} L_{i-1}'(x) - \int_{-1}^{1} L_N(x) L_{i-1}'(x)\,dx\Big) \\
&= 2\sqrt{i-1/2}\,,
\end{aligned}$$

if i is even, and zero, otherwise. Again by using the results of [3], $||L_N - 1||^2_{H_{00}^{1/2}(\Lambda)} \leq C\log(N)$. Therefore:

$$\frac{\sum_{i=2}^{N} a_i^2 ||\eta_i||^2_{H_{00}^{1/2}(\Lambda)}}{||u||^2_{H_{00}^{1/2}(\Lambda)}} \geq \frac{C}{\log(N)} \sum_{i\,even,\,i\geq 2}^{N} 1,$$

which implies the right inequality of (4.12). $\square$

Remark 4.3.1 *Theorem 4.3.1 also holds for the p-version applied to triangular substructures. Indeed, the analogue of (4.10) for triangles can be translated into inequalities involving the $H_{00}^{1/2}$-norm, by the extension theorem; see [5, Theorem 7.4]. The result is then implied by the proof of Theorem 4.3.1.*

4.4 Diagonal Edge Preconditioners for the Spectral Element Method in 2d

Theorem 4.4.1 *Let D_N be the diagonal of S_N. Then, $\forall u \in P_0^N(\Lambda)$,*

$$\lambda_{min}^N(\underline{u}^t D_N \underline{u}) \leq \underline{u}^t S_N \underline{u} \leq \lambda_{max}^N(\underline{u}^t D_N \underline{u}), \tag{4.17}$$

with

$$c \leq \lambda_{max}^N \leq C, \tag{4.18}$$

and

$$\frac{c}{N\log(N)} \leq \lambda_{min}^N \leq \frac{C}{N}. \tag{4.19}$$

Proof. Let $u(x) = \sum_{i=1}^{N-1} u(\xi_i)\ell_{i,N}(x)$, where $\{\ell_{i,N}\}$ is the Lagrange interpolation basis related to the GLL(N) points. By (4.9), we only need to estimate $||u||^2_{H_{00}^{1/2}(\Lambda)}$ in terms of $\sum_{i=2}^{N} u^2(\xi_i)||\ell_{i,N}||^2_{H_{00}^{1/2}(\Lambda)}$.

Let $v^N \in P_0^N(\Lambda)$. A consequence of (4.15) is that not only are the L^2- and H^1-norms of v_N and $v_h = I_N^h(v_N)$ equivalent, but also

$$||v_N||_{H_{00}^{1/2}(\Lambda)} \asymp ||v_h||_{H_{00}^{1/2}(\Lambda)}; \tag{4.20}$$

a detailed argument can be found in [37].

We start by showing that $||\ell_{i,N}||_{H_{00}^{1/2}(\Lambda)} \asymp 1$. Let $\ell_{i,h} = I_N^h(\ell_{i,N})$, $h_i = \xi_{i+1} - \xi_i$, and $\theta_i = \arccos(\xi_i)$. Then, for $1 \leq i \leq N-1$,

$$\frac{(N-i-(1/2))\pi}{N} \leq \theta_i \leq \frac{(N-i+1)\pi}{N}; \tag{4.21}$$

see [12, p. 76]. This relation implies that $h_{i+1} \asymp h_i$, for $0 \leq i \leq N-1$. A simple computation shows that for $1 \leq i \leq N-1$, $||\ell_{i,h}||^2_{L^2(\Lambda)} \leq Ch_i$ and $||\ell_{i,h}||^2_{H^1(\Lambda)} \leq C/h_i$. By interpolating between these two spaces and using (4.15), we obtain

$||\ell_{i,h}||_{H_{00}^{1/2}(\Lambda)} \leq C$. A rather tedious, yet elementary, computation using (4.8) shows that one of the positive terms which form $||\ell_{i,h}||^2_{H_{00}^{1/2}(\Lambda)}$ is greater than a positive constant, and this shows that $||\ell_{i,h}||_{H_{00}^{1/2}(\Lambda)} \asymp 1$. Then, by (4.20), we find that $||\ell_{i,N}||_{H_{00}^{1/2}(\Lambda)} \asymp 1$.

The left inequality of (4.18) follows by taking $u = \ell_{2,N}$, and using that $||\ell_{2,N}||_{H_{00}^{1/2}(\Lambda)} \asymp 1$. To prove the right inequality, we use (4.20) and restrict ourselves to piecewise linear functions. Let $E(u_h)(x,y) = \sum_{i=2}^{N} u_h(\xi_i)\ell_{i,h}(x)\psi_{i,h}(y)$, for some $\psi_{i,h}$ with $\psi_{i,h}(-1) = 1$ and $\psi_{i,h}(1) = 0$. We go through the same steps as in the proof of Theorem 4.3.1. Since the mass and stiffness matrices corresponding to the $\ell_{i,h}$ are tridiagonal, we obtain, as before:

$$|E(u_h)|^2_{H^1(\Lambda)} \leq C \sum_{i=1}^{N-1} (u_h(\xi_i))^2((1/h_i)||\psi_{i,h}||^2_{L^2(\Lambda)} + (h_i)||(\psi_{i,h})'||^2_{L^2(\Lambda)}).$$

By (4.15), we can choose the $\psi_{i,h}$ so that the coefficients of $(u_h(\xi_i))^2$ can be bounded above by a constant, thus proving (4.18).

Our remaining task is now to prove (4.19), and we start with the left inequality. For $u \in P_0^N(\Lambda)$, it is well known that

$$||u||^2_{L^\infty(\Lambda)} \leq C(1 + \log(N))||u||^2_{H_{00}^{1/2}(\Lambda)}; \tag{4.22}$$

see Theorem 6.2 of [5]. Then,

$$\sum_{i=1}^{N-1} |u(\xi_i)|^2 \leq C \sum_{i=1}^{N-1} (1 + \log(N))||u||^2_{H_{00}^{1/2}(\Lambda)},$$

and this, in turn, implies the left inequality of (4.19).

For the right inequality, let $u_h(x) = 1 - |x|$, and let $u_N = I_h^N(u_h)$. A standard argument of interpolation between $L^2(\Lambda)$ and $H_0^1(\Lambda)$ and a simple computation shows that $||u_N||_{H_{00}^{1/2}(\Lambda)} \asymp 1$. We also have

$$\begin{aligned}
\sum_{i=1}^{N-1} (u_N(\xi_i))^2 &\geq \sum_{i=\lceil N/2 \rceil}^{N-1} (u_N(\xi_i))^2 \\
&= \sum_{i=\lceil N/2 \rceil}^{N-1} (1 - \xi_i)^2 \\
&= N \sum_{i=\lceil N/2 \rceil}^{N-1} \frac{1}{N}(1 - \cos(\theta_i))^2,
\end{aligned}$$

where, by (4.21), the θ_i are asymptotically equidistant on the interval $[0, \pi/2]$. This last sum is then a Riemann sum for $(1/\pi)\int_0^{\pi/2}(1-\cos(\theta))^2\,d\theta$, and therefore,

$$\sum_{i=1}^{N-1}(u_N(\xi_i))^2 \geq CN,$$

completing the proof of (4.19). $\square$

Remark 4.4.1 *The Schur complement associated with an edge for a finite element space based on a quasi-uniform triangulation with a parameter h has a condition number on the order of $1/h$; see [15]. The techniques used to prove Theorem 4.4.1 can be used to establish that this condition number is between $1/h$ and $|\log(h)|/h$. Although ours is a slightly weaker result, our methods can be used in a context more general than for quasi-uniform meshes, e.g., for the GLL mesh.*

4.5 Numerical Experiments in 2d

We have performed numerical experiments to determine the actual values of the eigenvalues of Theorems 4.3.1 and 4.4.1. The results for $4 \leq N \leq 50$ are given in Figures 4.1 and 4.2. They agree, in a clear-cut way, with the theoretical results developed here. We remark that for these values of N, the approximate linear growth of the inverse of the smallest eigenvalue is clear, and that the graph of the largest eigenvalue appears to approach a horizontal asymptote. Table 4.1 gives the condition numbers for both $D_p^{-1}S_p$ and $D_N^{-1}S_N$, for $4 \leq N \leq 15$. We note that the relatively small condition numbers help explain the good convergence rates experienced with the algorithm implemented in [5], by the modular character of the Schwarz framework.

4.6 Diagonal Face Preconditioner for the Spectral Element Method in 3d

For three-dimensional problems, the preconditioner proposed in [5] has to be replaced by a preconditioner especially designed for three dimensional problems. A

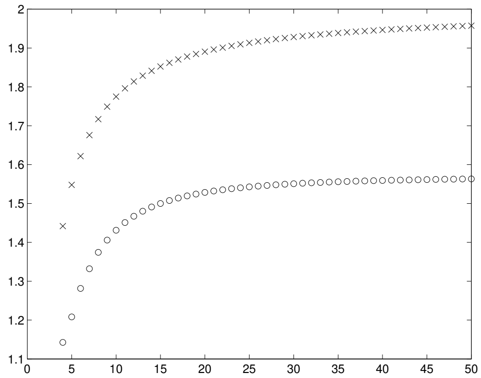

Figure 4.1: $\lambda_{max}(D_p^{-1}S_p) = \circ$, $\lambda_{max}(D_N^{-1}S_N) = \times$

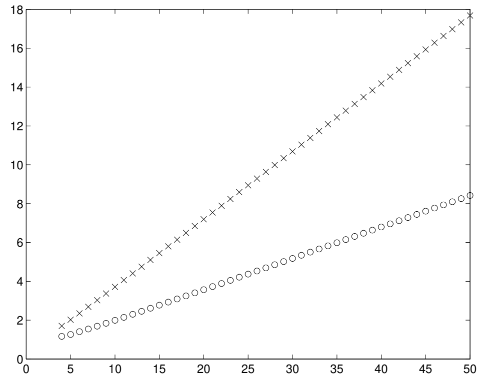

Figure 4.2: $1/\lambda_{min}(D_p^{-1}S_p) = \circ$, $1/\lambda_{min}(D_N^{-1}S_N) = \times$

straightforward generalization of that method results in a condition number increasing at least quadratically with the degree N. We note that Chapter 3 presents a variety of much better preconditioners for three-dimensional second-order elliptic problems. For the iterative substructuring algorithms proposed there, the factorization of the matrix related to the GLL finite element mesh in each subregion Ω_i is a preprocessing step, since back-substitution using this factored form is performed at every step of the algorithm.

In the next subsection, we propose another iterative substructuring algorithm in addition to those of Chapter 3, which can be traced back to [48], and which was proposed in a slightly different form in [96].

The weakest point of this type of preconditioners is its face component, since,

N	$\kappa(D_p^{-1}S_p)$	$\kappa(D_N^{-1}S_N)$
4	1.33	2.46
5	1.53	3.13
6	1.80	3.81
7	2.06	4.51
8	2.33	5.20
9	2.59	5.90
10	2.85	6.60
11	3.12	7.29
12	3.38	8.00
13	3.64	8.70
14	3.90	9.40
15	4.16	10.09

Table 4.1: Condition numbers of $D_p^{-1}S_p$ and $D_N^{-1}S_N$ for an edge

in each iteration and for each face, it requires the solution of a Dirichlet problem in the union of the two subregions sharing that face. This entails a new ordering of the nodes, and the factorization of the stiffness matrices corresponding to these unions. An alternative would be to use the inexact solver proposed by Couzy and Deville [44]; a tensor product basis can be used to solve a related problem in the union of two non-deformed subregions derived from the original ones by an averaging procedure.

In subsection 4.6.2, we propose a simplification of this component of the algorithm by introducing diagonal preconditioners of the face components.

The new algorithm has a condition number bounded from above by

$$C(1 + \log(N))^3 N.$$

We will present evidence that its growth is at least linear. However, as we will see in subsection 4.6.3, the actual condition numbers are relatively small. As before, numerically computed bounds for the conditioning of some of the modules of the algorithm (such as these diagonal face preconditioners) can be used to estimate the rate of convergence of the overall iterative process, by using the Schwarz framework of Section 2.3.

In subsection 4.6.3, we present some numerical results obtained by Einar Rønquist, of Nektonics Inc. These experiments show a linear growth of the condition numbers, and the small number of iterations encountered in practice. These three-dimensional results are obtained for a domain partitioned into 256 subdomains, and for a problem with 64 subdomains, with polynomial degrees up to 14, which appears to be the current limit of practical values of N. These are large problems, and as we will see, they can be solved with relatively modest computer resources.

4.6.1 An Iterative Substructuring Algorithm with a Standard Coarse Space

For the problem (1.16) in three dimensions, we describe an additive Schwarz preconditioner with a standard coarse space, as first proposed in [96].

The coarse space $V^H = P_0^1(\Omega)$ is the space of images via F_i of continuous piecewise trilinear functions, to which we associate the bilinear form $a(\cdot,\cdot)$.

The interior spaces are given by $V_{\Omega_i} = P_0^N(\Omega_i)$, and the corresponding bilinear form is $a_h(\cdot,\cdot)$, given, for $u, v \in V_{\Omega_i}$ by $a_h(u,v) = a(I_N^h(u), I_h^N(v))$.

To each face $\mathcal{F}_k$, shared by two subregions Ω_i and Ω_j, we associate the space $V_{\mathcal{F}_k} \subset P_0^N(\Omega)$ of piecewise h-discrete harmonic functions in Ω_i and Ω_j, vanishing on and outside $(\partial\Omega_i \cup \partial\Omega_j)\backslash\mathcal{F}_k$. The bilinear form is again $a_h(\cdot,\cdot)$.

The space $V_{\mathcal{W}} \subset P_0^N(\Omega)$ associated with the wirebasket $\mathcal{W}$ is formed by functions that are zero at all the GLL nodes that are not on an edge or vertex. The bilinear form $a_{\mathcal{W}}(u,v)$ is defined as $\int_{\mathcal{W}} u(x)v(x)\,dx$, calculated with the GLL(N) quadrature edge by edge. We remark that in the preconditioning step, this corresponds to the multiplication of the inverse of a diagonal matrix with the appropriate subvector of the residual.

For this preconditioner, each vertex degree of freedom belongs to two spaces, namely V^H and $V_{\mathcal{W}}$. This is very important for the performance of the algorithm, and can be seen to be necessary both from the theoretical point of view (see [48, Algorithm 6.3] for a discussion in a related context), and from practical considerations.

Let T_S be the preconditioned operator for the algorithm just described. By using the tools given in Chapter 3, we can easily get the following result, the proof of which we omit; see [96, Theorem 4.4].

Theorem 4.6.1

$$\kappa(T_S) \leq C(1 + \log(N))^2,$$

where the constant C is independent of N, the number of subregions, and the jumps of the coefficient across subdomain boundaries, provided these coefficients are quasi-monotone.

If the coefficients are not quasi-monotone, then we expect the bound to deteriorate when the jumps are very large. However, when this preconditioner is used in the context of CFD computations, the jumps are not a concern; see Chapters 5 and 6

4.6.2 A Diagonal Preconditioner for the Face Component

Apart from the factorization of the matrix corresponding to Dirichlet problems in the subregions, the computation of the $T_{\mathcal{F}_k}$ is the computationally intensive part of the preconditioner, because each of them involves the solution of a Dirichlet problem in the union of two subregions.

We propose to replace the bilinear form for each of these subspaces by a diagonal form, which dramatically reduces the number of floating point operations for these components of the preconditioner.

We consider one face $\mathcal{F}$ at a time, and denote by $S_{\mathcal{F}}$ the Schur complement of the degrees of freedom on $\mathcal{F}$ with respect to the nodes interior to the two subregions that share the face $\mathcal{F}$. We work on a reference configuration for which these two neighboring subregions are reference cubes, and $\mathcal{F}$ is the square $[-1,1]^2$. As in the derivation of (4.9), we can use the extension theorem for spectral element discretizations, [8, Theorem B.4], to show the uniform equivalence of the Schur complement norm and the $H_{00}^{1/2}$-norm. In other words, let $v_{\mathcal{F}} \in P_0^N(\mathcal{F})$, and

$\underline{v}_{\mathcal{F}}$ be the vector of degrees of freedom associated with the interior of $\mathcal{F}$. Then, $\forall v_{\mathcal{F}} \in P_0^N(\mathcal{F})$:

$$c||v_{\mathcal{F}}||^2_{H^{1/2}_{00}(\mathcal{F})} \leq \underline{v}^t_{\mathcal{F}} S_{\mathcal{F}} \underline{v}_{\mathcal{F}} \leq C||v_{\mathcal{F}}||^2_{H^{1/2}_{00}(\mathcal{F})}. \qquad (4.23)$$

Let $u \in P_0^N(\mathcal{F})$ be the restriction to $\mathcal{F}$ of any function in $V_{\mathcal{F}}$. Let

$$d(u,u) := \sum_{i=1}^{N-1} \sum_{i=1}^{N-1} (\rho_i + \rho_j)(u_{\mathcal{F}}(\xi_i, \xi_j))^2.$$

The diagonal matrix $D_{\mathcal{F}}$ is defined by

$$\underline{u}^t_{\mathcal{F}} D_{\mathcal{F}} \underline{u}_{\mathcal{F}} = d(u,u),$$

and will be used as an approximate solver on the space $V_{\mathcal{F}}$. We then have the following theorem.

Theorem 4.6.2

$$\lambda^{\mathcal{F},N}_{\min}(\underline{u}^t D_{\mathcal{F}} \underline{u}) \leq \underline{u}^t S_{\mathcal{F}} \underline{u} \leq \lambda^{\mathcal{F},N}_{\max}(\underline{u}^t D_{\mathcal{F}} \underline{u}),$$

with

$$\lambda^{\mathcal{F},N}_{\max} \leq C,$$

and

$$\lambda^{\mathcal{F},N}_{\min} \geq \frac{c}{N(1+\log(N))},$$

where C and c are independent of N.

Proof. By using the same notations as in the proof of Theorem 4.4.1, and dropping the subscript $\mathcal{F}$, we may write $u(x,y) = \sum_i \sum_j u(\xi_i, \xi_j) \ell_{i,N}(x) \ell_{j,N}(y)$. By (4.23), we only need to estimate $||u||^2_{H^{1/2}_{00}}$ in terms of $d(u,u)$.

To estimate $\lambda^{\mathcal{F},N}_{\min}$, we start by noting that, by (1.13),

$$\begin{aligned} d(u,u) &\leq C(\sum_i \int u^2(\xi_i, y)\, dy + \sum_j \int_I u^2(x, \xi_j)\, dx) \\ &= C(\sum_i ||u(\xi_i, \cdot)||^2_{L^2(I)} + \sum_j ||u(\cdot, \xi_j)||^2_{L^2(I)}), \end{aligned}$$

where I is the interval $[-1, 1]$.

By using Lemma 3.3.5, each term of these sums is bounded from above by $C(1+\log(N))||u||^2_{H^{1/2}_{00}}$, resulting in the estimate of $\lambda^{\mathcal{F},N}_{\min}$.

Our task now is to estimate $\lambda^{\mathcal{F},N}_{\max}$, and for that we use yet another alternative expression for the $H^{1/2}_{00}$-norm, which can be found for instance in [85, Lemma II.5.3]. The square of this equivalent norm is given by the sum of

$$I_1(u) = \int_I \int_I \frac{||u(s,\cdot) - u(t,\cdot)||^2_{L^2(I)}}{(s-t)^2}\, ds\, dt,$$

$$I_2(u) = \int_I \int_I \frac{||u(\cdot,s) - u(\cdot,t)||^2_{L^2(I)}}{(s-t)^2}\, ds\, dt,$$

and

$$I_3(u) = \int_I \int_I \frac{u^2(x,y)}{\sigma((x,y),\partial\mathcal{F})}$$

where $\sigma((x,y),\partial\mathcal{F})$ is the distance between the point (x,y) and $\partial\mathcal{F}$.

Let $P^h_0(\mathcal{F})$ be the space of piecewise bilinear functions, obtained from $P^N_0(\mathcal{F})$ by interpolation at the GLL(N) nodes. Let $u_h = I^h_N(u)$, where $u \in P^N_0(\mathcal{F})$. Then, to prove the estimate on $\lambda^{\mathcal{F},N}_{\max}$, it is enough to bound $||u||^2_{H^{1/2}_{00}(\mathcal{F})}$ from above by $d(u_h,u_h)$, since $d(u_h,u_h) = d(u,u)$ and

$$||u_h||^2_{H^{1/2}_{00}(\mathcal{F})} \asymp ||u||^2_{H^{1/2}_{00}(\mathcal{F})};$$

see (3.9) and (4.23).

Let $u_{i,h}$ be defined by $u_{i,h}(\xi_i,y) = u(\xi_j,y)\,\delta_{ij}$, i.e. $u_{i,h}$ agrees with u on the segment $\{\xi_i\}\times[-1,1]$, and is zero on $\{\xi_j\}\times[-1,1]$, for $j \neq i$. It is clear that $u_{i,h}(x,y) = u_{i,h}(\xi_i,y)\ell_{i,h}(x)$ has support in $[\xi_{j-1},\xi_{j+1}]\times[-1,1]$, and that $u_h(x,y) = \sum_i u_{i,h}(x,y)$.

We claim that, $\forall i$,

$$||u_{i,h}||^2_{H^{1/2}_{00}(\mathcal{F})} \leq \sum_j (\rho_i + \rho_j)(u(\xi_i,\xi_j))^2. \tag{4.24}$$

We start by estimating $I_1(u_h)$.

$$\begin{aligned} I_1(u_{i,h}) &= ||\ell_{i,h}||^2_{L^2(I)} |u_{i,h}(\xi_i,\cdot)|^2_{H^{1/2}(I)} \\ &\leq C\rho_i \sum_j u^2_{i,h}(\xi_i,\xi_j), \end{aligned}$$

by (1.13) applied to $\ell_{i,h}(x)$ and by (4.18) applied to $u_{i,h}(\xi_i,\cdot)$, since the diagonal elements of D_N (the diagonal of the edge Schur complement) are of order 1. Analogously, we have:

$$\begin{aligned} I_2(u_{i,h}) &= ||u_{i,h}(\xi_i,\cdot)||^2_{L^2(I)}|\ell_{i,h}|^2_{H^{1/2}(I)} \\ &\leq C\sum_j \rho_j u^2_{i,h}(\xi_i,\xi_j), \end{aligned}$$

since $|\ell_{i,h}|^2_{H^{1/2}(I)} \asymp 1$, by the proof of Theorem 4.4.1. By using the trivial inequality $(\sigma((x,y),\partial\mathcal{F}))^{-1} \leq 2((\sigma(x,\partial I))^{-1}+(\sigma(y,\partial I))^{-1})$, where $\sigma(x,\partial I)$ is the distance of x to ∂I, the estimate of $I_3(u_{i,h})$ is analogous to the estimates of $I_1(u_{i,h})$ and $I_2(u_{i,h})$. This completes the proof of (4.24), since $u_{i,h}$ and u_h agree on $\xi_i \times [-1,+1]$.

Next, we write $u_h(x,y) = \sum_i u_{i,h}(x,y)$, and arrive at

$$I_1(u_h) \leq C\int_I\int_I \frac{1}{(s-t)^2}\sum_i \int_I (u_{i,h}(x,s)-u_{i,h}(x,t))^2\, dx\, ds\, dt,$$

by applying the Cauchy-Schwarz inequality, and by noting that for fixed (s,t), the integrands $(u_{i,h}(x,s)-u_{i,h}(x,t))^2$ have disjoint support for any two values of i that differ by two or more from each other.

Then, by (4.24),

$$I_1(u_h) \leq C\sum_i I_1(u_{i,h}) \leq C\sum_i\sum_j(\rho_i+\rho_j)u^2_{i,h}(\xi_i,\xi_j).$$

The estimates for $I_2(u_h)$ and $I_3(u_h)$ are analogous. □

Let T_{SD} be the preconditioned operator defined in Section 4.6.1, except that the bilinear form for each $V_{\mathcal{F}_k}$ is replaced by $d(u_{\mathcal{F}_k},v_{\mathcal{F}_k}) = \underline{u}^t_{\mathcal{F}_k}D_{\mathcal{F}_k}\underline{u}_{\mathcal{F}_k}$. Then, by combining Theorems 4.6.2, 4.6.1, and the abstract Schwarz framework, we immediately obtain the following result.

Theorem 4.6.3

$$\kappa(T_{SD}) \leq (C_1(1+\log(N))^2)(C_2N(1+\log(N))),$$

where C_1 is the constant given by Theorem 4.6.1, and C_2 is the constant given by Theorem 4.6.2; they are both independent of N and H.

Remark 4.6.1 *We explicitly distinguished between C_1 and C_2 because they may be of very different magnitudes. If C_2 is estimated by solving a generalized eigenvalue problem in the union of two reference substructures, this numerically computed estimate can be entered directly into the estimate of the theorem. This is an advantage of the modularity of the Schwarz framework; see Section 1.1.*

4.6.3 Comments on Some Experiments by Einar Rønquist

We report on some numerical experiments performed by Einar Rønquist, of Nektonics Inc., to assess the actual value of the condition number of the operator T_{SD} of Theorem 4.6.3, and especially one of its variants. The Poisson problem with homogeneous Dirichlet boundary conditions was solved for different choices of the domain Ω.

Most of the results were obtained, in this preliminary study, with each of the diagonal matrices $D_{\mathcal{F}_k}$ replaced by $D'_{\mathcal{F}_k}$. Let $\mathcal{F}$ be a face of a reference substructure. The diagonal entry corresponding to $(\xi_i, \xi_j) \in \mathcal{F}$ is equal to $\rho_i + \rho_j$ for $D_{\mathcal{F}}$, while for $D'_{\mathcal{F}}$ it is set to $\left(\frac{\rho_i \rho_j}{\rho_i + \rho_j}\right)$; the actual matrices $D'_{\mathcal{F}_k}$ are obtained by mapping from the reference face. This was the preconditioner first proposed by us for the face subspaces, before we had a proof of Theorem 4.6.2. When this alternative is used for the face subspaces, the preconditioner is denoted by T'_{SD}. We present results for this alternative because this set of experiments is the most complete now available, although estimates that accurately explain its properties have not yet been found.

We first let Ω be the union of two unit cubes Ω_1 and Ω_2 sharing a face $\mathcal{F}$. In this case, the operators T_{SD} and T'_{SD} are generated by using three spaces: V_{Ω_1}, V_{Ω_2}, and $V_{\mathcal{F}}$. Table 4.2 shows the condition numbers for $T_{SD} = T_{\Omega_1} + T_{\Omega_2} + 4T_{\mathcal{F}}$ and $T'_{SD} = T_{\Omega_1} + T_{\Omega_2} + T_{\mathcal{F}}$. The factor 4 was found to further decrease the condition number of T_{SD}; it clearly does not affect by more than a factor 4 the bound of Theorem 4.6.3. If this factor is not used, an increase of 20% in $\kappa(T_{SD})$ is observed. We note that the values in the first column of Table 4.2 are consistent with the approximate linear growth of the condition number expected from Theorem 4.6.3.

We next let $\Omega = [-1, 1]^3$, and divide it into 64 equal cubic subdomains. Table 4.3 shows the condition number of T'_{SD}, with a piecewise trilinear (Q_1) or triquadratic

N	$\kappa(T_{SD})$	$\kappa(T'_{SD})$
4	5.74	5.79
6	7.25	7.70
8	8.25	8.47
10	9.16	9.41
12	10.2	10.4
14	12.8	12.9
16	15.8	15.8
18	19.1	19.1
20	22.8	22.7
22	26.9	26.7
24	-	31.1

Table 4.2: Condition numbers for the two subregion problem

(Q_2) coarse space; the last column gives the number of degrees of freedom of the system being solved.

Finally, in Table 4.4, we show the condition numbers of T'_{SD} on a non-cubical domain Ω' formed by deleting 32 subdomains out of the 64 into which the cube $[-1,1]^3$ had been decomposed, and then partitioning each of the remaining ones into eight equal cubes, to produce a decomposition into 256 cubes. The resulting region Ω' is connected, but it has no trivial geometrical symmetries.

From the results presented here, we conclude that T_{SD} (or T'_{SD}, for which there is no satisfactory theoretical explanation yet) can be expected to be very effective, even for domains partitioned into many substructures, and for realistic values of N; indeed, quite large problems were solved with modest computer resources.

N	$\kappa(T'_{SD})$ trilinear	$\kappa(T'_{SD})$ triquadratic	# D. O. F.
3	7.03	3.66	2,197
4	10.0	4.89	4,913
5	13.0	6.61	9,261
6	16.4	8.49	15,625
7	19.7	10.2	24,389
8	23.0	12.2	35,937
9	26.3	14.0	50,653
10	29.7	16.0	68,921
11	33.1	17.7	91,125
12	36.7	20.0	117,649

Table 4.3: Condition numbers of T'_{SD}, with trilinear and triquadratic coarse spaces, and 64 subdomains

N	$\kappa(T'_{SD})$ triquadratic	# D. O. F.
3	7.70	8,635
4	11.0	19,401
5	14.4	36,671
6	18.0	61,981
7	21.6	96,867
8	25.2	142,865

Table 4.4: Condition numbers of T'_{SD}, with a trilinear coarse space, and 256 subdomains

Chapter 5

Schwarz Methods for the Stokes Equation

5.1 Introduction

In this chapter, we consider fast methods of solving the linear system resulting from the discretization of the Stokes problem by the spectral element method; see (1.17). The Uzawa procedure has long been the prefered method of solution. The velocity is eliminated from the first equation of (1.18) and inserting into the second one, yielding:

$$BA^{-1}B^t\underline{p} = BA^{-1}\underline{\mathbf{f}}.$$

The coefficient matrix $BA^{-1}B^t$, preconditioned by the mass matrix for the pressure, corresponds, heuristically, to a zero-order operator, and indeed has a relatively small condition number. The PCG method is used to find $\underline{p}$; each iteration requires an exact solve with A. For large scale problems, a factorization of A is not an efficient choice, for well-known reasons, as pointed out, e.g., in Section 2.1. Each exact solve with A is performed by using a preconditioned iterative solver, which generates a costly inner iteration; see [61, 73]. Once $\underline{p}$ has been found to the required accuracy, $\underline{\mathbf{u}}$ is obtained from the first equation of (1.18).

In the finite element context, block-diagonal and block-triangular preconditioners have been used for the matrix

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix};$$

see [55, 69, 106]. This approach attempts to take full advantage of the power of the preconditioners developed for second-order elliptic equations (such as each of the diagonal blocks of A) and the many iterative methods available for indefinite matrices. These block methods have had considerable success in recent years, and we believe that a practical comparison with the type of algorithms introduced in this section would be a worthwhile undertaking.

Einar Rønquist has proposed an iterative substructuring method that is based on a decomposition of the domain into interiors of subregions, faces, edges, and vertices. The coarse problem is a Stokes problem approximated by a lower-dimensional pair of discrete spaces on the coarse mesh, such as the $Q_2 - Q_0$ element pair. Stokes problems are solved within the subregions, while a diagonal scaling using elements of the matrix A is performed on the interface velocity variables. Both velocities and pressures are used in a GMRES iteration. This scheme avoids costly inner iterations, and its built-in parallelism is certainly a very desirable feature. In [100], relatively large problems in three dimensions are solved with modest computer resources. The number of iterations is small (on the order of 40 for $M = 64$ and $N = 6$, with $44,501$ degrees of freedom). The small iteration count and the excellent approximation properties of the spectral element method for flow problems makes this a very efficient scheme.

Bramble and Pasciak [18] and Le Tallec and Patra [113] have proposed preconditioners for the h- and p- version, respectively, based on the elimination of the variables interior to the subregions, via the solution of a Stokes problem for each of them. By a standard extension theorem, the resulting bilinear form for the interface velocity degrees of freedom is positive definite and spectrally equivalent, up to a factor involving the Babuška-Brezzi constant, to the bilinear form of the Schur complement corresponding to the interface velocity components; see [60, Lemma I.2.2], [18, 113], or our Lemma 5.3.2. Although distinct in important respects, both of these methods use global pressure variables as Lagrange multipliers to constrain interface velocities and to guarantee that the discrete divergence free condition holds. The PCG method is used as the accelerator of the iterative process.

In an effort to more fully understand Rønquist's scheme, we have developed iterative substructuring methods, for which the velocities are restricted to the space of discretely divergence free functions in the spectral element sense; i.e. the velocities satisfy the last equation of (1.17). We note that the computation of a basis for this divergence free space in this context, is too expensive to be of practical value, and is not required here; to accomplish that, the Lagrange multiplier technique of [18] and [113] is used. The PCG method is applied to the resulting symmetric, positive definite linear system. The condition number of our algorithms grows at most like

$$\frac{C(1+\log(N))^3}{\beta_N},$$

where β_N is the Babuška-Brezzi constant introduced in Lemma 1.5.1.

This approach can be nicely adapted to the solution of the steady Navier-Stokes equation by Newton's method, since in each iteration only the velocity of the previous step is used. The pressure is computed only when required, typically after the velocity has been obtained to the prescribed accuracy; see Chapter 6.

The idea of developing domain decomposition preconditioners for mixed problems by restricting the velocity (or an analogous quantity) to its constrained space has already appeared in [81, 82, 83]. The problem is then reduced to a symmetric, positive definite one. Moreover, in contrast with our case, bases for discrete divergence free spaces are available for some finite element methods; this fact is used to design multigrid and domain decomposition methods in [22, 23, 87].

The next section introduces some notations. Section 5.3 gives a proof of an extension lemma that will be used throughout the chapter. In Section 5.4, we propose and analyze a Lagrange multiplier-based preconditioner that is similar to an algorithm proposed by Le Tallec and Patra [113]. In Section 5.5, the $Q_2 - Q_0$ pair is used to generate a coarse space for the Stokes problem; the construction of the coarse projection operator is intimately connected with the standard proof of the inf-sup condition for this pair; see Section 5.4. It will become clear that many of the stable element pairs with discontinuous pressures generate appropriate coarse spaces for the spectral element discretization. Moreover, from the point of view of analysis, the spectral element method may be replaced by any other finite element discretization with discontinuous pressures on a fine mesh. The theory carries over without any substantial change; see Remark 5.5.3. Several aspects of this analysis, which is analogous to the Schwarz theory now available for non-imbedded finite element spaces for elliptic problems, appear here for the first time. They involve the following well-known ingredients: L^2-approximation properties and H^1-stability of the coarse projection and of the interpolation from the coarse to the fine space; see (5.13) and Lemma 5.5.1.

Section 5.6 introduces yet another type of coarse space, based on the **curl** of C^1-finite element functions on the coarse mesh. The central new result of this section is Lemma 5.6.1, which allows us to analyze the resulting domain decomposition preconditioner in Theorem 5.6.1.

5.2 An Extension Operator

For a subregion Ω_i, we define an extension operator

$$E_i^{S,N} : (P^N(\partial\Omega_i))^3 \longrightarrow (P^N(\Omega_i))^3,$$

where $\mathbf{u}_i = E_i^{S,N}(\mathbf{g}_i)$ is the velocity component of the solution to the following Stokes problem:

Find $(\mathbf{u}_i, p_i) \in ((P^N(\Omega_i))^d, P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i))$, such that:

$$\begin{cases} a_Q(\mathbf{u}_i, \mathbf{v}_i) + b_{\Omega_i}(\mathbf{v}_i, p_i) = 0 \quad \forall \mathbf{v}_i \in (P_0^N(\Omega_i))^d, \\ b_{\Omega_i}(\mathbf{u}_i, q_i) = 0 \quad \forall q \in \bar{P}^{N-2}(\Omega_i) \cap L_0^2(\Omega_i), \\ \mathbf{u}_i|_{\partial\Omega_i} = \mathbf{g}_i. \end{cases} \tag{5.1}$$

Here, $b_{\Omega_i}(\cdot,\cdot)$ is the restriction of $b(\cdot,\cdot)$ to Ω_i, $a_Q(\cdot,\cdot)$ is the standard discrete form for the spectral element method; see subsection 1.5.2. In other words, $\mathbf{u}_i$ is the solution of a homogeneous Stokes problem with $\mathbf{g}_i$ as boundary data, and zero right hand side within Ω_i; $\mathbf{u}_i$ will also be called the Stokes-harmonic extension of the boundary values $\mathbf{g}_i$. We remark that $\mathbf{u}_i$ always exists, even if the outward fluxes $\int_{\partial\Omega_i} \mathbf{g}_i \cdot \mathbf{n}\, dS$ is not equal to zero, since the test space for the pressure does not include the constant function, and the Babuška-Brezzi condition is satisfied for the problems restricted to each subregion.

We remark that if $\mathbf{u}_i = E_i^{S,N}(\mathbf{g}_i)$, then

$$a_{Q,\Omega_i}(\mathbf{u}_i, \mathbf{u}_i) = \min_{\mathbf{v}_i|_{\partial\Omega_i} = \mathbf{g}_i} a_{Q,\Omega_i}(\mathbf{v}_i, \mathbf{v}_i) \quad \forall \mathbf{v}_i \in P_\nabla^N(\Omega_i), \tag{5.2}$$

where

$$P_{\nabla}^{N}(\Omega_i) = \{\mathbf{v}_i \in (P^N(\Omega_i))^d \mid \quad b_{\Omega_i}(\mathbf{v}_i, q_i) = 0 \quad \forall q_i \in P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i)\}.$$

We also define an analogous operator $E_i^{S,h}$ that in general is less expensive to compute than $E_i^{S,N}$, but still satisfies a minimizing property analogous to (5.2). Let

$$E_i^{S,h} : (P^N(\partial\Omega_i))^3 \longrightarrow (P^N(\Omega_i))^3,$$

where $\mathbf{u}_i = E_i^{S,h}(\mathbf{g}_i)$ is the velocity component of the solution to the following Stokes problem:
Find $(\mathbf{u}_i, p_i) \in ((P^N(\Omega_i))^d, P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i))$, such that:

$$\begin{cases} a_h(\mathbf{u}_i, \mathbf{v}_i) + b_{\Omega_i}(\mathbf{v}_i, p_i) = 0 \quad \forall \mathbf{v}_i \in (P_0^N(\Omega_i))^d, \\ b_{\Omega_i}(\mathbf{u}_i, q_i) = 0 \quad \forall q \in \bar{P}^{N-2}(\Omega_i) \cap L_0^2(\Omega_i), \\ \mathbf{u}_i|_{\partial\Omega_i} = \mathbf{g}_i. \end{cases} \tag{5.3}$$

Here, $a_h(\mathbf{u}_i, \mathbf{v}_i) = a(I_N^h(\mathbf{u}_i), I_N^h(\mathbf{v}_i))$, where the interpolation operator I_N^h is given in Section 3.2. The extension $E_i^{S,h}$ differs from $E_i^{S,N}$ only as far as the bilinear form for the velocity is concerned; the solution of (5.3) involves a sparse, finite difference-type matrix for each of the diagonal velocity blocks, whereas (5.1) involves the spectral element stiffness matrix, which is much less sparse. However, since the space $P^N(\hat{\Omega})$ is a tensor product space, the operator $E_i^{S,N}$ or variations thereof may also be used in practice; see Remark 5.4.1.

5.3 An Extension Lemma

The goal of this section is to compare the energy of the extension $\mathbf{u}_i$ given by (5.1) or (5.3) with the energy of its discrete harmonic extension. This is a standard result the proof of which we repeat here for completeness; cf. [18].

Let

$$P_{0,\nabla}^{N}(\Omega) := \{\mathbf{v} \in (P_0^N(\Omega))^d, b(\mathbf{v}, q) = 0 \quad \forall q \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega)\}, \tag{5.4}$$

and

$$P^N_{\Gamma,\nabla}(\Omega) := \{\mathbf{v} \in (P^N_0(\Omega))^d, \mathbf{v}|_{\Omega_i} = E^{S,N}_i(\mathbf{v}|_{\partial\Omega_i}) \text{ and } \int_{\partial\Omega_i} \mathbf{v}\cdot\mathbf{n}\, dS = 0,\ \forall i\}. \quad (5.5)$$

Lemma 5.3.1

$$P^N_{\Gamma,\nabla}(\Omega) \subset P^N_{0,\nabla}(\Omega).$$

This result easily follows from Green's Theorem, since the zero flux condition on $\partial\Omega_i$ translates into the constraints on $P^N_{0,\nabla}(\Omega)$ given by the pressure test functions which are constant in each Ω_i. The next result states that the Schur complement bilinear form is equivalent, up to a constant, to $a_Q(\cdot,\cdot)$ when restricted to $P^N_{\Gamma,\nabla}(\Omega)$. This lemma is crucial in our approach to the preconditioning of the Stokes problem.

Lemma 5.3.2 *Let* $\mathbf{u} \in (P^N_0(\Omega))^d$ *be of the form* $\mathbf{u}|_{\Omega_i} = E^{S,N}_i(\mathbf{u}|_{\partial\Omega_i})$*, and let* $\mathcal{H}\mathbf{u} \in (P^N(\Omega_i))^d$ *be the component-wise* Q*-discrete harmonic extension of* $\mathbf{u}|_{\partial\Omega_i}$*; see Section 3.2. Then,*

$$\beta_N c_0 a_{Q,\Omega_i}(\mathbf{u},\mathbf{u}) \le a_{Q,\Omega_i}(\mathcal{H}\mathbf{u},\mathcal{H}\mathbf{u}) \le a_{Q,\Omega_i}(\mathbf{u},\mathbf{u}) \quad \forall i.$$

The same conclusion holds if $E^{S,N}_i$ *is replaced by* $E^{S,h}_i$ *and/or if the* Q*-discrete harmonic extension* $\mathcal{H}$ *is replaced by the* h*-discrete harmonic extension* $\mathcal{H}_h$*.*

Proof. The right inequality is a consequence of the minimizing property of the Q-discrete harmonic extension applied to each component of $\mathbf{u}$.

For the left inequality, we note that, according to Lemma 1.5.1, the pair of spaces $(P^N_0(\Omega_i))^d \times \bar{P}^{N-2}(\Omega_i) \cap L^2_0(\Omega_i)$ satisfies a Babuška-Brezzi condition. Then, we can use [60, Lemma II.1.1] to find a $\mathbf{z} \in (P^N_0(\Omega_i))^d$ such that

$$b(\mathbf{z},q) = b(\mathcal{H}\mathbf{u},q) \quad \forall q \in P^{N-2}(\Omega_i) \cap L^2_0(\Omega_i),$$

and

$$a_{\Omega_i}(\mathbf{z},\mathbf{z}) \le \frac{1}{\beta_N} a_{\Omega_i}(\mathcal{H}\mathbf{u},\mathcal{H}\mathbf{u}).$$

Then, the trace of $\mathcal{H}\mathbf{u} - \mathbf{z} \in P^N_\nabla(\Omega_i)$ agrees with that of $\mathbf{u}$ on $\partial\Omega_i$, and satisfies the left inequality of the theorem. We can now conclude by using the minimizing property of $E^{S,N}_i(\cdot)$, i.e. equation (5.2).

An analogous proof applies to $E^{S,h}_i$ and/or to $\mathcal{H}_h$, by the FEM-SEM equivalence; see Section 3.2. □

5.4 A Lagrange Multiplier Based Algorithm

We propose an additive Schwarz preconditioner, induced by the following subspace decomposition, which is orthogonal with respect to the $a_Q(\cdot,\cdot)$ bilinear form:

$$P^N_{0,\nabla}(\Omega) = (\oplus_{i=1}^M P^N_{0,\nabla}(\Omega_i)), \bigoplus P^N_{\Gamma,\nabla}(\Omega) \tag{5.6}$$

where $P^N_{0,\nabla}(\Omega_i) = P^N_{\nabla}(\Omega_i) \cap H^1_0(\Omega_i)$.

To each space $(P^N_{0,\nabla}(\Omega_i))^d$, we associate the bilinear form $a_{Q,\Omega_i}(\cdot,\cdot)$, which defines an operator T_{Ω_i}, in the Schwarz framework.

Let the bilinear form $\tilde{a}(\cdot,\cdot)$ be defined by

$$\tilde{a}(\cdot,\cdot) = \sum_{j=1}^{d} \tilde{a}_j(\cdot,\cdot),$$

where the $\tilde{a}_j(\cdot,\cdot)$ are bilinear forms associated with good preconditioners for the Laplace operator restricted to the space of Q-discrete harmonic functions. We assume, for definiteness, that

$$c_{\min}\tilde{a}(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \le a_Q(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \le c_{\max}\tilde{a}(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma), \tag{5.7}$$

$\forall \mathbf{u}_\Gamma \in P^N_{\Gamma,\nabla}(\Omega) \subset (P^N_0(\Omega))^d$. Here, $\mathcal{H}\mathbf{u}_\Gamma$ is the component-wise Q-discrete harmonic extension of $\mathbf{u}_\Gamma$ to the interior of the subregions. Typically $c_{\max}$ is bounded independently of N, while $c_{\min}$ is bounded away from zero by $c(1+\log(N))^{-2}$, with both bounds being independent of the number of subregions. To the space $P^N_{\Gamma,\nabla}(\Omega)$, we associate a bilinear form

$$\tilde{s}(\cdot,\cdot) := \tilde{a}(\mathcal{H}\cdot, \mathcal{H}\cdot).$$

The Lagrange multiplier based preconditioned operator T_{lag} is defined by:

$$T_{\text{lag}} = T_\Gamma + \sum_{i=1}^{M} T_{\Omega_i};$$

the reason for its name will become clear in the next section.

5.4.1 Practical Computation of the Preconditioner

By definition, $T_{\Omega_i}\mathbf{u} \in P^N_{0,\nabla}(\Omega_i)$ is given by

$$a_Q(T_{\Omega_i}\mathbf{u}, \mathbf{v}) = a_Q(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in P^N_{0,\nabla}(\Omega_i).$$

However, the construction of a basis for the space $P^N_{0,\nabla}(\Omega_i)$ is too expensive for the algorithm to be of any practical value. We therefore need an alternative way of computing $T_{\Omega_i}\mathbf{u}$. It is easy to see that this function is the solution of a discrete Stokes problem in Ω_i. Indeed, $T_{\Omega_i}\mathbf{u} \in P^N_{0,\nabla}(\Omega_i)$ satisfies:

$$\left\{ \begin{array}{rcl} a_Q(T_{\Omega_i}\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & a_Q(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in (P^N_0(\Omega_i))^d, \\ \\ b(T_{\Omega_i}\mathbf{u}, q) & = & 0 \quad \forall q \in P^{N-2}(\Omega_i) \cap L^2_0(\Omega_i). \end{array} \right. \tag{5.8}$$

This is a standard Stokes problem that can be solved locally and in parallel for all subdomains Ω_i. The right hand side needed in the first equation of (5.8) is readily available in our iterative procedure, since we work with the nodal basis of $(P^N_0(\Omega))^d$ and the stiffness matrix A.

Similarly, $T_\Gamma\mathbf{u} \in P^N_{\Gamma,\nabla}(\Omega)$ is given by:

$$\tilde{s}(T_\Gamma\mathbf{u}, \mathbf{v}_\Gamma) = a_Q(\mathbf{u}, \mathbf{v}_\Gamma) \quad \forall \mathbf{v}_\Gamma \in P^N_{\Gamma,\nabla}(\Omega).$$

Again, it is not feasible to construct a basis of $P^N_{\Gamma,\nabla}(\Omega)$, and we must find a computationally more effective way of solving this problem. We first note that

$$\dot{P}_{\Gamma,\nabla}(\Omega) = \{\mathbf{v} \in (P^N_0(\Omega))^d \mid \quad \mathbf{v}|_{\Omega_i} = E^{S,N}_i(\mathbf{v}|_{\partial\Omega_i}) \ \ \forall i\},$$

is a space for which a standard nodal basis is easily available. We can select the functions that are one at one interface node and zero at the others, and which are properly extended to the interior of the Ω_i. The dot indicates that the flux constraints of $P^N_{\Gamma,\nabla}$, given in (5.5), are not enforced for the elements of $\dot{P}_{\Gamma,\nabla}(\Omega)$. We see, from (5.5), that $P^N_{\Gamma,\nabla}(\Omega)$ is formed exactly by the functions of $\dot{P}_{\Gamma,\nabla}(\Omega)$ that have zero flux across the boundaries of the Ω_i. By Green's formula:

$$\int_{\partial\Omega_i} \mathbf{u} \cdot \mathbf{n} \, dS = \int_{\Omega_i} \nabla \cdot \mathbf{u} \, dx = 0 \quad \forall i. \tag{5.9}$$

Therefore, we may introduce a space of Lagrange multipliers given by

$$P_0 = \{p \in L_0^2(\Omega), p \text{ is constant in each } \Omega_i\}, \tag{5.10}$$

and $(T_\Gamma \mathbf{u}, p) \in \dot{P}_{\Gamma,\nabla}(\Omega) \times P_0$ satisfies:

$$\begin{cases} \tilde{s}(T_\Gamma \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = a_Q(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \dot{P}_{\Gamma,\nabla}(\Omega), \\ \\ b(T_{\Omega_i} \mathbf{u}, q) = 0 \quad \forall q \in P_0. \end{cases} \tag{5.11}$$

By (5.9), it is clear that the solution $T_\Gamma \mathbf{u}$ of this saddle point problem belongs to $P_{\Gamma,\nabla}^N$.

By taking a basis for $\dot{P}_{\Gamma,\nabla}(\Omega)$ as described above (the degrees of freedom are the values at the interface nodes), and the trivial basis for P_0, this equation may be rewritten equivalently as:

$$\begin{cases} \tilde{S} \underline{T_\Gamma \mathbf{u}} + B_0^t \underline{p} = S \underline{\mathbf{u}} \\ \\ B_0 \underline{T_\Gamma \mathbf{u}} = \underline{0}. \end{cases} \tag{5.12}$$

Here $\tilde{S}$ is the Gram matrix with respect to $\tilde{s}(\cdot,\cdot)$ and the basis of $\dot{P}_{\Gamma,\nabla}(\Omega)$ just described, while B_0 is the Gram matrix with respect to the bilinear form $b(\cdot,\cdot)$ and the pair of spaces $\dot{P}_{\Gamma,\nabla}(\Omega) \times P_0$. $\underline{\mathbf{u}}_\Gamma$ is the vector of nodal values of $\mathbf{u}$ on Γ, and the matrix S is defined as $\mathbf{u}_\Gamma^t S \mathbf{u}_\Gamma = a_Q(\mathbf{u}, \mathbf{u})$, for $\mathbf{u} \in P_0^N(\Omega)$.

Then, by performing a block Gaussian elimination and back-substitution, we easily find that:

$$\underline{T_\Gamma \mathbf{u}} = (I - \tilde{S}^{-1} B_0^t (B_0 \tilde{S}^{-1} B_0^t)^{-1} B_0)(\tilde{S}^{-1} S \underline{\mathbf{u}}_\Gamma).$$

It is easy to see that the matrix $B_0 \tilde{S}^{-1} B_0^t$ is invertible, by proving an inf-sup condition for the pair of spaces $\dot{P}_{\Gamma,\nabla}(\Omega) \times P_0$; see [18, Theorem 4.1]. The matrix $(B_0 \tilde{S}^{-1} B_0^t)$ may be precomputed and factored in a preprocessing step of the algorithm, and then used within the iteration.

We may use a domain decomposition preconditioner of the form:

$$\tilde{S}^{-1} = \sum_{V_k} \tilde{S}_{V_k}^{-1} + \tilde{S}_{\text{global}}^{-1},$$

where the V_k are locally supported subspaces, as is the case for the iterative substructuring algorithms of Section 3.4. From (5.9), we observe that each column of B_0^t corresponds to the computation of the flux of a function across the boundary of a subregion Ω_i. Then, the product of $\tilde{S}_{V_k}^{-1}$ and this column of B_0^t vanishes if the support of V_k does not intersect $\partial\Omega_i$, and hence does not need to be performed. In other words, in the precomputation of $(B_0\tilde{S}^{-1}B_0^t)$, the local parts of the preconditioner need only be applied a number of times that is equal to the number of subdomains that intersect the support of each individual local space. The same reasoning also shows that the computation of this matrix involves the solution of a number of global problems that is equal to the number of subdomains.

The saddle point problem (5.12) may also be solved iteratively. Bramble and Pasciak [18] have proposed a preconditioned conjugate gradient method for this system. An alternative would be to use a block preconditioner; cf. [55, 69, 106].

5.4.2 Analysis of the Preconditioner

The operator T_{lag} has one global space, $P^N_{\Gamma,\nabla}$, while the local spaces are orthogonal to each other in the $a_Q(\cdot,\cdot)$-norm. Hence, hypothesis **H3** of Theorem 2.3.1 is trivially satisfied with $\mathcal{E}$ equal to the identity matrix. Note that $\tilde{s}(\cdot,\cdot)$ involves a partitioning into subspaces; however, this fact does not enter the analysis of T_{lag} since these subspaces are all grouped in $P^N_{\Gamma,\nabla}$, and we may use (5.7) directly.

Hypothesis **H2** follows immediately for the local spaces $P^N_{0,\nabla}(\Omega_i)$, $1 \leq i \leq M$. For the space $P^N_{\Gamma,\nabla}(\Omega)$, we have

$$a_Q(\mathbf{u},\mathbf{u}) \leq \frac{1}{\beta_N c_0} a_Q(\mathcal{H}\mathbf{u},\mathcal{H}\mathbf{u}) \leq \frac{c_{\max}}{\beta_N c_0}\tilde{s}(\mathbf{u},\mathbf{u}),$$

by Lemma 5.3.2 and (5.7), and we may take $\omega = \frac{c_{\max}}{\beta_N c_0}$.

The decomposition (5.6) is orthogonal in the $a_Q(\cdot,\cdot)$-inner product. Then, given $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$, we can easily define $\mathbf{u}_\Gamma \in P^N_{\Gamma,\nabla}$, and then the $\mathbf{u}_i \in P^N_{0,\nabla}(\Omega_i)$, such that

$$\mathbf{u} = \mathbf{u}_\Gamma + \sum_{i=1}^{M} \mathbf{u}_i$$

and

$$a_Q(\mathbf{u}_\Gamma,\mathbf{u}_\Gamma) + \sum_{i=1}^{M} a_Q(\mathbf{u}_i,\mathbf{u}_i) \leq a_Q(\mathbf{u}_\Gamma,\mathbf{u}_\Gamma).$$

Then, by (5.7), we obtain:

$$\begin{aligned}\tilde{a}(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) &\leq \frac{1}{c_{\min}} a_Q(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \\ &\leq \frac{1}{c_{\min}} a_Q(\mathbf{u}, \mathbf{u}).\end{aligned}$$

Therefore, we have:

$$\tilde{a}(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) + \sum_{i=1}^{M} a_Q(\mathbf{u}_i, \mathbf{u}_i) \leq \frac{1}{c_{\min}} a_Q(\mathbf{u}, \mathbf{u}),$$

which proves hypothesis **H1** of Theorem 2.3.1 with $C_0^2 = 1/c_{\min}$, and hence the following result.

Theorem 5.4.1 *The operator T_{lag} satisfies*

$$\kappa(T_{lag}) \leq C \frac{c_{\max}}{\beta_N c_0 c_{\min}}.$$

The condition number of this method appears to be worse than the corresponding algorithm for the Laplace equation only by a factor β_N^{-1}; the same will be true for all the preconditioners of this chapter. Although this factor grows algebraically with N, as predicted by the theory (Lemma 1.5.1), a much more moderate growth is observed for the practical range of values of N; see [75]. One could use a modification of the algorithm of subsection 4.6.2 to generate the bilinear form $\tilde{s}(\cdot,\cdot)$; we would then expect an iteration count of the same order as those encountered in the experiments described in subsection 4.6.3.

We note that the computation of $T_\Gamma \mathbf{u}$ requires the evaluation of $\tilde{S}^{-1}\underline{\mathbf{v}}$ twice per iteration, where $\underline{\mathbf{v}}$ is a generic vector of the same size as $\underline{\mathbf{u}}$.

Remark 5.4.1 *By the FEM-SEM equivalence and Lemma 5.3.2, we may replace the extension operator $E_i^{S,N}$ by $E_i^{S,h}$, and still maintain the condition number estimate of Theorem 5.4.1. The same is true of the algorithms proposed in the next two sections.*

Similarly, the operator $E_i^{S,N}$ may be replaced by $E_i^{S,N}|_{\hat{\Omega}_i}$, where $\hat{\Omega}_i$ is a right parallelepiped which is derived from Ω_i This extension can be computed exactly and efficiently by using a fast method based on the tensor product form of the Laplace operator on $\hat{\Omega}_i$. This approach has been used very efficiently by Couzy and Deville [44].

5.5 Another Domain Decomposition Preconditioner

If we consider only iterative methods with velocities in the discrete divergence free space $P^N_{0,\nabla}$, then the design and analysis of T_{lag} from the standard result given in Lemma 5.3.2 is relatively straightforward. However, it has two serious drawbacks: two applications of $\tilde{S}^{-1}$ to a vector are necessary in each iteration, and it requires the factorization of $B_0\tilde{S}^{-1}B_0^t$, or at least the exact solution of a problem with this coefficient matrix in each step of the iteration.

In this section, we present a first attempt to reduce these difficulties. We propose a different method by partitioning $P^N_{0,\nabla}(\Omega)$ into a coarse space and local subspaces. In each of these subspaces the constraints are easier to deal with.

In our view, such an approach gives more insight into the resulting method and why it works, while the approach of Section 5.4 relies, maybe too heavily, on the equivalence (5.7) and the use of Lagrange multipliers. It can also be argued that the preconditioners proposed in this section and in the next give more insight into an iterative solution of the Navier-Stokes equation.

The analysis of our preconditioner is closely related to the study of Schwarz methods with non-embedded spaces, which has attracted considerable attention in the past few years; see [22, 23, 28, 17, 42]

There is no obvious basis of locally supported functions for $P^N_{0,\nabla}$, and we will therefore work with a coarse space that is not contained in this space. We note that there are many finite element methods for which the choice of a coarse space and of local spaces for the discrete divergence free space is trivial by using a curl-like isomorphism between a space of discrete stream functions and the discrete divergence free ; see [22, 23, 87]. We are far from this situation here, and our construction has to be essentially different.

We describe the construction in detail for two dimensions. The three dimensional case is analogous; see [60, Section II.3.1], and Remark 5.5.2. For a reference square $\hat{\Omega} = [-1, 1]^2$, let

$$V^H_{\mathbf{n}}(\hat{\Omega}) = (Q_1(\hat{\Omega}))^2 \oplus \text{span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\},$$

where $\mathbf{p}_i \in (Q_2(\hat{\Omega}))^2$ vanishes on the edges $\mathcal{E}_j$ for $j \neq i$, and is normal to $\mathcal{E}_i$. For example, for the edge $\mathcal{E}_1$ given by $x = 1$, $\mathbf{p}_1 = ((1 + x)(1 - y^2), 0)$.

The space $V_{\mathbf{n}}^{H}(\Omega) \subset (H_0^1(\Omega))^2$ is the space whose restrictions to each Ω_i is the image of $V_{\mathbf{n}}^{H}(\hat{\Omega})$ under the mapping F_i, which is here taken to be isoparametric with respect to the space $(Q_1(\hat{\Omega}))^2$; see [60, Section A.2]. There are 12 degrees of freedom per element, namely the nodal values at each vertex and the fluxes across each of the edges.

Let $Q_0^H(\Omega)$ be the space of functions of zero mean on Ω that are constant within each subdomain Ω_i. It is well-known that for the discretization of the Stokes problem on the coarse mesh, the pair $V_{\mathbf{n}}^H - Q_0^H$ yields a stable discretization in the Babuška-Brezzi sense, with a stability constant bounded away from zero independently of H. The proof of this result involves the construction of an operator $\pi_H : (H_0^1(\Omega))^3 \rightarrow V_{\mathbf{n}}^H(\Omega)$ such that:

$$\begin{cases} \int_{\mathcal{E}}(\pi_H \mathbf{v} - \mathbf{v}) \cdot \mathbf{n} \, dS = 0 \quad \forall \text{ edges } \mathcal{E} \text{ of } \mathcal{T}_H, \\ ||\mathbf{v} - \pi_H \mathbf{v}||_{L^2(\Omega_i)} \leq CH|\mathbf{v}|_{H^1(\mathcal{R}(\Omega_i))}, \\ |\pi_H \mathbf{v}|_{H^1(\Omega_i)} \leq C|\mathbf{v}|_{H^1(\mathcal{R}(\Omega_i))}, \end{cases} \tag{5.13}$$

where $\mathcal{R}(\Omega_i)$ is the union of the subdomains adjacent to Ω_i; see [60, Lemma 3.2]. The operator π_H plays the role of the fine-to-coarse operator in our algorithm; for standard elliptic problems, the L^2-projection on the coarse grid space is often used for that purpose.

Let $V_{\mathbf{n},\nabla_H}^H(\Omega)$ be defined by:

$$V_{\mathbf{n},\nabla_H}^H(\Omega) = \{\mathbf{u} \in V_{\mathbf{n}}^H \mid \int_{\Omega_i} \nabla \cdot \mathbf{u} \, dx = \int_{\partial\Omega_i} \mathbf{u} \cdot \mathbf{n} \, dS = 0\}.$$

This space plays the role of our coarse space, but it is clearly not contained in $P_{0,\nabla}^N(\Omega)$, since a function $\mathbf{u} \in V_{\mathbf{n},\nabla_H}^H(\Omega)$ in general fails to have a divergence orthogonal to the space $P^{N-2}(\Omega_i)$ in $L^2(\Omega_i)$; see (5.4). We define a transfer operator $I_H^h : V_{\mathbf{n},\nabla_H}^H(\Omega) \rightarrow P_{0,\nabla}^N(\Omega)$ by:

$$\begin{cases} I_H^h(\mathbf{u}_H)|_{\partial\Omega_i} = \mathbf{u}_H|_{\partial\Omega_i} \\ I_H^h(\mathbf{u}_H)|_{\Omega_i} = E_i^{S,N}(\mathbf{u}_H|_{\partial\Omega_i}). \end{cases} \tag{5.14}$$

This operator has the following stability and approximation properties.

Lemma 5.5.1

$$|I_H^h(\mathbf{u}_H)|_{H^1(\Omega_i)} \le C|\mathbf{u}_H|_{H^1(\Omega_i)},$$

$$||\mathbf{u}_H - I_H^h(\mathbf{u}_H)||_{L^2(\Omega_i)} \le CH|\mathbf{u}_H|_{H^1(\Omega_i)}.$$

Here, C is independent of N and H.

Proof. The constant C can easily be bounded from above by β_N^{-1}, by using Lemma 5.3.2. However, $\mathbf{u}_H$ has degree at most 2, and is not, therefore, a generic function that is being extended as in the second equation of (5.14).

We first note that for $\mathbf{u}_H \in V_{\mathbf{n}}^H$, $q := \nabla \cdot \mathbf{u}_H$ belongs to $P^2(\Omega_i)$. We also note that it is enough to consider one subregion at a time, and that by a scaling argument, these estimates can be obtained from similar inequalities for the reference substructure $\hat{\Omega}$.

Let $\Lambda = [-1, 1]$, and let π_{N-2} be the orthogonal L^2-projection from $L^2(\Lambda)$ into $P^{N-2}(\Lambda)$. We claim that, $\forall \varphi_N \in P^N(\Lambda) \cap H_0^1(\Lambda)$, if $\pi_{N-2}(\varphi_N)$ is of degree at most 2, then

$$||\varphi_N||_{L^2(\Lambda)} \le C_1||\pi_{N-2}(\varphi_N)||_{L^2(\Lambda)}, \tag{5.15}$$

where C_1 is independent of N.

We follow the proof of Lemma 1.5.1, given in [12, Proposition IV.7.2] for two dimensions. Indeed, let $\varphi_N = \sum_{k=0}^N \alpha_k L_k$ be the orthogonal decomposition of φ_N; see Section 4.2. Then, $\pi_{N-2}(\varphi_N) = \sum_{k=0}^{N-2} \alpha_k L_k$, which implies that $\alpha_k = 0$, for $3 \le k \le N-2$. Moreover, it is clear that

$$||\varphi_N||^2_{L^2(\Lambda)} = ||\pi_{N-2}(\varphi_N)||^2_{L^2(\Lambda)} + |\alpha_{N-1}|^2 \frac{1}{(N-1/2)} + |\alpha_N|^2 \frac{1}{(N+1/2)}, \tag{5.16}$$

by (4.3) and the orthogonality of the decomposition. Since $\varphi_N(\pm 1) = 0$ and the Legendre polynomials have values ± 1 at the endpoints, we have:

$$\begin{gathered}\alpha_0 + \alpha_1 + \alpha_2 + \alpha_{N-1} + \alpha_N = 0 \\ \alpha_0 - \alpha_1 + \alpha_2 + (-1)^{N-1}\alpha_{N-1} + (-1)^N \alpha_N = 0.\end{gathered}$$

This in turn implies that, if N is even,

$$|\alpha_N|^2 \le C(|\alpha_0|^2 + |\alpha_2|^2) \le C||\pi_{N-2}(\varphi_N)||^2_{L^2(\Lambda)}.$$

It is clear that a similar argument applies for N odd and for α_{N-1}. These inequalities, together with (5.16), clearly imply (5.15).

Next, (5.15) and the proof of Proposition IV.7.2 in [12] can be applied straightforwardly to obtain a polynomial $\hat{\mathbf{v}}_N \in P_0^N(\hat{\Omega})$ such that

$$b(\hat{\mathbf{v}}_N, \hat{q}) = ||\hat{q}||^2_{L^2(\Lambda)},$$

and

$$||\hat{\mathbf{v}}_N||^2_{H^1(\hat{\Omega})} \leq CC_1||\hat{q}||^2_{L^2(\Lambda)},$$

where C is independent of N, C_1 comes from (5.15), and $\hat{q}$ corresponds to q in the reference structure. Let $\hat{\mathbf{v}}_H$ be defined by $\hat{\mathbf{v}}_H = \hat{\mathbf{u}}_H - \hat{\mathbf{v}}_N$. Then, $\hat{\mathbf{v}}_H$ agrees with $\hat{\mathbf{u}}_H$ (and with $\hat{I}_H^h\hat{\mathbf{u}}_H$) on $\partial\hat{\Omega}$, belongs to $P_\nabla^N(\hat{\Omega})$, and satisfies:

$$|\hat{\mathbf{v}}_H|^2_{H^1(\hat{\Omega})} \leq C(|\hat{\mathbf{u}}_H|^2_{H^1(\hat{\Omega})} + ||\hat{q}||^2_{L^2(\Lambda)}) \leq C|\hat{\mathbf{u}}_H|^2_{H^1(\hat{\Omega})},$$

where the new C incorporates both C and C_1, above.

By the definition of $\hat{I}_H^h$ and the minimizing property of $E_i^{S,N}$, equation (5.2), we have:

$$|\hat{I}_H^h\hat{\mathbf{u}}_H|^2_{H^1(\hat{\Omega})} \leq |\hat{\mathbf{v}}_H|^2_{H^1(\hat{\Omega})} \leq C|\hat{\mathbf{u}}_H|^2_{H^1(\hat{\Omega})},$$

which is the H^1-stability on $\hat{\Omega}$.

The L^2-stability follows from the Poincaré-Friedrichs inequality. Indeed, $\hat{\mathbf{u}}_H - \hat{I}_H^h\hat{\mathbf{u}}_H$ has zero trace on $\partial\hat{\Omega}$. Then, (1.1) applied to this function gives

$$||\hat{\mathbf{u}}_H - I_H^h(\hat{\mathbf{u}}_H)||_{L^2(\hat{\Omega})} \leq C|\mathbf{u}_H|_{H^1(\hat{\Omega})}.$$

The proof of the lemma now follows from a straightforward scaling argument.

□

For $\mathbf{u}, \mathbf{v} \in V^H$, we define the coarse space bilinear form by $a_H(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v})$. The coarse solver T_h^H is given by

$$a_H(T_h^H\mathbf{u}, \mathbf{w}) = a_Q(\mathbf{u}, I_H^h\mathbf{w}) \quad \forall \mathbf{w} \in V^H_{\mathbf{n},\nabla_H}(\Omega).$$

For each edge $\mathcal{E}_k$ shared by two subregions Ω_i and Ω_j, let Ω_{ij} be the union of Ω_i, Ω_j, and $\mathcal{E}_k$. The local space $V_{\mathcal{E}_k} \subset P_{0,\nabla}^N(\Omega)$ consists of functions $\mathbf{u}_{\mathcal{E}_k}$ with support in $\bar{\Omega}_{ij}$, and whose values in the interior of Ω_i and Ω_j are given by $E_i^{S,N}$ and $E_j^{S,N}$, respectively. This definition implies that $\forall \mathbf{u}_{\mathcal{E}_k} \in V_{\mathcal{E}_k} \subset P^N_{0,\nabla}(\Omega)$, $\int_{\mathcal{E}_k} \mathbf{u}_{\mathcal{E}_k} \cdot \mathbf{n}\, dS = 0$. The bilinear form associated with $V_{\mathcal{E}_k}$ is $a_Q(\cdot,\cdot)$.

For each interior vertex v_n, let $\mathcal{E}(v_n)$ be the collection of all edges having v_n as an endpoint. We define $\phi_{v_n,x} \in P^N_{0,\nabla}(\Omega)$ by assigning values at the interface nodes, and using the $E_i^{S,N}$ to extend these values to the interior of the subdomains. We let $\phi_{v_n,x}(v_n) = (1,0)$, let $\phi_{v_n,x}$ be equal to zero at all the interface nodes not adjacent to v_n, and $\forall \mathcal{E}_k \in \mathcal{E}(v_n)$, we let $\phi_{v_n,x}$ be equal to a constant vector at the node v'_n next to v_n on the edge $\mathcal{E}_k$. This constant vector is taken to be normal to the interface at v'_n, and so that

$$\int_{\mathcal{E}_k} \phi_{v_n,x} \cdot \mathbf{n}\, dS = 0.$$

We define $\phi_{v_n,y}$ analogously. The one-dimensional vertex spaces are given by:

$$V_{v_n,x} = \operatorname{span}\{\phi_{v_n,x}\} \text{ and } V_{v_n,y} = \operatorname{span}\{\phi_{v_n,y}\}.$$

The bilinear form associated with the vertex spaces is $a_Q(\cdot,\cdot)$.

The interior spaces are $V_{\Omega_i} = P^N_{0,\nabla}(\Omega_i)$, and the bilinear form associated with all of them is $a_Q(\cdot,\cdot)$.

The preconditioned operator is now

$$T_{\mathbf{n}} = I_H^h T_h^H + \sum_{v_n} (T_{v_n,x} + T_{v_n,y}) + \sum_{\mathcal{E}_k} T_{\mathcal{E}_k} + \sum_{i=1}^{M} T_{\Omega_i}. \tag{5.17}$$

This operator does not exactly fit the Schwarz framework given by Theorem 2.3.1, but the following extension of that result for our particular case is immediately seen to hold by following the steps of the standard proof and making the appropriate changes; cf. [22, 23, 28, 17, 42]. Let V_0 denote the coarse space $V^H_{\mathbf{n},\nabla_H}(\Omega)$, and let V_s and $b_s(\cdot,\cdot)$, $s \geq 1$ denote all the other (local) spaces and bilinear forms, respectively.

Theorem 5.5.1 *Suppose that $T_{\mathbf{n}}$ satisfies*

H1': *There exists a constant C_0^2 such that every $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$ can be decomposed into a sum $I_H^h \mathbf{u}_H + \sum_{s\geq 1} \mathbf{u}_s$ such that*

$$a_H(\mathbf{u}_H, \mathbf{u}_H) + \sum_{s\geq 1} b_s(\mathbf{u}_s, \mathbf{u}_s) \leq C_0^2 a_Q(\mathbf{u}, \mathbf{u}).$$

H2'*:*

$$a_Q(I_H^h \mathbf{u}_H, I_H^h \mathbf{u}_H) \le C a_H(\mathbf{u}_H, \mathbf{u}_H) \quad \forall \mathbf{u}_H \in V^H_{\mathbf{n},\nabla_H}$$

and for $s \ge 1$*,*

$$a_Q(\mathbf{u}_s, \mathbf{u}_s) \le C b_s(\mathbf{u}_s, \mathbf{u}_s).$$

Then, $\lambda_{\max}(T_{\mathbf{n}}) \le C$ *and* $\lambda_{\min}(T_{\mathbf{n}}) \ge C/C_0^2$*.*

We next apply this theorem to derive a condition number estimate of $T_{\mathbf{n}}$. Hypothesis **H2**' clearly holds by Lemma 5.5.1.

To verify **H1**', we have to decompose a function $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$. Let $\tilde{\mathbf{u}}_H = \pi_H \mathbf{u}$, and let $\mathbf{w} = \tilde{\mathbf{u}}_H - \mathbf{u}$. By (5.13), $\mathbf{w}$ has flux zero across all the edges of the Ω_i, and satisfies $|\mathbf{w}|_{H^1(\Omega_i)} \le |\mathbf{u}|_{H^1(\mathcal{R}(\Omega_i))}$ and $||\mathbf{w}||_{L^2(\Omega_i)} \le CH|\mathbf{u}|_{H^1(\mathcal{R}(\Omega_i))}$.

For each interior vertex v_n, let $\tilde{\phi}_{v_n,x}$ coincide with $\phi_{v_n,x}$ on the interface Γ, and let $\tilde{\phi}_{v_n,x}$ be the zero vector at all the GLL nodes interior to the Ω_i; $\tilde{\phi}_{v_n,y}$ is defined analogously. Let

$$\tilde{\mathbf{u}}_{v_n,x} = w_1(v_n)\tilde{\phi}_{v_n,x} \qquad \tilde{\mathbf{u}}_{v_n,y} = w_2(v_n)\tilde{\phi}_{v_n,y}.$$

Then, $\mathbf{z} = \mathbf{w} - \sum_{v_n}(\tilde{\mathbf{u}}_{v_n,x} + \tilde{\mathbf{u}}_{v_n,y})$ vanishes at all the vertices and has zero flux across the edges, since $\mathbf{w}$, $\tilde{\phi}_{v_n,x}$, and $\tilde{\phi}_{v_n,y}$ have this property. We claim that

$$|\mathbf{z}|^2_{H^1(\Omega_i)} \le C(1 + \log(N))|\mathbf{u}|^2_{H^1(\mathcal{R}(\Omega_i))},$$

and

$$||\mathbf{z}||^2_{L^2(\Omega_i)} \le CH^2|\mathbf{u}|^2_{H^1(\mathcal{R}(\Omega_i))}.$$

Indeed, by using the definition of the $\tilde{\phi}_{v_n,x}$ and the FEM-SEM equivalence, we have:

$$||\tilde{\phi}_{v_n,x}||^2_{L^2(\Omega_i)} \le \frac{CH^2}{N^4} \quad \text{and} \quad |\tilde{\phi}_{v_n,x}|^2_{H^1(\Omega_i)} \le C;$$

the same inequalities also hold for $\tilde{\phi}_{v_n,y}$. These in turn imply, together with a standard Sobolev-like inequality, that

$$||\mathbf{z}||^2_{L^2(\Omega_i)} \;\le\; C(||\mathbf{w}||^2_{L^2(\Omega_i)} + ||\mathbf{w}||^2_{L^\infty(\Omega_i)} \sum_{v_n \in \partial\Omega_i} (||\tilde{\phi}_{v_n,x}||^2_{L^2(\Omega_i)} + ||\tilde{\phi}_{v_n,y}||^2_{L^2(\Omega_i)}))$$

$$\begin{aligned}
&\leq C(||\mathbf{w}||^2_{L^2(\Omega_i)} + C(1+\log(N))(||\mathbf{w}||^2_{H^1(\Omega_i)})(\frac{CH^2}{N^4})) \\
&\leq CH^2(1+\frac{1+\log(N)}{N^4})||\mathbf{w}||^2_{H^1(\Omega_i)} \\
&\leq CH^2|\mathbf{u}|^2_{H^1(\mathcal{R}(\Omega_i))},
\end{aligned}$$

where the last inequality follows from the inequalities satisfied by $\mathbf{w}$, and $||\cdot||_{H^1(\Omega_i)}$ refers to the scaled H^1-norm; see Section 1.2 A similar argument shows that

$$\begin{aligned}
|\mathbf{z}|^2_{H^1(\Omega_i)} &\leq C(1+\log(N))||\mathbf{w}||^2_{H^1(\Omega_i)} \\
&\leq C(1+\log(N))|\mathbf{w}|^2_{H^1(\Omega_i)}.
\end{aligned}$$

We may now partition $\mathbf{z}$ into $\mathbf{z} = \sum_{\mathcal{E}_k} \tilde{\mathbf{u}}_{\mathcal{E}_k}$, where $\tilde{u}_{\mathcal{E}_k} = I^h(\vartheta_{\mathcal{E}_k}\mathbf{z})$, and the $\vartheta_{\mathcal{E}_k}$ are the two dimensional analogues of the partition of unity functions of Lemma 3.3.7. This same lemma implies, together with the estimates just obtained for $\tilde{\mathbf{u}}_H$, $\tilde{\mathbf{u}}_{v_n,x}$, $\tilde{\mathbf{u}}_{v_n,y}$, and $\mathbf{z}$, that

$$|\tilde{\mathbf{u}}_H|^2_{H^1(\Omega)} + \sum_{v_n}(|\tilde{\mathbf{u}}_{v_n,x}|^2_{H^1(\Omega)} + |\tilde{\mathbf{u}}_{v_n,y}|^2_{H^1(\Omega)}) + \sum_{\mathcal{E}_k} |\tilde{\mathbf{u}}_{\mathcal{E}_k}|^2_{H^1(\Omega)} \leq C(1+\log(N))^3|\mathbf{u}|^2_{H^1(\Omega)},$$

since the union of the $\mathcal{R}(\Omega_i)$ or the union of the regions adjacent to the vertices cover the region a finite (and small) number of times.

This decomposition of $\mathbf{u}$ is such that

$$(\tilde{\mathbf{u}}_H + \sum_{v_n}(\tilde{\mathbf{u}}_{v_n,x} + \tilde{\mathbf{u}}_{v_n,y}) + \sum_{\mathcal{E}_k} \tilde{\mathbf{u}}_{\mathcal{E}_k})|_\Gamma = \mathbf{u}|_\Gamma.$$

We next modify this decomposition, keeping the values on Γ, and extending all the components to the interior of the Ω_i as Stokes-harmonic functions, by using the $E_i^{S,N}$. The modified components are now denoted by $\mathbf{u}_H$, $\mathbf{u}_{v_n,x}$, $\mathbf{u}_{v_n,y}$, and $\mathbf{u}_{\mathcal{E}_k}$. By Lemma 5.3.2, and since the edge components are now $a_Q(\cdot,\cdot)$-orthogonal to the functions in $P^N_{0,\nabla}(\Omega_i)$, we can choose $\mathbf{u}_{\Omega_i} \in P^N_{0,\nabla}(\Omega_i)$ such that

$$\mathbf{u} = \mathbf{u}_H + \sum_{v_n}(\mathbf{u}_{v_n,x} + \mathbf{u}_{v_n,y}) + \sum_{\mathcal{E}_k} \mathbf{u}_{\mathcal{E}_k} + \sum_{\Omega_i} \mathbf{u}_{\Omega_i},$$

and

$$a_Q(\mathbf{u}_H,\mathbf{u}_H) + \sum_{v_n}(a_Q(\mathbf{u}_{v_n,x},\mathbf{u}_{v_n,x}) \quad + \quad a_Q(\mathbf{u}_{v_n,y},\mathbf{u}_{v_n,y})) + \sum_{\mathcal{E}_k} a_Q(\mathbf{u}_{\mathcal{E}_k},\mathbf{u}_{\mathcal{E}_k}) +$$

$$\sum_{\Omega_i} a_Q(\mathbf{u}_{\Omega_i}, \mathbf{u}_{\Omega_i}) \;\leq\; \frac{C(1+\log(N))^3}{\beta_N} a_Q(\mathbf{u}, \mathbf{u}).$$

The following result easily follows from Theorem 5.5.1.

Theorem 5.5.2 *The condition number of $T_{\mathbf{n}}$ satisfies:*

$$\kappa(T_{\mathbf{n}}) \leq \frac{C(1+\log(N))^3}{\beta_N}.$$

Remark 5.5.1 *In this version of the algorithm, we use exact solvers for all the local subspaces. We can also use inexact solvers, and we mention here just one possibility. For the edge spaces, we can use the bilinear form*

$$\tilde{a}(\mathbf{u}, \mathbf{u}) = a_Q(\mathcal{H}\mathbf{u}, \mathcal{H}\mathbf{u}),$$

or some other form that is spectrally equivalent to $\tilde{a}(\cdot, \cdot)$, e.g., a tensor product based inexact solver for the Dirichlet problem in the union of two subregions; see Couzy and Deville [44].

Remark 5.5.2 *In three dimensions, edge and face functions play the role of the vertex and edge functions of the two dimensional version, respectively. For each edge, the edge function is the analogue of the ϕ_{v_n} above; it is nonzero for the interface nodes adjacent to the edge, and have zero flux across all the faces of the subregions. The condition number estimate is the same as in Theorem 5.5.2, where β_N is now the Babuška-Brezzi constant for the three dimensional discretization.*

Remark 5.5.3 *The same analysis would apply for a finite element discretization of the Stokes problem, where the spaces $(P_0^N(\Omega))^d$ and $\bar{P}^{N-2}(\Omega)$ are replaced by a standard stable pair of spaces, for which the Babuška- Brezzi condition is independent of the discretization parameter h. We require the pressure space to be formed by functions discontinuous across subdomain boundaries. The upper bound of the theorem is then replaced by $C(1+\log(H/h))^3$. It is also clear that an analogous result holds if a different stable finite element pair replaces the pair $V_{\mathbf{n}}^H \times P_0$ in the construction of the coarse space, as long as the coarse pressure space is formed by functions discontinuous across subdomain boundaries.*

5.6 A Stream Function Based Coarse Space

One advantage of using the operator $T_{\mathbf{n}}$ of Section 5.5 is that the coarse space for this operator is a finite element space commonly used for the solution of the Stokes problem, except here it is used for the coarse space solution. We now propose another coarse space that makes it possible to avoid the use of the Stokes-harmonic extensions $E_i^{S,N}$ in the definition of the coarse component. It relies on a finite element space $\tilde{V}^H \in C^1(\Omega)$ formed by piecewise polynomials and provides a coarse space of even lower dimension. We restrict ourselves to two dimensions, and remark that an analogous, although much more involved theory could be developed for three-dimensional problems, by using vector potentials and a different formulation of the Stokes equations; see [60, Section III.5].

The main characteristic of the new coarse space $V^H = \mathbf{curl}(\tilde{V}^H)$ is that $\mathbf{u}_H \in V^H$ is exactly divergence free, and therefore belongs to $P^N_{0,\nabla}(\Omega)$. However, this requires the use of C^1 elements, and we have only been able to satisfy this requirement if the mappings $F_i : \hat{\Omega} \to \Omega_i$ are affine; in the remainder of this section we assume that this relatively restrictive condition holds. We also assume that Ω is a connected polygonal region, though not necessarily simply connected. Let $\partial\Omega^{(0)}, \partial\Omega^{(1)}, \ldots, \partial\Omega^{(p)}$ be the connected components of $\partial\Omega$, where $\partial\Omega^{(0)}$ is also the boundary of the unbounded component of the complement of Ω.

For the reference square $\hat{\Omega}$, the Bogner-Fox-Schmit finite element space is the space of polynomials $\hat{p} \in Q_2(\hat{\Omega})$; see [43, Section 2.2]. The degrees of freedom are

$$\begin{cases} \hat{p}(\hat{a}_i), & \hat{a}_i \text{ a vertex,} \\ D\hat{p}(\hat{a}_i)(\hat{a}_j - \hat{a}_i), & \hat{a}_j \text{ adjacent to } \hat{a}_i, \\ D^2\hat{p}(\hat{a}_i)(\hat{a}_j - \hat{a}_i)(\hat{a}_{j'} - \hat{a}_i), & \hat{a}_j \neq \hat{a}_{j'}, \text{ both adjacent to } \hat{a}_i; \end{cases}$$

see Figure 5.1, where the notation for the degrees of freedom is clear.

For our purposes, we will not need the degrees of freedom associated with the mixed second derivative at the vertices. Let $\tilde{V}_b^H(\hat{\Omega})$ be the Bogner-Fox-Schmit space of bicubic functions (hence the subscript b) such that the mixed second derivatives at the vertices are zero; this space has been used by Zhang [118, 119], in a different context. The mappings F_i now define the space $\tilde{V}_b^H(\Omega) \subset H_0^2(\Omega)$, obtained by

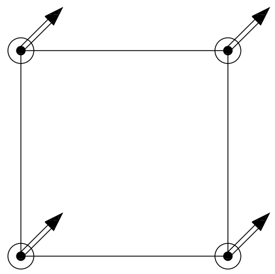

Figure 5.1: The Bogner-Fox-Schmit element

setting all the degrees of freedom on $\partial\Omega$ to zero. The coarse space used for our algorithm is given by

$$V_b^H(\Omega) = \mathbf{curl}(\tilde{V}_b^H(\Omega)),$$

where $\mathbf{curl}(\psi) = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x})$. The **curl** operator is well defined since all the functions in $\tilde{V}_b^H(\Omega)$ have continuous first derivatives.

For $\mathbf{u} \in P_{0,\nabla}^N(\Omega)$, the coarse projection $T_b^H\mathbf{u} \in V_b^H(\Omega)$ is given by

$$a(T_b^H\mathbf{u}, \mathbf{v}_H) = a_Q(\mathbf{u}, \mathbf{v}_H) \quad \forall \mathbf{v}_H \in V_b^H(\Omega). \tag{5.18}$$

This problem may be viewed as a finite element approximation to a biharmonic equation, in terms of stream functions in $\tilde{V}_b^H(\Omega)$. Indeed, if $\psi_H, \phi_H \in \tilde{V}_b^H(\Omega)$ are such that $\mathbf{curl}(\psi_H) = T_b^H\mathbf{u}$ and $\mathbf{curl}(\phi_H) = \mathbf{v}_H$, ψ_H may be found as the solution to the equations:

$$\begin{cases} \nu(\Delta\psi_H, \Delta\phi_H)_{L^2(\Omega)} = a_Q(\mathbf{u}, \mathbf{curl}(\phi_H)) \quad \forall \phi_H \in \tilde{V}_b^H(\Omega), \\ \psi_H = 0 \text{ on } \partial\Omega^{(0)}, \quad \psi_H = c_j \text{ on } \partial\Omega^{(j)}, \quad 1 \le j \le p, \\ \dfrac{\partial\psi_H}{\partial\mathbf{n}} = \mathbf{0} \text{ on } \partial\Omega. \end{cases} \tag{5.19}$$

We note that the c_j are unknown constants. In the general non-homogeneous case $g|_{\partial\Omega} \not\equiv 0$ as in (1.9), an analogous formulation is available; see [60, Theorem I.5.5].

In the finite element literature, many different ways are described for solving the biharmonic equation as a way of solving the Stokes problem. These methods are designed for solving the problem on a fine mesh, for which the construction of a

C^1 element is very costly. That has motivated several formulations of mixed type, for which the C^1 requirement is relaxed; see [60, Chapter III]. Here, we insist on this requirement, so that the operator I_h^H is given by the trivial operator, which is allowed since $V_b^H(\Omega) \subset P_{0,\nabla}^N(\Omega)$.

The other spaces are the same as before. Therefore, our new preconditioned operator is given by:

$$T_b = T_b^H + \sum_{v_n}(T_{v_n,x} + T_{v_n,y}) + \sum_{\mathcal{E}_k} T_{\mathcal{E}_k} + \sum_{\Omega_i}^{M} T_{\Omega_i}.$$

Our goal now is to prove a condition number estimate for the operator T_b. We start by analyzing the new coarse space in detail. For a vertex $\mathbf{a}_j$ of the coarse triangulation $\mathcal{T}_H$, let $\mathcal{R}(\mathbf{a}_j)$ be the union of the subdomains sharing $\mathbf{a}_j$. We define a projection operator by:

$$\begin{aligned} \tilde{\pi}_b^H : P_{0,\nabla}^N(\Omega) &\longmapsto \tilde{V}_b^H(\Omega) \\ \mathbf{u} &\longmapsto \psi = \psi(\mathbf{u}), \end{aligned}$$

where for a given $\mathbf{u}$, ψ is defined by:

$$\begin{cases} (\dfrac{\partial \psi}{\partial y}, -\dfrac{\partial \psi}{\partial x})(\mathbf{a}_j) = (0,0) \quad \forall \mathbf{a}_j \in \partial\Omega, \\ (\dfrac{\partial \psi}{\partial y}, -\dfrac{\partial \psi}{\partial x})(\mathbf{a}_j) = \dfrac{1}{|\mathcal{R}(\mathbf{a}_j)|}\displaystyle\int_{\mathcal{R}(\mathbf{a}_j)} \mathbf{u}(x)\,dx \quad \forall \mathbf{a}_j \in \Omega, \\ \dfrac{\partial^2 \psi}{\partial x_1 \partial x_2}(\mathbf{a}_j) = 0 \quad \forall \mathbf{a}_j \in \overline{\Omega}, \\ \psi(\mathbf{a}_0) = 0 \quad \text{for a fixed } \mathbf{a}_0 \in \partial\Omega^{(0)}, \\ \psi(\mathbf{a}_\ell) - \psi(\mathbf{a}_j) = \displaystyle\int_{\mathcal{E}_k} \mathbf{u}\cdot\mathbf{n}\,dS \quad \forall \mathbf{a}_j, \mathbf{a}_\ell \in \overline{\Omega}, \end{cases} \tag{5.20}$$

where $\mathbf{a}_j$ and $\mathbf{a}_\ell$ are connected by an edge $\mathcal{E}_k$. We note that these equations assign values to degrees of freedom which are independent of each other.

The Bogner-Fox-Schmit element has the following standard Hermite basis functions:

- $\phi_j(x)$, associated with the interpolated value at $\mathbf{a}_j$,
- $\phi_{x_1,j}(x)$ and $\phi_{x_2,j}(x)$, associated with the x_1- and x_2-derivatives at $\mathbf{a}_j$, respectively, and
- $\phi_{x_1x_2,j}(x)$, associated with the second mixed derivative at $\mathbf{a}_j$.

The mixed derivative degrees of freedom need not be considered here, since by the definition of $\tilde{V}_b^H(\Omega)$, they are set to zero.

By writing the basis functions explicitly, we easily obtain:

$$|\phi_j|^2_{H^1(\Omega)} \leq C, \quad |\phi_{x_1,j}|^2_{H^1(\Omega)} \leq CH^2, \text{ and } \quad |\phi_{x_2,j}|^2_{H^1(\Omega)} \leq CH^2. \tag{5.21}$$

The key result of this section is the following lemma.

Lemma 5.6.1 *Given* $\mathbf{v} \in H_0^2(\Omega)$*, the function* $\psi = \psi(\mathbf{v})$ *satisfies:*

$$\begin{cases} \int_{\mathcal{E}}(\mathbf{curl}(\psi(\mathbf{v})) - \mathbf{v}) \cdot \mathbf{n}\, dS = 0 \quad \forall \textit{ edges } \mathcal{E} \textit{ of } \mathcal{T}_H, \\ ||\mathbf{v} - \mathbf{curl}(\psi(\mathbf{v}))||^2_{L^2(\Omega_i)} \leq CH^2|\mathbf{v}|^2_{H^1(\mathcal{R}(\Omega_i))}, \\ |\mathbf{curl}(\psi(\mathbf{v}))|^2_{H^1(\Omega_i)} \leq C|\mathbf{v}|^2_{H^1(\mathcal{R}(\Omega_i))}. \end{cases} \tag{5.22}$$

Proof. The first statement is clear from the identity $\mathbf{curl}(\psi) \cdot \mathbf{n} = \frac{\partial\psi}{\partial\tau}$ and the last line of (5.20).

We start with the L^2-approximation property. If $\hat{\mathbf{c}} = (\hat{c}_1, \hat{c}_2)$ is an arbitrary constant in the reference substructure $\hat{\Omega}$, $\hat{\psi}(\hat{\mathbf{c}}) \in \tilde{V}_b^H(\hat{\Omega})$ satisfies $\hat{\psi}(\hat{\mathbf{c}}) = -\hat{c}_2\hat{x}_1 + \hat{c}_1\hat{x}_2 + \hat{c}_3$, where $\hat{c}_3$ is an arbitrary constant. Indeed, as is easily checked, both sides have the same values for the degrees of freedom of $\tilde{V}_b^H(\hat{\Omega})$. Therefore, $\mathbf{curl}(\psi(\mathbf{c})) = \mathbf{c}$.

Since the **curl** of a constant function is zero, we may assume, in order to estimate the norms of $\mathbf{curl}(\psi)$ over an individual subregion, that $\psi(\mathbf{a}_1) = 0$, where the $\mathbf{a}_j$, $1 \leq j \leq 4$ are the vertices of Ω_i. Hence, for $x \in \Omega_i$,

$$\psi(x) = \sum_{j=2}^{4} \psi(\mathbf{a}_j)\phi_j(x) + \sum_{j=1}^{4}(\frac{\partial\psi}{\partial x_1}(\mathbf{a}_j)\phi_{x_1,j}(x) + \frac{\partial\psi}{\partial x_2}(\mathbf{a}_j)\phi_{x_2,j}(x)).$$

Then, by (5.21),

$$||\mathbf{u}-\mathbf{curl}(\psi)||^2_{L^2(\Omega_i)} \leq ||\mathbf{u}||^2_{L^2(\Omega_i)}+\sum_{j=2}^{4} C|\psi(\mathbf{a}_j)|^2+\sum_{j=1}^{4} CH^2(|\frac{\partial\psi}{\partial x_1}(\mathbf{a}_j)|^2+|\frac{\partial\psi}{\partial x_2}(\mathbf{a}_j)|^2).$$

Let $\mathcal{E}_k$ be the edge connecting $\mathbf{a}_j$ to $\mathbf{a}_1$. The definition of ψ, (5.20), gives

$$|\psi(\mathbf{a}_j)|^2 \leq \int_{\mathcal{E}_k} |\mathbf{u}|^2\, dS \int_{\mathcal{E}_k} 1\, dS \leq CH^2||\mathbf{u}||^2_{H^1(\Omega_i)},$$

by the Cauchy-Schwarz inequality, and a trace theorem coupled with a scaling argument. The same inequality is easily seen to hold for the vertex diagonally opposite to $\mathbf{a}_1$, by using a triangle inequality. Moreover, $\forall \mathbf{a}_j \in \partial\Omega_i$,

$$|\frac{\partial\psi}{\partial x_1}(\mathbf{a}_j)|^2 \leq \frac{1}{|\mathcal{R}(\mathbf{a}_j)|^2} CH^2 \int_{\mathcal{R}(\mathbf{a}_j)} u_2^2\, dx \leq \frac{C}{H^2}||\mathbf{u}||^2_{L^2(\mathcal{R}(\mathbf{a}_j))},$$

with an analogous estimate for $\frac{\partial\psi}{\partial x_2}(\mathbf{a}_j)$. Here, u_2 is the second component of $\mathbf{u}$, and we have used the Cauchy-Schwarz inequality. Then,

$$||\mathbf{u}-\mathbf{curl}(\psi)||^2_{L^2(\Omega_i)} \leq CH^2||\mathbf{u}||^2_{H^1(\mathcal{R}(\mathbf{a}_j))}.$$

Since $\mathbf{u} \mapsto \psi(\mathbf{u})$ is linear and $\mathbf{curl}(\psi(\mathbf{c})) = \mathbf{c}$, the use of a Poincaré inequality completes the proof of the second statement of Lemma 5.6.1.

The H^1-stability property follows from the L^2-approximation and an inverse inequality in a standard way:

$$\begin{aligned}
|\mathbf{curl}(\psi)|^2_{H^1(\Omega_i)} &\leq \frac{C}{H^2}||\mathbf{curl}(\psi)||^2_{L^2(\Omega_i)} \\
&\leq \frac{C}{H^2}(||\mathbf{u}-\mathbf{curl}(\psi)||^2_{L^2(\Omega_i)} + ||\mathbf{u}||^2_{L^2(\Omega_i)}) \\
&\leq C||\mathbf{u}||^2_{H^1(\mathcal{R}(\Omega_i))},
\end{aligned}$$

and the proof is completed by again using a quotient space argument. □

These properties of $\psi(\mathbf{u})$ yield the following theorem, the proof of which is the same as Theorem 5.5.2, setting I_H^h to be the injection operator, and using Lemma 5.6.1 instead of (5.13).

Theorem 5.6.1 *The condition number of T_b satisfies:*

$$\kappa(T_b) \leq \frac{C(1+\log(N))^3}{\beta_N}.$$

Chapter 6

Schwarz Methods for the Stationary Navier-Stokes Equations

6.1 Statement of the Problem and Newton's Method

The stationary Navier-Stokes equations with homogeneous boundary data can be written in the form:

$$\begin{cases} -\nu\Delta\mathbf{u} + \sum_{j=1}^{d} \mathbf{u}_j(D_j\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{in } \partial\Omega; \end{cases} \tag{6.1}$$

see [60, Section I.5].

Following [100], we consider a Galerkin spectral element discretization of 6.1, given by:
Find $(\tilde{\mathbf{u}}_N, \tilde{p}_{N-2}) \in P_0^N(\Omega) \times (\bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega))$ such that

$$\begin{cases} a(\tilde{\mathbf{u}}_N, \mathbf{v}_N) + c(\tilde{u}_N; \tilde{u}_N, \mathbf{v}_N) + b(\mathbf{v}_N, \tilde{p}_{N-2}) = \int_\Omega \mathbf{f}\mathbf{v}_N \, dx \quad \forall \mathbf{v}_N \in P_0^N(\Omega), \\ b(\tilde{\mathbf{u}}_N, q_{N-2}) = 0 \quad \forall q_{N-2} \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega). \end{cases} \tag{6.2}$$

The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ have been defined in subsection 1.8. For $\mathbf{u}, \mathbf{v}$, and $\mathbf{w} \in H^1(\Omega)$, the trilinear form $c(\cdot;\cdot,\cdot)$ is defined as:

$$c(\mathbf{u};\mathbf{v},\mathbf{w}) := \sum_{i,j=1}^{d} \int_\Omega u_j(D_j v_i) w_i \, dx.$$

As far as we know, an analysis of the discretization error incurred when approximating (6.1) by (6.2) is not yet available. However, numerical computations show that $\tilde{\mathbf{u}}_N$ is a good approximation for $\mathbf{u}$, at least for Reynolds number $Re = 1/\nu$ on the order of 50; see [100].

In practice, the bilinear and trilinear forms appearing in (6.2) have to be computed by using a quadrature formula of good quality, e.g., GLL quadrature. For simplicity, we only analyze (6.2), but remark that the solution method and estimates that will be presented in this chapter extend straightforwardly to the quadrature case.

Following Rønquist [100], we solve the non-linear system of equations (6.2) with Newton's method; see [60, subsection IV.6.3], where a now classical framework for the Navier-Stokes equations is given. From an initial guess $\mathbf{u}_N^0$, Newton's method produces a sequence of iterates $\mathbf{u}_N^k$, $k = 0, 1, \ldots$. Given $\mathbf{u}_N^{k-1}$, we find $\mathbf{u}_N^k \in P_0^N(\Omega)$ by solving (see [60, equation IV.6.52]):

$$\left\{ \begin{array}{l} a(\mathbf{u}_N^k, \mathbf{v}_N) + c(\mathbf{u}_N^{k-1}; \mathbf{u}_N^k, \mathbf{v}_N) + c(\mathbf{u}_N^k; \mathbf{u}_N^{k-1}, \mathbf{v}_N) \\ \qquad\qquad + b(\mathbf{v}_N, p_{N-2}^k) = (\mathbf{f}^k, \mathbf{v}_N) \quad \forall \mathbf{v}_N \in P_0^N(\Omega), \\ b(\mathbf{u}_N^k, q_{N-2}) = 0 \quad \forall q_{N-2} \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega), \end{array} \right. \tag{6.3}$$

where

$$(\mathbf{f}^k, \mathbf{v}_N) := \int_\Omega \mathbf{f} \mathbf{v}_N \, dx + c(\mathbf{u}_N^{k-1}; \mathbf{u}_N^{k-1}, \mathbf{v}_N).$$

We assume that $\tilde{\mathbf{u}}_N$ is a non-singular solution of (6.2) in the sense that the system (6.3) is uniquely solvable if we set $\mathbf{u}_N^{k-1} = \tilde{\mathbf{u}}_N$. If the Reynolds number Re= $1/\nu$ is small enough, this can be proved by classical arguments (see [60, Theorem IV.2.4]); our analysis does not assume that, although the iteration count of the method may deteriorate when Re increases. By choosing $\mathbf{u}_N^0$ sufficiently close to $\tilde{\mathbf{u}}_N$, we can conclude that the system (6.3) is solvable, for $k \geq 1$, and that $\mathbf{u}_N^k \to \tilde{\mathbf{u}}_N$ as $k \to \infty$, with a quadratic convergence rate. Moreover, it is clear that there exists a constant $\bar{C}$, depending only on $\tilde{\mathbf{u}}_N$, such that

$$|\mathbf{u}_N^k|_{H^1(\Omega)} \leq \bar{C} |\tilde{\mathbf{u}}_N|_{H^1(\Omega)} \quad (k \geq 0). \tag{6.4}$$

In this chapter, we will develop Schwarz preconditioners for the system representing the Newton step (6.3). We fix k, and to simplify notations, set $\mathbf{u}_N := \mathbf{u}_N^k$, $\mathbf{w} := \mathbf{u}_N^{k-1}$, and $\mathbf{g} := \mathbf{f}^k$. Then, $\mathbf{u}_N$ is the solution of the following problem:
Find $\mathbf{u}_N \in P_{0,\nabla}^N(\Omega)$ such that

$$B_{\mathbf{w}}(\mathbf{u}_N, \mathbf{v}_N) = (\mathbf{g}, \mathbf{v}_N) \quad \mathbf{v}_N \in P_{0,\nabla}^N(\Omega), \tag{6.5}$$

where

$$B_{\mathbf{w}}(\mathbf{u}_N, \mathbf{v}_N) = a(\mathbf{u}_N, \mathbf{v}_N) + c(\mathbf{w}; \mathbf{u}_N, \mathbf{v}_N) + c(\mathbf{u}_N; \mathbf{w}, \mathbf{v}_N). \tag{6.6}$$

The form $B_{\mathbf{w}}(\cdot,\cdot)$ depends on $\mathbf{w}$ (the previous iterate). However, we will exploit that the H^1-norm of $\mathbf{w}$ is uniformly bounded; see (6.4). In what follows, we drop the subscript $\mathbf{w}$.

The next section briefly presents two Schwarz methods for non-symmetric problems, adapted from [32, 33, 116], where a theory is developed for scalar second-order equations. After these modifications it can be applied to (6.5). The preconditioning step involves the solution of a coarse problem and a number of local problems. In addition to a decomposition lemma analogous to hypothesis **H1'** of Theorem 5.5.1, two crucial hypothesis are used in this theory:

- Gårding's inequality for $B(\cdot,\cdot)$, and
- an approximation property for the coarse subspace.

We prove Gårding's inequality for $B(\cdot,\cdot)$ in Section 6.3. We establish this inequality assuming only that $\mathbf{w} \in H_0^1(\Omega)$; the constants of this inequality will depend on $|\mathbf{w}|_{H^1(\Omega)}$, the viscosity ν and the domain Ω.

In Section 6.4, we introduce a coarse space $\tilde{V}_{\mathbf{n},\nabla_H}^H(\Omega)$, which satisfies the requirements of the Schwarz theory. The restriction of $\tilde{\mathbf{u}}_H \in \tilde{V}_{\mathbf{n},\nabla_H}^H(\Omega)$ to the interface Γ is the same as that of a function $\mathbf{u}_H \in V_{\mathbf{n},\nabla_H}^H(\Omega)$, which was defined in Section 5.5; the normal flux across each $\partial\Omega_i$ is required to be zero. The extension of $\tilde{\mathbf{u}}_H$ to the interior of the Ω_i is given by a linear operator which preserves the trace on Γ, and is chosen so that $\tilde{\mathbf{u}}_H \in P_{\nabla}^N(\Omega_i)$ is the best approximation of $\mathbf{u}_H$ in the $a_{\Omega_i}(\cdot,\cdot)$-norm.

6.2 Schwarz Preconditioners

We will design Schwarz preconditioners for $B(\cdot,\cdot)$, by viewing $B(\cdot,\cdot)$ restricted to $P^N_{0,\nabla}(\Omega)$ as a perturbation of the symmetric bilinear form $a(\cdot,\cdot)$. We first present a preconditioner that is closely related to one of the methods proposed in [32] for standard second-order problems; it is based on a decomposition of $P^N_{0,\nabla}(\Omega)$. As before, we assume that τ_H is a shape regular triangulation, not necessarily quasi-uniform, and set $H = \max_i H_i$, where H_i is the diameter of Ω_i.

Let $V_0 \subset P^N_{0,\nabla}(\Omega)$ be a coarse space, usually of dimension much smaller than $P^N_{0,\nabla}(\Omega)$, and let $V_s \subset P^N_{0,\nabla}(\Omega)$, $s \geq 1$, be the local spaces. The operator $Q_0 : P^N_{0,\nabla}(\Omega) \to V_0$ is defined by

$$B(Q_0\mathbf{u}, \mathbf{v}_0) = B(\mathbf{u}, \mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0. \tag{6.7}$$

We remark that although $B(\cdot,\cdot)$ is not necessarily positive definite, (6.7) is guaranteed to have solutions for sufficiently small values of H; see property **P3** below. Let V_s, $s \geq 1$ be the local spaces used to define the operator $T_{\mathbf{n}}$; see (5.17). In three dimensions, there is one local space associated with the interior of each Ω_i, one space related to each face, and one for each edge. For $s \geq 1$, the operator $P_s : P^N_{0,\nabla}(\Omega) \to V_s$ is defined by

$$a(P_s\mathbf{u}, \mathbf{v}_s) = B(\mathbf{u}, v_s) \quad \forall \mathbf{v}_s \in V_s. \tag{6.8}$$

Following [32, 33, 116], we need to verify that $B(\cdot,\cdot)$ and the decomposition $\{V_s\}_{s\geq 0}$ satisfy the following properties:

P1: *(Gårding's inequality)*

$$B(\mathbf{u},\mathbf{u}) \geq C_1 a(\mathbf{u},\mathbf{u}) - C_2 ||u||^2_{L^2(\Omega)} \quad \forall \mathbf{u} \in (H^1_0(\Omega))^d.$$

P2: *There exists a constant C_0^2, bounded from above by*

$$C\frac{(1+\log(N))^3}{\beta_N},$$

such that any $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$ can be decomposed into a sum

$$\mathbf{u} = \sum_{s\geq 0} \mathbf{u}_s,$$

for which

$$\sum_{s\geq 0} a(\mathbf{u}_s,\mathbf{u}_s) \leq C_0^2 a(\mathbf{u},\mathbf{u}).$$

P3: *Given any $\epsilon > 0$, there exist constants $H_0 > 0$ and $C(H_0) > 0$ such that if $H \leq H_0$, and $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$, (6.7) has a unique solution $Q_0\mathbf{u}$,*

$$\|Q_0\mathbf{u}\| \leq C(H_0)||\mathbf{u}||_{H^1(\Omega)},$$

and

$$||Q_0\mathbf{u}-\mathbf{u}||_{L^2(\Omega)} \leq C(H_0)H^\gamma||Q_0\mathbf{u}-\mathbf{u}||_{H^1(\Omega)}.$$

P4: *Let $N_{\mathbf{w}}(\cdot,\cdot)$ be given by*

$$N_{\mathbf{w}}(\cdot,\cdot) = c(\mathbf{w};\mathbf{u},\mathbf{v}) + c(\mathbf{u};\mathbf{w},\mathbf{v}).$$

There exists a constant $H_0 > 0$ such that if $H \leq H_0$,

$$\sup_{u,v\in H_0^1(\Omega)} \frac{|N((I-Q_0)u,v)|}{|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}} \leq C(H_0)H^\gamma,$$

for a constant $\gamma > 0$.

Let $B^*(\mathbf{u},\mathbf{v}) = B(\mathbf{v},\mathbf{u}) \quad \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega)$. We consider the problem: Find $\varphi_{\mathrm{g}} \in H_0^1(\Omega)$ such that $\nabla\cdot\varphi_{\mathrm{g}} = 0$ and

$$B^*(\varphi_{\mathrm{g}},\mathbf{v}) = \int_\Omega \mathbf{g}\mathbf{v}\,dx \quad \forall \mathbf{v} \in H_0^1(\Omega) \text{ with } \nabla\cdot\mathbf{v} = 0. \tag{6.9}$$

By hypothesis, the system (6.3) is uniquely solvable. Then a classical argument can be used to show that

$$\text{(6.9) has a unique solution } \varphi_{\mathrm{g}} \in H_0^1(\Omega) \quad \forall \mathbf{g} \in (L^2(\Omega))^d. \tag{6.10}$$

We assume that the solution of (6.9) satisfies the following regularity estimate, $\forall \mathbf{g} \in (L^2(\Omega))^d$:

$$\varphi_{\mathrm{g}} \in (H^{1+\gamma}(\Omega))^d, \text{ and } ||\varphi_{\mathrm{g}}||_{H^{1+\gamma}(\Omega)} \leq C||\mathbf{g}||_{L^2(\Omega)}, \tag{6.11}$$

for a constant $\gamma > 0$.

Theorem 6.2.1 *Assume that (6.11) is valid. Then, properties* **P1, P2, P3**, *and* **P4** *hold.*

Property **P1** will be proved in Section 6.3. A coarse space for which **P2, P3**, and **P4** are valid will be introduced in Section 6.4, where these properties will also be proved.

Remark 6.2.1 *We have chosen to use (6.11) for clarity, but analogous results can be obtained, by using instead a relatively recent result of Schatz and Wang [104]. Properties* **P3** *and* **P4** *would then become:*

P3': *For any $\epsilon > 0$, there exist constants $H_0 > 0$ and $C(H_0) > 0$ such that if $H \leq H_0$, and $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$, (6.7) has a unique solution $Q_0\mathbf{u}$,*

$$\|Q_0\mathbf{u}\| \leq C(H_0)||\mathbf{u}||_{H^1(\Omega)},$$

and

$$||Q_0\mathbf{u} - \mathbf{u}||_{L^2(\Omega)} \leq C(H_0)\,\epsilon\,||Q_0\mathbf{u} - \mathbf{u}||_{H^1(\Omega)}.$$

P4': *Let $N_{\mathbf{w}}(\cdot,\cdot)$ be given by*

$$N_{\mathbf{w}}(\cdot,\cdot) = c(\mathbf{w};\mathbf{u},\mathbf{v}) + c(\mathbf{u};\mathbf{w},\mathbf{v}).$$

Then, for any $\epsilon > 0$, there exists a constant $H_0 > 0$ such that if $H \leq H_0$,

$$\sup_{u,v\in H^1_0(\Omega)} \frac{|N((I - Q_0)u, v)|}{|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}} \leq C(H_0)\,\epsilon.$$

The proofs of **P3'** *and* **P4'** *are analogous to the proofs of* **P3** *and* **P4** *that we will present in Section 6.4; inequality (6.21) is replaced by a similar relation derived by Schatz and Wang [104, Theorem 2]. We note that if (6.11) is valid, H_0 can be bounded from below by an explicit constant depending on γ, C_1 and C_2. If (6.11) does not hold, an explicit bound cannot be obtained.*

Our main results (theorems 6.2.2 and 6.2.3) are still valid, if we replace **P3** *and* **P4** *by* **P3'** *and* **P4'**, *respectively.*

Theorem 6.2.2 *([32]) There exist positive constants H_0, $c(H_0)$, and $C(H_0)$ such that the operator*

$$Q_a = Q_0 + \sum_{s \geq 1} P_s$$

satisfies, $\forall \mathbf{u} \in P^N_{0,\nabla}(\Omega)$,

$$a(Q_a\mathbf{u}, Q_a\mathbf{u}) \leq C(H_0)a(\mathbf{u}, \mathbf{u}),$$

and

$$c(H_0)C_0^{-2}a(\mathbf{u}, \mathbf{u}) \leq a(Q_a\mathbf{u}, \mathbf{u}).$$

We omit the proof of this theorem, which uses the properties **P1, P2**, and **P4**, since it is very similar to the proof of Theorem 1 in [32].

This result immediately implies an upper bound on the iteration count of the GMRES method applied to the preconditioned system

$$Q_{\mathrm{a}}\underline{\mathbf{u}}_N = b,$$

where b is chosen so that $\underline{\mathbf{u}}_N$ is the vector of nodal values of $\mathbf{u}_N$.

Following [33] and [116], we also define a multiplicative variant of this algorithm. The error propagation operator is given by:

$$I - Q_{\mathrm{m}} := \prod_{s \geq 1}(I - P_s)(I - Q_0).$$

A perturbation argument analogous to that leading to Theorem 3.2 in [116] can be used to prove the following result; see also [33]. The proof uses property **P4**, which is itself proved by using **P1** and **P3**; see Section 6.4.

Theorem 6.2.3 *([116]) Let δ_0 be defined by*

$$\delta_0 = \sup_{u,v \in H_0^1(\Omega)} \frac{|S((I - Q_0)u, v)|}{|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}}.$$

If δ_0 is sufficiently small, achievable by virtue of property **P4**, *then*

$$||I - Q_m||_a \leq 1 - \frac{C}{C_0^2} + \frac{6\delta_0}{1 - \delta_0}.$$

This result again provides an upper bound on the iteration count of the GMRES method applied to the preconditioned system

$$Q_{\mathrm{m}}\underline{\mathbf{u}}_N = b,$$

where b is chosen so that $\underline{\mathbf{u}}_N$ is the vector of nodal values of $\mathbf{u}_N$.

6.3 A proof of Gårding's inequality

In this section, we prove Gårding's inequality (property **P1**), using a method that only requires an upper bound for $|\mathbf{w}|_{H^1(\Omega)}$, which by (6.4) remains uniformly bounded throughout the Newton iteration (6.3). The dimension d may be 2 or 3, and we will use a classical Sobolev imbedding theorem, and the Gagliardo-Nirenberg inequality.

A simple computation shows that

$$B(\mathbf{u},\mathbf{u}) = \nu \int_\Omega |\nabla \mathbf{u}|^2 \, dx + \sum_{i,j=1}^{d} \int_\Omega w_j (D_j u_i) u_i + (D_j w_i) u_j u_i \, dx \quad \forall \mathbf{u}, \mathbf{w} \in H_0^1(\Omega).$$

We observe that, by integration by parts,

$$\begin{aligned} \int_\Omega w_j (D_j u_i) u_i \, dx &= -\int_\Omega u_i D_j (w_j u_i) \, dx \\ &= -\int_\Omega u_i (D_j w_j) u_i + w_j u_i (D_j u_i) \, dx. \end{aligned}$$

Hence,

$$B(\mathbf{u},\mathbf{u}) = \nu \int_\Omega |\nabla \mathbf{u}|^2 \, dx + \sum_{i,j=1}^{d} \int_\Omega -\frac{1}{2} (D_j w_j) u_i^2 + (D_j w_i) u_j u_i \, dx.$$

An upper bound on the L^∞-norms of the derivatives of $\mathbf{w}$, if available, could easily be combined with this inequality to prove **P1**. Instead, we give a more general argument. We derive a bound for a typical term of the last integral. By applying Hölder's inequality and a Sobolev imbedding argument, Theorem 1.4.4.1 in [64], we find that

$$\begin{aligned} \left| \int_\Omega (D_j w_i) u_j u_i \, dx \right| &\leq ||D_j w_i||_{L^2(\Omega)} ||u_j||_{L^4(\Omega)} ||u_i||_{L^4(\Omega)} \\ &\leq C(\Omega) |\mathbf{w}|_{H^1(\Omega)} ||\mathbf{u}||^2_{H^{3/4}(\Omega)} \\ &\leq C(\Omega) |\mathbf{w}|_{H^1(\Omega)} \left(\eta |\mathbf{u}|_{H^1(\Omega)} + C(\Omega) \eta^{-3} ||\mathbf{u}||_{L^2(\Omega)} \right)^2. \end{aligned}$$

In the last inequality, we have used the Gagliardo-Nirenberg inequality; see Theorem 1.4.3.3 in [64]. Here $\eta > 0$ is arbitrary. By choosing η small enough, we obtain

$$B(\mathbf{u},\mathbf{u}) \geq \frac{\nu}{2}|\mathbf{u}|^2_{H^1(\Omega)} - C_2||\mathbf{u}||^2_{L^2(\Omega)},$$

where C_2 is a constant that depends on $|\mathbf{w}|_{H^1(\Omega)}$, Ω, and ν. This completes the proof of **P1**.

6.4 A Coarse Space

In this section, we define a new coarse space $\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)$, and show that properties **P2, P3**, and **P4** hold.

We first define an extension operator $\tilde{I}^h_H$, similar to the operator I^h_H, defined in (5.14). Let $\tilde{I}^h_H : V^H_{\mathbf{n},\nabla_H}(\Omega) \longrightarrow P^N_{0,\nabla}(\Omega)$, and let $\tilde{\mathbf{u}}_H = \tilde{I}^h_H(\bar{\mathbf{u}}_H)$ for $\bar{u}_H \in V^H_{\mathbf{n},\nabla_H}(\Omega)$. The restriction of $\tilde{\mathbf{u}}_H$ to a subregion Ω_i is the solution of the following non-homogeneous Stokes problem:

Find $\tilde{\mathbf{u}}_H \in P^N_\nabla(\Omega_i)$, with $\tilde{\mathbf{u}}_H = \bar{\mathbf{u}}_H$ on $\partial\Omega_i$, and $\tilde{p}_H \in P^{N-2}(\Omega_i) \cap L^2_0(\Omega_i)$ such that

$$\begin{cases} a(\tilde{\mathbf{u}}_H, \mathbf{v}_N) + b(\mathbf{v}_N, \tilde{p}_H) = a(\bar{\mathbf{u}}_H, \mathbf{v}_N) \quad \forall \mathbf{v}_N \in P^N_0(\Omega_i), \\ \\ b(\tilde{\mathbf{u}}_H, q_{N-2}) = 0 \quad \forall q_{N-2} \in P^{N-2}(\Omega_i) \cap L^2_0(\Omega_i). \end{cases} \tag{6.12}$$

By restricting the test function $\mathbf{v}_N$ to have zero discrete divergence, i.e. $\mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i)$, $\tilde{\mathbf{u}}_H$ can also be determined by:

Find $\tilde{\mathbf{u}}_H \in P^N_\nabla(\Omega_i)$, $\tilde{\mathbf{u}}_H = \bar{\mathbf{u}}_H$ on $\partial\Omega_i$, and such that

$$a(\tilde{\mathbf{u}}_H, \mathbf{v}_N) = a(\bar{\mathbf{u}}_H, \mathbf{v}_N) \quad \forall \mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i). \tag{6.13}$$

The new coarse space is defined by:

$$\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega) = \tilde{I}^h_H(V^H_{\mathbf{n},\nabla_H}(\Omega)).$$

A reasoning analogous to that of the proof of Lemma 5.3.1 easily shows that $\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega) \subset P^N_{0,\nabla}(\Omega)$. This is our coarse space V_0; cf. Section 6.2. The next lemma shows that $\tilde{\mathbf{u}}_H$ is the function of $P^N_{0,\nabla}(\Omega)$ which coincides with $\bar{\mathbf{u}}_H$ on Γ, and which is the best approximation of $\bar{\mathbf{u}}_H$ in the $a(\cdot,\cdot)$-semi-norm (and in the H^1-semi-norm, since they differ only by a fixed factor ν).

Lemma 6.4.1 *Let* $\bar{\mathbf{u}}_H \in V^H_{\mathbf{n},\nabla_H}(\Omega_i)$ *be given, and let* $\tilde{\mathbf{u}}_H = \tilde{I}^h_H(\bar{\mathbf{u}}_H)$. *Then*

$$|\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H|_{H^1(\Omega_i)} = \inf_{\mathbf{w}_N \in P^N_\nabla(\Omega_i),\, \mathbf{w}_N = \bar{\mathbf{u}}_H \text{ on } \partial\Omega_i} |\mathbf{w}_N - \bar{\mathbf{u}}_H|_{H^1(\Omega_i)}.$$

Proof. Any $\mathbf{w}_N$ can be written as $\mathbf{w}_N = \tilde{\mathbf{u}}_H + \mathbf{v}_N$, where $\mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i)$. Then,

$$\begin{aligned} a(\mathbf{w}_N - \bar{\mathbf{u}}_H, \mathbf{w}_N - \bar{\mathbf{u}}_H) &= a(\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H + \mathbf{v}_N, \tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H + \mathbf{v}_N) \\ &= a(\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H, \tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H) + 2a(\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H, \mathbf{v}_N) + a(\mathbf{v}_N, \mathbf{v}_N) \\ &\geq a(\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H, \tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H), \end{aligned}$$

since the second term vanishes, by (6.13). □

The next lemma shows an approximation property of $\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)$, which is essential in establishing properties **P3** and **P4**.

Lemma 6.4.2 *There exists a constant* C, *independent of* N *and* H, *such that* $\forall \mathbf{u} \in H^1_0(\Omega)$ *with* $\nabla \cdot \mathbf{u} = 0$,

$$\inf_{\tilde{\mathbf{v}}_H \in \tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)} |\tilde{\mathbf{v}}_H - \mathbf{u}|_{H^1(\Omega)} \leq C \inf_{\mathbf{v}_H \in V^H_{\mathbf{n}}(\Omega)} |\mathbf{v}_H - \mathbf{u}|_{H^1(\Omega)}.$$

Proof. Let $\mathbf{u}_H \in V^H_{\mathbf{n}}(\Omega)$ realize the minimum in the right hand side. The pair $V^H_{\mathbf{n}}(\Omega) \times \bar{P}^0(\Omega) \cap L^2_0(\Omega)$ yields a stable discretization in the Babuška-Brezzi sense, with a stability constant $\beta_H \geq c > 0$; see (5.13) and the discussion preceding it. Then, we can find $\mathbf{z}_H \in V^H_{\mathbf{n}}(\Omega)$ such that

$$b(\mathbf{z}_H, q_H) = b(\mathbf{u} - \mathbf{u}_H, q_H) \quad \forall q_H \in \bar{P}^0(\Omega) \cap L^2_0(\Omega),$$

and

$$|\mathbf{z}_H|_{H^1(\Omega)} \leq \frac{C}{\beta_H} |\mathbf{u} - \mathbf{u}_H|_{H^1(\Omega)};$$

see [60, Equation (1.16)]. Let $\bar{\mathbf{u}}_H = \mathbf{u}_H + \mathbf{z}_H$. Then, $\bar{\mathbf{u}}_H \in V^H_{\mathbf{n},\nabla_H}(\Omega)$, and since $\nabla \cdot \mathbf{u} = 0$,

$$\begin{aligned} |\mathbf{u} - \bar{\mathbf{u}}_H|_{H^1(\Omega)} &\leq |\mathbf{z}_H|_{H^1(\Omega)} + |\mathbf{u}_H - \mathbf{u}|_{H^1(\Omega)} \\ &\leq (\frac{C}{\beta_H} + 1)|\mathbf{u}_H - \mathbf{u}|_{H^1(\Omega)}. \end{aligned} \tag{6.14}$$

We now choose $\tilde{u}_H \in \tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)$ as $\tilde{u}_H = \tilde{I}^h_H(\bar{\mathbf{u}}_H)$. Then, from Lemma 6.4.1,

$$|\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H|_{H^1(\Omega)} \leq |\mathbf{w}_N - \bar{\mathbf{u}}_H|_{H^1(\Omega)} \quad \forall \mathbf{w}_N \in P^N_\nabla(\Omega_i) \text{ with } \mathbf{w}_N = \bar{\mathbf{u}}_H \text{ on } \partial\Omega_i. \tag{6.15}$$

Let $\mathbf{w}_N = \bar{\mathbf{u}}_H + \mathbf{v}_N$, where $\mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i)$ is yet to be chosen. Since $\bar{\mathbf{u}}_H|_{\Omega_i}$ has degree at most 2, the argument in the proof of Lemma 5.5.1 applies, and hence we may choose $\mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i)$ such that

$$b_{\Omega_i}(\mathbf{v}_N, q_{N-2}) = -b_{\Omega_i}(\bar{\mathbf{u}}_H, q_{N-2}),$$

and

$$|\mathbf{v}_N|_{H^1(\Omega_i)} \leq C||\nabla \cdot \bar{\mathbf{u}}_H||_{L^2(\Omega_i)},$$

with C independent of N. Therefore,

$$|\mathbf{w}_N - \bar{\mathbf{u}}_H|_{H^1(\Omega_i)} = |\mathbf{v}_N|_{H^1(\Omega_i)} \leq C||\nabla \cdot \bar{\mathbf{u}}_H||_{L^2(\Omega_i)}. \tag{6.16}$$

Since $\nabla \cdot \mathbf{u} = 0$, we have

$$|\mathbf{w}_N - \bar{\mathbf{u}}_H|_{H^1(\Omega_i)} \leq C||\nabla \cdot (\bar{\mathbf{u}}_H - \mathbf{u})||_{L^2(\Omega_i)} \leq C|\bar{\mathbf{u}}_H - \mathbf{u}|_{H^1(\Omega_i)}. \tag{6.17}$$

Finally, by using the triangle inequality, (6.15), and (6.17), and summing over all the subdomains, we find that

$$\begin{aligned}
|\tilde{\mathbf{u}}_H - \mathbf{u}|_{H^1(\Omega)} &\leq |\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H|_{H^1(\Omega)} + |\bar{\mathbf{u}}_H - \mathbf{u}|_{H^1(\Omega)} \\
&\leq |\mathbf{w}_N - \bar{\mathbf{u}}_H|_{H^1(\Omega)} + |\bar{\mathbf{u}}_H - \mathbf{u}|_{H^1(\Omega)} \\
&\leq C|\bar{\mathbf{u}}_H - \mathbf{u}|_{H^1(\Omega)} + |\bar{\mathbf{u}}_H - \mathbf{u}|_{H^1(\Omega)} \\
&\leq C(1 + \frac{C}{\beta_H})|\mathbf{u}_H - \mathbf{u}|_{H^1(\Omega)}.
\end{aligned}$$

In the last inequality, we have used (6.14). □

*Proof of **P3**.* Let $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$, and let $\mathbf{e} = \mathbf{u} - Q_0\mathbf{u}$. We follow the steps of a classical argument of Schatz [103]. From **P1** and (6.7), we obtain

$$\begin{aligned}
C_1\nu|\mathbf{e}|^2_{H^1(\Omega)} - C_2||\mathbf{e}||^2_{L^2(\Omega)} &\leq B(\mathbf{e}, \mathbf{e}) \\
&= B(\mathbf{e}, \mathbf{u}) \\
&\leq |\mathbf{e}|_{H^1(\Omega)}|\mathbf{u}|_{H^1(\Omega)}.
\end{aligned}$$

Assuming that $\mathbf{e} \neq 0$, dividing through by $|\mathbf{e}|_{H^1(\Omega)}$, and using Lemma 1.2.2, we find that

$$C_1 \nu |\mathbf{e}|_{H^1(\Omega)} - C_2' ||\mathbf{e}||_{L^2(\Omega)} \leq |\mathbf{u}|_{H^1(\Omega)}. \tag{6.18}$$

We next estimate the L^2-norm of the error. From (6.9), the definition of $B^*(\cdot,\cdot)$, (6.7), and the continuity of $B(\cdot,\cdot)$, we obtain, $\forall \tilde{\varphi}_H \in \tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)$,

$$\begin{aligned} ||\mathbf{e}||_{L^2(\Omega)} &= \frac{\int_\Omega \mathbf{e}\cdot\mathbf{e}\,dx}{||\mathbf{e}||_{L^2(\Omega)}} = \frac{B^*(\varphi_{\mathrm{e}},\mathbf{e})}{||\mathbf{e}||_{L^2(\Omega)}} \\ &= \frac{B(\mathbf{e},\varphi_{\mathrm{e}})}{||\mathbf{e}||_{L^2(\Omega)}} = \frac{B(\mathbf{e},\varphi_{\mathrm{e}}-\tilde{\varphi}_H)}{||\mathbf{e}||_{L^2(\Omega)}} \\ &\leq C|\mathbf{e}|_{H^1(\Omega)}\frac{|\varphi_{\mathrm{e}}-\tilde{\varphi}_H|_{H^1(\Omega)}}{||\mathbf{e}||_{L^2(\Omega)}} \end{aligned} \tag{6.19}$$

Since $V^H_{\mathbf{n}}(\Omega)$ is an unconstrained finite element space which contains the piecewise linear functions, a standard argument (see, e.g., [43]) and (6.11) show that there exists a $\varphi_H \in V^H_{\mathbf{n}}(\Omega)$ such that

$$|\varphi_H - \varphi_{\mathrm{e}}|_{H^1(\Omega)} \leq CH^\gamma |\varphi_{\mathrm{e}}|_{H^{1+\gamma}(\Omega)} \leq CH^\gamma ||\mathbf{e}||_{L^2(\Omega)}. \tag{6.20}$$

We now combine (6.19), (6.20), and Lemma 6.4.2 with $\mathbf{u} = \varphi_{\mathrm{e}}$ and obtain that

$$||\mathbf{e}||_{L^2(\Omega)} \leq CH^\gamma |\mathbf{e}|_{H^1(\Omega)}. \tag{6.21}$$

Therefore, by introducing (6.21) into (6.18), and using Lemma 1.2.2, we find that

$$(C_1\nu - C_2' CH^\gamma)|\mathbf{e}|_{H^1(\Omega)} \leq |\mathbf{u}|_{H^1(\Omega)}. \tag{6.22}$$

We choose $H_0 > 0$ such that for $H \leq H_0$, we have $C_1\nu - C_2CH^\gamma > 0$. If $\mathbf{u} = 0$, then inequality (6.22) implies that $\mathbf{e} = 0$. Therefore, the finite-dimensional linear system (6.7) has a unique solution, and hence is solvable $\forall \mathbf{u} \in H^1_0(\Omega)$, with $\nabla\cdot\mathbf{u} = 0$. The use of (6.21) and (6.22) completes the proof of **P3**. □

*Proof of **P2**.* We claim that the analog of Lemma 5.5.1 holds for $\tilde{I}^h_H$. For $\bar{u}_H \in V^H_{\mathbf{n},\nabla_H}(\Omega)$,

$$|\tilde{I}^h_H(\bar{\mathbf{u}}_H)|_{H^1(\Omega_i)} \leq C|\bar{\mathbf{u}}_H|_{H^1(\Omega_i)}, \tag{6.23}$$

and

$$||\bar{\mathbf{u}}_H - \tilde{I}_H^h(\bar{\mathbf{u}}_H)||_{L^2(\Omega_i)} \leq CH|\bar{\mathbf{u}}_H|_{H^1(\Omega_i)}. \tag{6.24}$$

Indeed, let $\tilde{\mathbf{u}}_H = \tilde{I}_H^h(\bar{\mathbf{u}}_H)$. By combining (6.15) and (6.16), we find that

$$|\tilde{\mathbf{u}}_H - \bar{\mathbf{u}}_H|_{H^1(\Omega_i)} \leq C|\bar{\mathbf{u}}_H|_{H^1(\Omega_i)},$$

which clearly implies (6.23). Lemma 1.2.2 and a scaling argument can now be used to prove (6.24); see the end of the proof of Lemma 5.5.1.

The estimate of C_0^2 is analogous to the analysis given in the verification of hypothesis **H1'** for the operator $T_{\mathbf{n}}$; see Theorem 5.5.2. Let π_H be the operator given in 5.13. Then, given $\mathbf{u} \in P_{0,\nabla}^N(\Omega)$, the coarse space component is defined as

$$\tilde{\mathbf{u}}_H = \tilde{I}_H^h(\pi_H(\mathbf{u})).$$

The remainder of the proof is similar; (6.23) and (6.24) are used instead of Lemma 5.5.1. We obtain

$$C_0^2 \leq C\frac{(1+\log(N))^3}{\beta_N}.$$

□

*Proof of **P4**.* Property **P3** and a standard interpolation argument easily imply that

$$||Q_0\mathbf{u} - \mathbf{u}||_{H^{1/2}(\Omega)} \leq CH^{\gamma/2}C(H_0)|\mathbf{u}|_{H^1(\Omega)}. \tag{6.25}$$

Therefore, **P4** will be proved (with γ replaced by $\gamma/2$) if we can show that $N(\cdot,\cdot)$ satisfies

$$|N(\mathbf{u},\mathbf{v})| \leq C||\mathbf{u}||_{H^{1/2}(\Omega)}|\mathbf{v}|_{H^1(\Omega)} \quad \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega), \tag{6.26}$$

where C depends only on Ω, $|\mathbf{w}|_{H^1(\Omega)}$, and ν.

By definition, we have that

$$N(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{d}\left(\int_\Omega w_j(D_ju_i)v_i\,dx + \int_\Omega u_j(D_jw_i)v_i\,dx\right).$$

Each term of the first integral can be estimated by integrating by parts, using Hölder's inequality, and Sobolev's imbedding theorem [64, Theorem 1.4.4.1]:

$$\left|\int_\Omega w_j(D_ju_i)v_i\,dx\right| \quad = \quad \left|-\int_\Omega u_i(D_jw_j)v_i + u_iw_j(D_jv_i)\,dx\right|$$

$$
\begin{aligned}
&\leq \; ||u_i||_{L^3(\Omega)}||D_j w_j||_{L^2(\Omega)}||v_i||_{L^6(\Omega)} \\
&\qquad\qquad +||u_i||_{L^3(\Omega)}||w_j||_{L^6(\Omega)}||D_j v_i||_{L^2(\Omega)} \\
&\leq \; ||\mathbf{u}||_{H^{1/2}(\Omega)}|\mathbf{w}|_{H^1(\Omega)}|\mathbf{v}|_{H^1(\Omega)}.
\end{aligned}
$$

The second integral can also be estimated similarly, and (6.26) is established. □

6.5 Remarks on the Implementation

In this section, we comment on some implementation issues specific to our method, and point to some alternatives that may further enhance its efficiency.

In each application of the preconditioner, the computation of $Q_0\mathbf{u}$ for $\mathbf{u} \in P^N_{0,\nabla}(\Omega)$ appears to require the solution of a linear system with a coefficient matrix generated by a basis of the space $\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)$. To avoid constructing such a basis, we can work with the space $\tilde{V}^H_{\mathbf{n}}(\Omega) = \tilde{I}^h_H(V^H_{\mathbf{n}}(\Omega))$, and use pressure Lagrange multipliers in $\bar{P}^0(\Omega)$; this is analogous to the procedure used in Section 5.4. The vector of nodal values of $Q_0\mathbf{u}$ is the velocity component of the solution of the system:

$$
\begin{cases}
\mathcal{B}_0\underline{Q_0\mathbf{u}} + B_0^t\underline{p_0} = \underline{\mathbf{u}}_0, \\
\\
B_0\underline{Q_0\mathbf{u}} = 0.
\end{cases}
$$

Here, $\underline{\mathbf{u}}_0$ is the residual vector of (6.7). The matrix B_0 is defined in (5.12), while $\mathcal{B}_0$ is generated by a basis of $\tilde{V}^H_{\mathbf{n}}(\Omega)$ and the form $B(\cdot,\cdot)$. This matrix is the sum of the stiffness matrix generated by $a(\cdot,\cdot)$ (which is independent of $\mathbf{w} = \mathbf{u}_N^{k-1}$), and a matrix $N_{\mathbf{w}}$ generated $N_{\mathbf{w}}(\cdot,\cdot)$, which has to be recomputed in each Newton step, since it changes with $\mathbf{w}$.

The construction of a basis of $\tilde{V}^H_{\mathbf{n}}(\Omega)$ involves the solution of a small number of inhomogeneous Stokes problems within each Ω_i; a factorization of these local matrices can be precomputed, and used throughout the Newton process.

In practice, the construction of such basis can conceivably be avoided. $Q_0\mathbf{u}$ may be replaced by $\tilde{I}^h_H(\bar{Q}_0\mathbf{u})$, where $\bar{Q}_0\mathbf{u}$ is the solution of (6.7), with $V_0 = V^H_{\mathbf{n},\nabla_H}(\Omega)$; we do not currently have a full theoretical justification of this variant of the algorithm. We intend to perform numerical experiments in the near future, to settle this and other issues mentioned in what follows.

As briefly explained in Remark 5.5.1, in the context of the Stokes operator, inexact solvers may be used for the solution of the local problems, i.e. the bilinear forms may be replaced by forms computationally more convenient. Such variants can easily be analyzed by our techniques.

A number of alternatives to Newton's method can be considered, in an attempt to decrease the work associated with the recomputation of $N_{\mathbf{w}}(\cdot,\cdot)$ in each step. This recomputation affects not only the coarse solver, as explained above, but also the residual computation in each step of the GMRES method applied to (6.5), since the matrices are seldom assembled in the spectral element method.

A well-known method is described in [60, Equation IV.6.53], for which the modified Newton's step is given by:

$$\left\{\begin{aligned} & a(\mathbf{u}_N^k,\mathbf{v}_N)+c(\mathbf{u}_N^0;\mathbf{u}_N^k,\mathbf{v}_N)+c(\mathbf{u}_N^k;\mathbf{u}_N^0,\mathbf{v}_N) \\ & \qquad +b(\mathbf{v}_N,p_{N-2}^k)=(\tilde{\mathbf{f}}^k,\mathbf{v}_N) \quad \forall \mathbf{v}_N\in P_0^N(\Omega), \\ & b(\mathbf{u}_N^k,q_{N-2})=0 \quad \forall q_{N-2}\in \bar{P}^{N-2}(\Omega)\cap L_0^2(\Omega), \end{aligned}\right. \tag{6.27}$$

for a suitable $\tilde{\mathbf{f}}^k$, which depends explicitly on $\mathbf{u}_N^0$ and $\mathbf{u}_N^{k-1}$. It is clear that each step requires much less setup than (6.3); the trade-off is that the convergence $\mathbf{u}_N^k\to\tilde{\mathbf{u}}_N$ is linear. Again, $\mathbf{u}_N^0$ has to be chosen sufficiently close to $\tilde{\mathbf{u}}_N$.

We next describe a form of the Picard iteration to find an approximate solution to (6.1). In each step, we find $\mathbf{u}_N^k\in P_{0,\nabla}^N(\Omega)$ by solving

$$\left\{\begin{aligned} & a(\mathbf{u}_N^k,\mathbf{v}_N)+\tilde{c}(\mathbf{u}_N^{k-1};\mathbf{u}_N^k,\mathbf{v}_N)+b(\mathbf{v}_N,p_{N-2}^k) \\ & \qquad =(\mathbf{f}^k,\mathbf{v}_N) \quad \forall \mathbf{v}_N\in P_0^N(\Omega), \\ & b(\mathbf{u}_N^k,q_{N-2})=0 \quad \forall q_{N-2}\in \bar{P}^{N-2}(\Omega)\cap L_0^2(\Omega); \end{aligned}\right. \tag{6.28}$$

see [60, Equation IV.1.19]. Following a now standard trick due to Temam [114, Equation I.3.23], we set

$$\tilde{c}(\mathbf{u};\mathbf{v},\mathbf{w})=\frac{1}{2}(c(\mathbf{u};\mathbf{v},\mathbf{w})-c(\mathbf{u};\mathbf{w},\mathbf{v})).$$

By integration by parts, it is easy to see that

$$\tilde{c}(\mathbf{u};\mathbf{v},\mathbf{w})=c(\mathbf{u};\mathbf{v},\mathbf{w}) \quad \forall \mathbf{u},\mathbf{v}, \text{ and } \mathbf{w}\in H_0^1(\Omega) \text{ with } \nabla\cdot\mathbf{u}=0.$$

Moreover,

$$\tilde{c}(\mathbf{u};\mathbf{v},\mathbf{w}) = -\tilde{c}(\mathbf{u};\mathbf{w},\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v},\ \text{and } \mathbf{w} \in H_0^1(\Omega).$$

In the context of the time-dependent Navier Stokes equations, these relations are very important to obtain meaningful discretizations. They also appear to be necessary to guarantee that the scheme (6.28) will converge to an approximation of the solution of (6.1). For the continuous problem, the Picard iteration is known to converge to the solution, for any initial guess, if ν is sufficiently large; we expect the same behavior here.

With this choice of trilinear form, (6.28) corresponds to a form $B_{\mathbf{w}}(\cdot,\cdot)$ which can be written as

$$B_{\mathbf{w}}(\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}) + \tilde{c}(\mathbf{w};\mathbf{u},\mathbf{v}),$$

where $a(\cdot,\cdot)$ is symmetric, and $\tilde{c}(\cdot;\cdot,\cdot)$ is anti-symmetric. In this case, the estimates of the lower and upper bounds relevant to the GMRES method (see (2.3)) are improved and greatly simplified.

Chapter 7

Hierarchical Preconditioners for the Mortar Methods

7.1 Introduction

Mortar finite element methods were introduced about seven years ago by Bernardi, Maday, and Patera; see [14]. The discretization of an elliptic, second order problem starts by partitioning the computational domain Ω into the union of nonoverlapping subregions (substructures), $\{\Omega_i\}_{i=1}^{I}$, and an interface Γ, which is the set of points which belong to the boundaries of at least two subregions. For convenience, we use the notation I for the total number of substructures, and reserve M for other purposes. Here, we restrict ourselves to the geometrically conforming case in two dimensions; the intersection between the closure of two different subregions is either empty, a vertex, or a whole edge. We note that mortar element methods have also been developed for geometrically nonconforming decompositions of the given region, i.e. for decompositions which violate this rule, as well as for problems in three dimensions.

The restriction of the mortar finite element space considered here to any subregion Ω_i is just a standard piece-wise linear finite element space. We can adopt a strategy of successive refinement to obtain flexible, geometrically conforming, and shape regular triangulations of each of the subregions. The meshes of two neighboring subregions do not necessarily match on their common interface and the elements of the discrete space V^h are typically discontinuous across the interface Γ. Instead of pointwise continuity, the interface jumps are made orthogonal to a carefully chosen space of trial functions. In our work, we exclusively consider the second generation mortar element methods for which continuity is not even imposed at the vertices of the substructures; even if the meshes match across the interface between adjacent subregions, the mortar finite element functions will not, generally, be pointwise continuous.

Similarly as when working with other nonconforming methods, the original bilinear form $a(\cdot,\cdot)$ is replaced by $a^{\Gamma}(\cdot,\cdot)$, defined as the sum of contributions from the individual subregions to $a(\cdot,\cdot)$:

$$a^{\Gamma}(u_h, v_h) = \sum_{i=1}^{I} a_{\Omega_i}(u_h, v_h). \tag{7.1}$$

For $u_h = v_h$, we obtain the square of what is often called a broken norm. Here the norm has been broken along Γ and it is finite for any element of the mortar space even if it is discontinuous across Γ.

It is known that the resulting discrete variational problem gives rise to a linear system with a symmetric, positive definite matrix, and that its solution is an accurate approximation to the exact solution of the continuous problem; see [8, 9, 14] where error bounds of the same type as for standard conforming methods are derived.

In this chapter, we address the issue of solving this linear system efficiently. We note that direct methods and classical, unpreconditioned iterative methods have well-known limitations. Domain decomposition algorithms, which form a special family of preconditioned conjugate gradient methods, have been developed extensively for standard conforming finite elements. The present study is part of an effort to extend the applicability of these methods to a wider family of discretizations. Here, we have chosen to use a hierarchical preconditioner modeled on an algorithm developed by Smith and Widlund [110]. That work, in turn, was based on a result of Yserentant [117]. We note that, in the conforming case, this was found to be an effective preconditioner with certain advantages over some similar iterative methods because of being relatively simple, and as effective as the others; cf. [107] for motivation and a comparative study.

Our algorithm is a preconditioned conjugate gradient method with a condition number bounded from above by $C(1+\ell)^2$. Here ℓ is the maximum number of successive refinements of any individual subregion Ω_i into elements, and C a constant which depends on the minimal angle of the triangulations into subregions and elements, but which is otherwise independent of ℓ, and the number and size of the substructures and elements. Our method is an iterative substructuring algorithm, i.e. the linear system is first reduced in size by implicitly eliminating all the nodal variables interior to the subregions. The nodal values on $\partial\Omega_N$, the part of the boundary where a Neumann condition is imposed, are also classified as being interior. In each step of the iteration, we solve a local boundary value problem for each subregion, perform very fast local transformations between the nodal and hierarchical bases restricted to each individual edge, and solve a global problem of a dimension equal to the number of crosspoints of the partitioning of the region into substructures. We note that the global coarse space of our algorithm is the same as for the conforming case. This is in contrast with those proposed in Achdou, Maday, and Widlund [4] and Dryja [46], which are of higher dimension.

Other iterative substructuring methods for mortar finite elements have been described and analyzed by Achdou, Kuznetsov, and Pironneau [1, 2] and Le Tallec [70]. Ongoing work in the field also includes Maday and Widlund [76]. We also note that certain technical issues related to extending the algorithm of this paper to higher order elements are discussed in a recent paper by the first author; cf. [36].

In the next section, we introduce the mortar space V^h, and establish some properties of certain special vertex basis functions. In Section 3, we introduce the hierarchical structure, and describe and analyze our algorithm. In Section 4, we report on some numerical experiments that demonstrate the effectiveness of the algorithm in a relatively wide range of situations.

7.2 The Elliptic Problem and the Mortar Finite Element Method

Let Ω be a bounded polygonal region in $\mathbb{R}^2$ with a diameter on the order of 1. For simplicity, we consider only Poisson's equation as a model problem. The boundary of Ω, $\partial\Omega$, is the union of $\partial\Omega_N$ and $\partial\Omega_D$ on which Neumann and homogeneous Dirichlet conditions are imposed, respectively. We assume that $\partial\Omega_D$ is a closed set of positive measure. Let

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad (7.2)$$

define an elliptic and continuous bilinear form on $H_0^1(\Omega, \partial\Omega_D)$, the subspace of $H^1(\Omega)$ with elements which vanish on $\partial\Omega_D$. Let $f(\cdot)$ be a continuous linear functional on $H_0^1(\Omega, \partial\Omega_D)$; it includes a contribution from the nonhomogeneous Neumann boundary values, if any, in the form of a line integral. Then, by the Lax-Milgram lemma, there is a unique $u \in H_0^1(\Omega, \partial\Omega_D)$ satisfying

$$a(u,v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D). \qquad (7.3)$$

7.2.1 Triangulation of the region and the subregions

We assume that Ω can be partitioned into nonoverlapping, shape regular triangular substructures, $\{\Omega_i\}_{i=1}^I$; we will focus on the analysis of the case of triangular substructures but we note that a similar theory can be developed for the quadrilateral case. As noted before, the intersection between the closure of any two distinct substructures is either empty, a vertex, or a whole edge; this coarse triangulation is geometrically conforming. We also assume that if $\partial\Omega_i \cap \partial\Omega$ is nonempty, then the boundary condition does not change type in the interior of any edge of Ω_i. We note that we are primarily interested in the case of a large number of subregions, since the potential for parallelizing our method depends on having enough subproblems. Our analysis will only involve individual subregions and their next neighbors. The subregions are assumed to be shape regular but there is no need to assume that the coarse triangulation is quasi-uniform. To simplify our analysis, we assume that the triangulation of each subregion is quasi-uniform. We will denote the diameter of the subregion Ω_i by H_i, and the smallest diameter of any of its elements by h_i. Our results depend only on the minimal angle of the overall triangulation, and ℓ, the maximum of the number of refinement levels $\ell(i)$ of the substructures Ω_i.

We start the detailed description of the finite element space V^h by defining a multi-level triangulation within each substructure; see [117]. Each Ω_i is subdivided by a nested family of standard conforming finite element triangulations

$\mathcal{T}_0^i = \{\Omega_i\}, \mathcal{T}_1^i, \mathcal{T}_2^i, \dots, \mathcal{T}_{\ell(i)}^i$. The quasi-uniform triangulation $\mathcal{T}_{k+1}^i$ is obtained from the next coarser triangulation, $\mathcal{T}_k^i$, by subdividing each of its triangles into four, not necessarily equal, but shape-regular triangles. In particular, triangles of level $k+1$ have diameters of an order approximately one half of the diameter of those of level k.

7.2.2 The mortar finite elements

The interface Γ is defined by the coarse triangulation and is given by

$$\Gamma = \overline{\cup_{i=1}^{I} \partial\Omega_i \setminus \partial\Omega}.$$

A set of *mortars* $\{\gamma_m\}_{m=1}^{M}$ is obtained by selecting open edges of the substructures such that

$$\Gamma = \cup_{m=1}^{M} \overline{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset \ \text{ if } m \neq n.$$

Our view is that a mortar γ_m belongs to just one substructure, denoted by $\Omega_{i(m)}$, while the other edge, which geometrically occupies the same place, is denoted by δ_m. We refer to it as a *nonmortar*, and the subregion to which it belongs is denoted by $\Omega_{j(m)}$. The restrictions of the triangulations of $\Omega_{i(m)}$ and $\Omega_{j(m)}$ to this common edge will typically differ and are denoted by γ_m^h and δ_m^h, respectively. Discontinuous mortar finite element functions have different traces on γ_m and δ_m given by one-sided limits with respect to the two subregions $\Omega_{i(m)}$ and $\Omega_{j(m)}$. An important component of our preconditioner will be related to the union of the two subregions $\Omega_{i(m)}$ and $\Omega_{j(m)}$ and the edge in between, and we will denote this set by $\mathcal{R}(\gamma_m)$. Similarly, $\mathcal{R}(\Omega_i)$ is the union of all subregions Ω_j the closure of which intersects the closure of Ω_i. We also introduce the notation $V^h(\Sigma)$ to mean the restriction of V^h to a set Σ which, in this paper, will always be a single subregion or the union of a few of them. Finally, we denote by $V_0^h(\Sigma)$ the subspace of $V^h(\Sigma)$ of functions which vanish on $\partial\Sigma$.

Even though we will use a hierarchical basis in the design of our preconditioner, we can primarily work with a nodal basis. We will use a nodal basis of the mortar finite element space associated with the following sets of nodes:

- all nodes interior to the substructures and on $\partial\Omega_N$,

- all nodes interior to the mortars, and
- all nodes of vertices of subregions except those on $\partial\Omega_D$.

We denote by $\mathcal{V}$ the set of vertices of the substructures that are associated with degrees of freedom of V^h, i.e. those for which the values are not given by the Dirichlet data on $\partial\Omega_D$. Each crosspoint of Γ corresponds to several nodes of $\mathcal{V}$ and to one degree of freedom for each of the subregions that meet at that point; these nodes are in the same geometrical position, but are assigned to different subregions.

For each m, we further define a space of test functions $\mathcal{W}^h(\delta_m)$ given by the restriction to the nonmortar δ_m of the original finite element space defined on $\Omega_{j(m)}$ subject to the constraints that these continuous, piece-wise linear functions are constant in the first and last mesh intervals of δ_m^h.

The *mortar projection* π_m maps the space of finite element functions defined on γ_m^h into that of δ_m^h. Given $u^{i(m)}$ in $\Omega_{i(m)}$, and boundary values of $u^{j(m)}$ at the two endpoints v_{n_1} and v_{n_2} of δ_m, we determine the values of $\pi_m(u^{i(m)}, u^{j(m)}(v_{n_1}), u^{j(m)}(v_{n_2}))$ at the interior nodes of δ_m^h by

$$\int_{\delta_m} (u^{i(m)} - \pi_m(u^{i(m)}, u^{j(m)}(v_{n_1}), u^{j(m)}(v_{n_2}))\psi ds = 0 \quad \forall \psi \in \mathcal{W}^h(\delta_m). \qquad (7.4)$$

After these preparations, the mortar finite element space V^h can now be fully defined. The restriction of V^h to Ω_i, $V^h(\Omega_i)$, is a regular conforming finite element space as described above. For each nonmortar there is a set of constraints,

$$u^{j(m)}|_{\delta_m} = \pi_m(u^{i(m)}, u^{j(m)}(v_{n_1}), u^{j(m)}(v_{n_2})), \qquad (7.5)$$

which replace the pointwise continuity of conforming spaces.

The discrete problem is then:
Find $u \in V^h$ such that

$$a^\Gamma(u, v) = f^\Gamma(v) \quad \forall v \in V^h, \qquad (7.6)$$

where $a^\Gamma(u, v)$ is defined in formula (7.1) and, similarly, $f^\Gamma(v)$ is the sum of contributions from the different subregions.

The rate of convergence of the solution of (7.6) to the solution of (7.3) is comparable to that of a conforming discretization; cf. [10], [14], and references therein for theoretical and experimental results.

At the expense of an exact solution of a finite element problem per subregion, with homogeneous Dirichlet data, we reduce problem (7.6) to that of finding the piece-wise discrete harmonic part of the solution. We recall that a finite element function u is discrete harmonic in the subregion Ω_i if

$$a^{\Gamma}(u,v) = 0 \quad \forall v \in V^h \cap H_0^1(\Omega_i),$$

and that a discrete harmonic function provides the unique minimal energy extension of finite element boundary data given on the boundary $\partial\Omega_i$. In what follows, we will work exclusively with piece-wise discrete harmonic functions, without restricting the generality of our discussion; from now on V^h will denote this subspace.

We next formulate a basic result proven in [14].

Lemma 7.2.1 *The mapping π_m is stable:*

$$|\pi_m(u,0,0)|_{H_{00}^{1/2}(\delta_m)} \leq C|u|_{H_{00}^{1/2}(\gamma_m)} \quad \forall u \in H_{00}^{1/2}(\gamma_m). \tag{7.7}$$

We end the subsection by proving a Poincaré inequality and formulating a Friedrichs inequality. They will be formulated for a region $\mathcal{R}(c_r)$ which is the union of the substructures which have a crosspoint c_r in common. In order to obtain a result that is independent of the mesh, we will establish the inequality for a space $\hat{V}(\mathcal{R}(c_r))$, which contains all possible $V^h(\mathcal{R}(c_r))$ as a subspace but which itself is not a finite element space. The restriction of $\hat{V}(\mathcal{R}(c_r))$ to any of the substructures $\Omega_i \subset \mathcal{R}(c_r)$ equals $H^1(\Omega_i)$. As in the case of mortar finite element functions, we potentially have two traces on any edge Γ_{ij} between any pair of substructures Ω_i and Ω_j. We only impose one constraint per edge, namely that $\int_{\Gamma_{ij}}[u]ds = 0$, where $[u]$ is the jump of u across Γ_{ij}.

Lemma 7.2.2 *Let $u \in \hat{V}(\mathcal{R}(c_r))$. Then,*

$$\inf_{c\in\Re} ||u-c||^2_{L^2(\mathcal{R}(c_r))} \leq C \sum_{\Omega_j \subset \mathcal{R}(c_r)} H_j^2 |u|^2_{H^1(\Omega_j)}. \tag{7.8}$$

Here C depends only on the minimal angle of the substructures that form the coarse triangulation of $\mathcal{R}(c_r)$ and is independent of the diameters of the substructures and their triangulations.

Proof. We note that we can confine our study to a finite number of configurations allowed by the minimal angle condition on the coarse mesh. Each configuration corresponds to a specific number of substructures that have the crosspoint c_r in common. We assume in our discussion that c_r is an interior crosspoint; the extension of our argument to cases when $c_r \in \partial\Omega_N$ poses no problems. The problem can be further specialized by noticing that a piece-wise affine map can be found that maps the triangulation of $\mathcal{R}(c_r)$ onto a regular polygon $\mathcal{R}(0)$ of diameter 1, centered at 0, the image of c_r. This mapping is benign under the assumption of shape regularity. It also accounts for the factors H_j^2 in the estimate.

Thus, what remains is to prove the Poincaré inequality (7.8) for this finite number of special reference regions. We use a variant of a well-known argument given, e.g. in Ciarlet [43, Theorem 3.1.1]. Our result follows by proving a bound

$$\sum_{\Omega_j \subset \mathcal{R}(0)} \|u\|^2_{L^2(\Omega_j)} \leq C\Big(\sum_{\Omega_j \subset \mathcal{R}(0)} |u|^2_{H^1(\Omega_j)} + \Big(\int_{\mathcal{R}(0)} u(x)dx\Big)^2\Big) \quad \forall u \in \hat{V}(\mathcal{R}(0)). \tag{7.9}$$

Following Ciarlet, we use a proof by contradiction. We assume that there is a sequence of $u_k \in \hat{V}(\mathcal{R}(0))$ with unit L^2-norm, for which all the terms of the right hand side of (7.9) go to zero. By applying Rellich's theorem, one subregion at the time, selecting a subsequence of the previous subsequence every time we move on to a new subregion, we find a limit function which is locally in $H^1(\Omega_j)$, and which, because of the continuity of the trace mappings, satisfies the jump conditions of $\hat{V}(\mathcal{R}(0))$. Thus, the limit function belongs to $\hat{V}$, and is a constant, which must vanish since the last term of (7.9) vanishes in the limit. □

We note that a proof of the following Friedrichs inequality can be found in [13]; a proof can also be given using the same techniques as above.

Lemma 7.2.3 *Let $u \in \hat{V}(\mathcal{R}(c_r))$ vanish on at least one of the edges of the substructures that form $\mathcal{R}(c_r)$. Then,*

$$||u||^2_{L^2(\mathcal{R}(c_r))} \leq C \sum_{\Omega_j \subset \mathcal{R}(c_r)} H_j^2 |u|^2_{H^1(\Omega_j)}. \tag{7.10}$$

Here C depends only on the minimal angle of the substructures that form the coarse triangulation of $\mathcal{R}(c_r)$ and is independent of the diameters of the substructures and their triangulations.

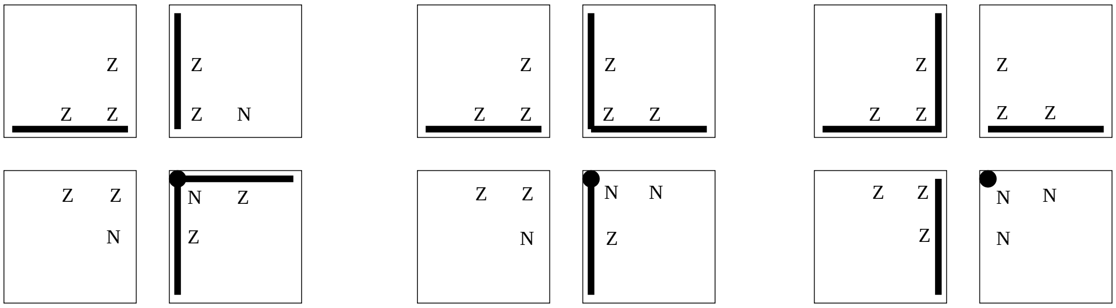

Figure 7.1: Three cases of vertex basis functions

7.2.3 Vertex basis functions

As we have already pointed out, the mortar finite element functions are typically multi-valued at the crosspoints of the subregions. In order to describe and analyze our algorithm, we need to define a special vertex basis function for each of these degrees of freedom and derive estimates of their norms. These special functions are piece-wise discrete harmonic functions.

For each vertex v_n of $\mathcal{V}$, let $\phi_{v_n} \in V^h(\Omega)$ be defined by the value 1 at v_n, with all other nodal values on Γ set to zero. This completely defines ϕ_{v_n} since the interior nodal values on the nonmortars are given by the mortar projections, and those in the interior of the Ω_i by discrete harmonic extensions.

As indicated in Fig. 1, ϕ_{v_n} differs from zero at the interior nodes of some of the nonmortar edges associated with the same crosspoint as v_n. The marked node is the vertex v_n, and it touches two, one, or no mortars. In the figure, we distinguish between values at the vertices and at the interior nodes of the edges. The bold lines represent mortars, and Z and N stand for zero and nonzero values of the vertex function ϕ_{v_n} at the vertices or at the interior nodes of the edges. The figure displays the three basic configurations in the case of four subregions meeting at a crosspoint.

In the first case, v_n is the left endpoint of a horizontal mortar γ_m. Then, ϕ_{v_n} coincides with the standard nodal basis function φ_{v_n} on γ_m. Across the edge, on the nonmortar, $\phi_{v_n}|_{\delta_m} = \pi_m(\varphi_{v_n}, 0, 0)$.

In the second and third cases, v_n is the left endpoint of a horizontal nonmortar

δ_m. By construction, ϕ_{v_n} vanishes on the mortar γ_m across the edge. Therefore, $\phi_{v_n}|_{\delta_m} = \pi_m(0,1,0)$.

The following two lemmas provide estimates that will be used in Lemma 7.2.6 to estimate the $a^\Gamma(\cdot,\cdot)$- and L^∞-norms of ϕ_{v_n}.

Lemma 7.2.4 *Let δ be a nonmortar and let π be the mortar projection associated with it. Then,*

$$||\pi(0,1,0)||_{L^\infty(\delta)} \leq C, \tag{7.11}$$

and

$$||\pi(0,1,0)||_{H^{1/2}(\delta)} \leq C. \tag{7.12}$$

Proof. Let $\underline{u}_\delta$ be the vector of nodal values, interior to δ, of $\pi(0,1,0)$. A nodal basis of $\mathcal{W}^h(\delta)$ is formed from the standard nodal basis on δ^h by combining two basis functions at each end to create two special basis functions which are constant in the mesh intervals of δ^h that touch the endpoints of δ^h.

By using (7.4), we obtain a tridiagonal system of linear equations

$$\mathcal{M}\underline{u}_\delta = \underline{b}.$$

It is easy to show that only the first and last diagonal elements of $\mathcal{M}$ differ from those of the mass matrix with respect to the space of piece-wise linear functions on δ^h that vanish at the endpoints. The differences between these matrix elements are positive and therefore

$$||\pi(0,1,0) - \varphi_{v_n}||^2_{L^2(\delta)} \leq \underline{u}_\delta^T \mathcal{M}\underline{u}_\delta \leq \underline{u}_\delta^T b \leq (\underline{u}_\delta^T \mathcal{M}\underline{u}_\delta)^{1/2}(\underline{b}^T \mathcal{M}^{-1}\underline{b})^{1/2},$$

where φ_{v_n} is the nodal basis function on δ^h associated with the left endpoint. By examining the right hand side $\underline{b}$, which has only one non-zero, we easily find, using the quasi-uniformity of δ^h, that

$$||\pi(0,1,0)||^2_{L^2(\delta)} \leq Ch_\delta,$$

from which (7.11) and (7.12) follow by using inverse inequalities. $\square$

Lemma 7.2.5 *Let v_n be the left endpoint of a mortar γ, and let φ_{v_n} be the standard nodal basis function on γ^h corresponding to v_n. Then,*

$$||\pi(\varphi_{v_n}, 0, 0)||_{L^\infty(\delta)} \leq C, \tag{7.13}$$

and

$$||\pi(\varphi_{v_n}, 0, 0)||_{H^{1/2}(\delta)} \leq C. \tag{7.14}$$

Proof. We first prove the lemma for $h_\delta \leq h_\gamma$; these are the minimal mesh sizes of δ^h and γ^h, respectively. In addition, we first also assume that the second leftmost point of γ^h, Q, coincides with a mesh point of δ^h. Using (7.4), we can easily check that

$$\pi(\varphi_{v_n}, 0, 0) - \varphi_{v_n} = -\pi(0, 1, 0),$$

since both sides are finite element functions on δ^h. The L^∞-norm estimate (7.13) now easily follows from (7.11). The $H^{1/2}$-norm estimate (7.14) is a consequence of $||\varphi_{v_n}||^2_{H^{1/2}} \leq C$ and (7.12).

If the second leftmost point of γ^h, Q, does not coincide with a mesh point of δ^h, we denote by R the mesh point of δ^h that is the right next neighbor of Q. Let $\tilde{\varphi}$ be the piece-wise linear function that equals 1 at v_n and vanishes at R, and to the right of R. Then, the argument just given can be used for $\tilde{\varphi}$, since it is now a finite element function on the mesh δ^h. There remains to estimate $\varphi_{v_n} - \tilde{\varphi}$. From the definition of $\tilde{\varphi}$, we find that $||\varphi_{v_n} - \tilde{\varphi}||^2_{L^2(\delta)} \leq Ch_\delta$. An argument similar to that of the proof of Lemma 7.2.4 shows that $||\pi(\varphi_{v_n} - \tilde{\varphi}, 0, 0)||^2_{L^2(\delta)} \leq Ch_\delta$, and we can conclude the proof of the result for $h_\delta \leq h_\gamma$ by using two inverse inequalities.

If $h_\delta > h_\gamma$, we can use that $||\varphi_{v_n}||_{L^2(\gamma)} \leq Ch_\gamma$ and an argument similar to that of the proof of Lemma 7.2.4, to conclude that

$$||\pi(\varphi_{v_n}, 0, 0)||_{L^2(\delta)} \leq Ch_\gamma^2/h_\delta \leq Ch_\gamma.$$

The two estimates (7.13) and (7.14) now follow as before. $\square$

Lemma 7.2.6 *For any $v_n \in \mathcal{V}$, we have:*

$$||\phi_{v_n}||_{L^\infty(\Gamma)} \leq C, \tag{7.15}$$

and

$$a^\Gamma(\phi_{v_n}, \phi_{v_n}) \leq C(1 + \ell). \tag{7.16}$$

Proof. The first bound, (7.15), follows immediately from Lemmas 7.2.4 and 7.2.5. The bound on the square of the trace norm, proportional to $(1+\ell)$, follows from (7.15) and an argument in the proof of Lemma 3.2 in [115]. $\square$

7.3 Algorithm and Analysis

Our solution procedure starts with the static condensation of all degrees of freedom interior to the different substructures, reducing the size of the discrete system. We note that it is not necessary to compute the Schur complement. All that is needed is to carry out a matrix-vector multiplication with the Schur complement. After finding sufficiently accurate values on Γ, the solution of (7.6) is then computed everywhere by solving a finite element problem for each subregion Ω_i with Dirichlet data given on $\partial\Omega_i \setminus \partial\Omega_N$.

7.3.1 Schwarz methods

We solve (7.6) with a preconditioned conjugate gradient method, using an additive Schwarz method determined by a finite family of subspaces $\{V_s\}$ whose sum spans V^h, and bilinear forms $\{b_s(\cdot,\cdot)\}_s$ defined on $V_s \times V_s$. Using the Schwarz framework described in Dryja and Widlund [52], we define approximate projections $T_s : V^h \rightarrow V_s$, by

$$b_s(T_s u, v_s) = a^\Gamma(u, v_s) \quad \forall v_s \in V_s.$$

The preconditioned operator T is given by

$$T = \alpha T_0 + \sum_{s\geq 1} T_s,$$

where α is a positive parameter that is used to tune the algorithm; see Section 7.4.

Let C_0^2 be a constant such that, for all $u \in V^h$, there exists $\{u_s\}_s$, $u_s \in V_s$, with

$$\sum_s b_s(u_s, u_s) \leq C_0^2 a^\Gamma(u,u) \quad \text{where} \quad u = \sum_s u_s,$$

and let ω be a constant such that

$$a^\Gamma(u,u) \leq \omega b_s(u,u) \quad \forall u \in V_s.$$

Since our algorithm is a two-level algorithm, the third hypothesis of Theorem 2.2 in [52] is trivially satisfied. This theorem then provides a bound on the condition number of T:

$$\kappa(T) \leq C C_0^2 \omega. \tag{7.17}$$

In subsection 7.3.3, we will introduce our algorithm and establish bounds for C_0 and ω.

7.3.2 A hierarchical basis

Before we can introduce our preconditioner in detail, we need to review some aspects of Yserentant's hierarchical basis method; cf. [117]. We denote by $\mathcal{N}_k^i$, $k = 0, 1, \ldots, \ell(i)$, the set of vertices of the triangles of $\mathcal{T}_k^i$, by V_k^i the space of continuous functions on $\overline{\Omega}_i$ that are linear in the triangles of $\mathcal{T}_k^i$, and by V^i the most refined space $V_{\ell(i)}^i$. All elements of V_k^i vanish on $\partial\Omega_i \cap \partial\Omega_D$. An interpolation operator $I_k^i : V^i \rightarrow V_k^i$, is defined by

$$I_k^i u(x) = u(x) \quad \forall x \in \mathcal{N}_k^i.$$

Following Yserentant [117], we define a discrete norm, for any set $\Lambda \subset \Omega_i$, by

$$|||u|||_\Lambda^2 = \sum_{k=1}^{\ell(i)} \sum_{x \in \mathcal{N}_k^i \backslash \mathcal{N}_{k-1}^i \cap \overline{\Lambda}} |(I_k^i u - I_{k-1}^i u)(x)|^2; \tag{7.18}$$

cf. Yserentant [117]. Let W_k^i be the image of $I_k^i - I_{k-1}^i$; this is the subspace of functions of V_k^i that vanish on $\mathcal{N}_{k-1}^i$. A hierarchical basis of V^i can now be defined recursively. The hierarchical basis of V_0^i is the standard finite element nodal basis restricted to the single triangle Ω_i. It is clear that $V_k^i = V_{k-1}^i + W_k^i$, $k \geq 1$. In each step, we augment the hierarchical basis of V_{k-1}^i by the level k nodal basis functions which span $W_k^i \subset V_k^i$. For a function u represented in this basis, the discrete norm $|||u|||_\Lambda^2$ is simply the Euclidean norm and thus very easy to compute. Moreover, the transformation between the standard nodal basis and the hierarchical basis is very fast and easy to implement; see [110] and [117].

We first describe some results that have motivated the definition of our preconditioner. For one substructure, we have:

$$|I_0^i u|_{H^1(\Omega_i)}^2 + |||u|||_{\Omega_i}^2 \leq C(1 + \ell(i))||u||_{L^\infty(\Omega_i)}^2 \quad \forall u \in V^i, \tag{7.19}$$

and

$$|u|^2_{H^1(\Omega_i)} \leq C(|I_0^i u|^2_{H^1(\Omega_i)} + |||u|||^2_{\Omega_i}) \quad \forall u \in V^i. \tag{7.20}$$

Equation (7.19) is an easy consequence of the definition of the interpolation operators. Equation (7.20) results from a strengthened Cauchy-Schwarz inequality; see Yserentant [117].

For our purposes, we need a variant of (7.20):

Lemma 7.3.1 *For $u \in V^h(\Omega_i)$,*

$$|u|^2_{H^1(\Omega_i)} \leq C(|I_0^i u|^2_{H^1(\Omega_i)} + |||u|||^2_{\partial\Omega_i}). \tag{7.21}$$

Proof. We first define an extension $E(u) \in V^i$ of u, normally not discrete harmonic, such that $E(u)$ agrees with u on $\partial\Omega_i$, and when written in the hierarchical representation, has all its degrees of freedom in the open set Ω_i equal to zero. Equation (7.20), applied to $E(u)$, implies

$$|E(u)|^2_{H^1(\Omega_i)} \leq C(|I_0^i u|^2_{H^1(\Omega_i)} + |||u|||^2_{\partial\Omega_i}),$$

since $|||E(u)|||_{\Omega_i} = |||u|||_{\partial\Omega_i}$. We conclude by noting that the discrete harmonic function u has the smallest energy among all extensions in V^i of the boundary values of u. $\square$

7.3.3 The algorithm

We are now in a position to describe and analyze our algorithm. The coarse space, which is conforming, is given by

$$V_0 = \{u \in V^h \cap H_0^1(\Omega, \partial\Omega_D)| \ u \text{ is linear on each } \ \Omega_i\}.$$

The bilinear form associated with V_0 is $a^\Gamma(\cdot,\cdot)$ which coincides with $a(\cdot,\cdot)$ on this subspace.

A one-dimensional vertex space is associated with each $v_n \in \mathcal{V}$:

$$V_{v_n} = \text{span of } \ \phi_{v_n}.$$

We use the exact bilinear form $a^\Gamma(\cdot,\cdot)$ for these spaces.

A subspace $V_{\gamma_m} = V_0^h(\mathcal{R}(\gamma_m))$ is associated with each mortar. The bilinear forms for these spaces are given by $b_{\gamma_m}(\cdot,\cdot) = |||\cdot|||^2_{\gamma_m}$.

The Schwarz framework provides a preconditioned equation $Tu = b$ in terms of these spaces and bilinear forms and the solution of this equation is the same as that of (7.6). The main result of this paper is the following theorem. We note that a more straightforward approach to the proof would lead to a bound with a fourth power of ℓ.

Theorem 7.3.1 *The condition number of T satisfies*

$$\kappa(T) \leq C(1+\ell)^2.$$

Proof. We first partition $u \in V^h$ and obtain the estimate $C_0^2 \leq C(1+\ell)^2$. To do so, we select $u_0 \in V_0$, in the representation of $u = \sum u_s$, by making $u_0(c_r) = \bar{u}_{c_r}$, where $\bar{u}_{c_r}$ is the average value of u at the vertices of $\mathcal{V}$ that coincide geometrically with c_r. A standard Sobolev-like inequality for finite elements, see e.g. [16], gives:

$$\begin{aligned} |u(v_n) - u_0(v_n)|^2 &\leq C \sum_{\Omega_j \subset \mathcal{R}(c_r)} (1+\log(H_j/h_j))(|u|^2_{H^1(\Omega_j)} + \frac{1}{H_j^2}||u||^2_{L^2(\Omega_j)}) \qquad (7.22)\\ &\leq C(1+\ell) \sum_{\Omega_j \subset \mathcal{R}(c_r)} (|u|^2_{H^1(\Omega_j)} + \frac{1}{H_j^2}||u||^2_{L^2(\Omega_j)}), \end{aligned}$$

since $\log(H_j/h_j)$ is proportional to $\ell(j)$.

If $\mathcal{R}(c_r)$ has a whole edge on $\partial\Omega_D$, then the last sum above can be bounded by $a^{\Gamma}_{\mathcal{R}(c_r)}(u,u)$, the restriction of $a^{\Gamma}(\cdot,\cdot)$ to $\mathcal{R}(c_r)$, by using Lemma 7.2.3. If $\mathcal{R}(c_r)$ has only one point in common with $\partial\Omega_D$, we consider its union with an additional subregion, chosen so that this new $\mathcal{R}(c_r)$ has a whole edge on $\partial\Omega_D$, and use Lemma 7.2.3 again. If $\partial\Omega_i \cap \partial\Omega_D = \emptyset$, we add a constant to u, which does not change the left hand side, and use Lemma 7.2.2. For any of these three cases, we have:

$$|u(v_n) - u_0(v_n)|^2 \leq C(1+\ell) \sum_{\Omega_j \subset \mathcal{R}(c_r)} |u|^2_{H^1(\Omega_j)}, \qquad (7.23)$$

which in turn implies, by a standard argument, that

$$|u_0|^2_{H^1(\Omega_i)} \leq C(1+\ell) \sum_{\Omega_j \subset \mathcal{R}(c_r)} |u|^2_{H^1(\Omega_j)}. \qquad (7.24)$$

We next define the vertex space components of u. For each vertex $v_n \in \mathcal{V}$, let $u_{v_n} = (u - u_0)(v_n)\phi_{v_n}$. Using equation (7.16) and (7.23), we obtain

$$\begin{aligned} a^{\Gamma}(u_{v_n}, u_{v_n}) &\leq C(1+\ell)|(u-u_0)(v_n)|^2 \\ &\leq C(1+\ell)^2 a^{\Gamma}_{\mathcal{R}(c_r)}(u,u), \end{aligned} \tag{7.25}$$

where c_r is the crosspoint that coincides geometrically with v_n.

Let $w = u - u_0 - \sum_{v_n} u_{v_n}$. Then, w vanishes at all the vertices. For each mortar γ_m, let $u_{\gamma_m} \in V_{\gamma_m}$ coincide with w on γ_m. It is easy to see that $u = u_0 + \sum_n u_{v_n} + \sum_m u_{\gamma_m}$, and that

$$\sum_{m=1}^{M} |||u_{\gamma_m}|||^2_{\gamma_m} \leq \sum_{i=1}^{I} |||w|||^2_{\partial\Omega_i}.$$

By the argument used to derive (7.19), we have

$$|||w|||^2_{\partial\Omega_i} \leq C(1+\ell)\|w\|^2_{L^\infty(\partial\Omega_i)}.$$

By (7.15), we know that $\|\phi_{v_n}\|_{L^\infty(\Gamma)} \leq C$ for all $v_n \in \mathcal{V}$. Hence,

$$\begin{aligned} |||w|||^2_{\partial\Omega_i} &\leq C(1+\ell)||u-u_0||^2_{L^\infty(\partial\Omega_i)} \\ &\leq C(1+\ell)(||u||^2_{L^\infty(\Omega_i)} + \sum_{c_r\in\Omega_i} |u_0(c_r)|^2) \\ &\leq C(1+\ell)^2(\sum_{\Omega_j\subset\mathcal{R}(\Omega_i)} |u|^2_{H^1(\Omega_j)} + \frac{1}{H_j^2}||u||^2_{L^2(\Omega_j)}), \end{aligned}$$

since the value of u_0 at c_r depends only on the values of u at the vertices that coincide geometrically with c_r, and by using the same Sobolev-like inequality used to derive (7.22).

We now repeat the quotient space argument of (7.23), and obtain

$$|||w|||^2_{\partial\Omega_i} \leq C(1+\ell)^2 a^{\Gamma}_{\mathcal{R}(\Omega_i)}(u,u).$$

Summing over all subregions, we find

$$\sum_{m=1}^{M} |||u_{\gamma_m}|||^2_{\gamma_m} \leq C(1+\ell)^2 \sum_{i=1}^{I} a^{\Gamma}_{\mathcal{R}(\Omega_i)}(u,u). \tag{7.26}$$

Every point of Ω is covered only a small number of times by $\{\mathcal{R}(c_r)\}$, c_r a crosspoint, and by $\{\mathcal{R}(\Omega_i)\}$. We use (7.24), sum (7.25) over all c_r, and (7.26) to obtain

$$\begin{aligned} |u_0|^2_{H^1(\Omega)} & + \textstyle\sum_{v_n \in \mathcal{V}} a^\Gamma(u_{x_n}, u_{x_n}) + \sum_{m=1}^{M} |||u_{\gamma_m}|||^2_{\gamma_m} \\ & \leq C(1+\ell)^2 a^\Gamma(u,u), \end{aligned}$$

which completes the estimate of C_0^2.

Our next task is to show that ω can be bounded by a constant. Fortunately, this is a very simple matter. For the coarse and vertex spaces, $\omega = 1$, since we use exact solvers for these spaces. Let $u \in V_{\gamma_m}$. Then,

$$a^\Gamma(u,u) = |u|^2_{H^1(\Omega_{i(m)})} + |u|^2_{H^1(\Omega_{j(m)})}.$$

The stability of the mortar projection, Lemma 7.2.1, the standard trace theorem, and an extension theorem for finite element functions, [15, Lemma 5.1], allow us to bound the second term of the right hand side by the first. Then (7.21) can be used to obtain

$$a^\Gamma(u,u) \leq C|||u|||^2_{\gamma_m} = C b_{\gamma_m}(u,u) \quad \forall u \in V_{\gamma_m},$$

since the elements of V_{γ_m} vanish at the subdomain vertices. Hence, $\omega \leq C$. □

7.4 Numerical Experiments

Our method has been implemented in MATLAB, and the code is general enough to treat regions that can be decomposed into the union of rectangles aligned with the axes; the mesh inside each subregion can be any tensor product mesh, and the meshes do not necessarily match on the interface between the subregions. We only report results for a very simple region; we note that even for more general ones, the algorithm appears to be insensitive to quite different mesh sizes in adjacent regions.

In a first set of experiments, the region Ω is the unit square, divided uniformly into $M \times M$ substructures, where M is $2, 4, 8$, or 16. The substructures are squares, and V_0 is the space of continuous, piece-wise bilinear functions on the coarse triangulation. For every $N \in \{4, 8, 16, 32\}$, each substructure is divided into an

TABLE 1

Condition numbers for the $N \times (N+4)$ case

Refinement levels	3	4	5	6	7	8	9
N	4	8	16	32			
M=2	7.84	7.26	8.30	10.11			
N		4	8	16	32		
M=4		9.95	8.24	8.58	10.35		
N			4	8	16	32	
M=8			10.01	8.22	8.88	11.23	
N				4	8	16	32
M=16				10.94	8.27	9.09	N/A

$N \times (N+4)$ grid of smaller rectangles, if the substructure is in an odd row, and into an $(N+4) \times N$ grid if it is in an even row. These small rectangles are then divided into two triangles by drawing the diagonals from bottom left to top right. The meshes do not match at the interfaces of the substructures; we assign mortars and nonmortars in an arbitrary fashion. The results are summarized on Table 1, which is organized in the same way as Table 2 in [110] to facilitate a comparison. The number of refinement levels is approximately equal to $\log_2(MN)$ starting from the entire region.

In a second set of experiments, odd rows of subregions have uniform grids of $N \times N$ squares divided into two triangles each, and even lines have $(N+4) \times (N+4)$ squares also divided in two triangles each. Table 2 summarizes the results for this case.

As in [110], the coarse space V_0 generates a separate contribution to the preconditioner, which may be multiplied by a constant α in order to improve the overall condition number; cf. Subsection 7.3.1. In our experiments, we found that $\alpha = 5$ is close to the optimal parameter value for a large range of N and M, and all the results reported have been obtained with this value of α.

We remark that the growth of the condition number is virtually independent of M^2, the number of substructures, if we fix the value of $H/h = N$. Our results are quantitatively slightly better than the results obtained in the conforming case.

TABLE 2

Condition numbers for the $N \times N$ and $(N+4) \times (N+4)$ case

Refinement levels	3	4	5	6	7	8	9
N	4	8	16	32			
M=2	5.48	6.32	7.80	9.81			
N		4	8	16	32		
M=4		9.30	7.95	8.36	10.28		
N			4	8	16	32	
M=8			9.46	8.36	9.01	11.23	
N				4	8	16	32
M=16				9.52	8.55	9.21	11.48

This appears to be due to a larger overlap of the subspaces in a neighborhood of the crosspoints, since several subspaces are nonzero there. It can also be an effect of using parameters, different from those of the conforming case, to scale the contribution of the coarse problem to the preconditioner.

The growth is also consistent with the estimate $\log^2(N)$ given by Theorem 1. For small values of N, N and $N+4$ differ substantially, and this appears to be the reason for the variations in this pattern found in the tables.

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