# DIAGONAL EDGE PRECONDITIONERS IN p-VERSION AND SPECTRAL ELEMENT METHODS

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**Abstract.** Domain decomposition preconditioners for high-order Galerkin methods in two dimensions are often built from modules associated with the decomposition of the discrete space into subspaces of functions related to the interior of elements, individual edges, and vertices. The restriction of the original bilinear form to a particular subspace gives rise to a diagonal block of the preconditioner, and the action of its inverse on a vector has to be evaluated in each iteration. Each block can be replaced by a preconditioner in order to decrease the cost. Knowledge of the quality of this local preconditioner can be used directly in a study of the convergence rate of the overall iterative process.

The Schur complement of an edge with respect to the variables interior to two adjacent elements is considered. The assembly and factorization of this block matrix are potentially expensive, especially for polynomials of high degree. It is demonstrated that the diagonal preconditioner of one such block has a condition number that increases approximately linearly with the degree of the polynomials. Numerical results demonstrate that the actual condition numbers are relatively small.

**Key words.** p-version finite element, spectral element method, Schur complement, diagonal preconditioner.

**AMS(MOS) subject classifications.** 65F35, 65N22, 65N30, 65N35, 65N55

**1. Introduction.** Polynomials of high degree have been used extensively to approximate second order elliptic partial differential equations in the plane. Two well-known discretization schemes are the p-version finite element method [22], and the spectral element method [12, 13].

In the conforming formulation of these schemes, the domain is partitioned into a union of elements so that the intersection between two distinct elements is either empty, one vertex, or a whole edge. In each element, the discretization space consists of polynomials of degree $N$; the discrete solution approaches the exact one when $N$ increases. Previous theoretical and practical work shows that these methods take full advantage of the regularity of the solution of the partial differential equation; see [2, 3, 10, 13, 22]. The basis of this polynomial space is usually chosen so that it can be partitioned into sets of functions associated with the interior of the element, individual edges, or the vertices.

Let the stiffness matrices corresponding to the p-version and spectral element methods for the homogeneous Dirichlet problem defined in one element be denoted by $K_p$ and $K_N$, respectively. Let the usual bases for these methods, which will be described in Section 2, be used to generate these matrices. Then, the condition numbers satisfy:

$$\kappa(K_p) \asymp N^4 \quad \text{and} \quad \kappa(K_N) \asymp N^3; \tag{1}$$

see [5] and [21]. Here, and in what follows, $\asymp$ means that the ratio of the quantities being compared is bounded from above and below by constants independent of the degree $N$. These conditioning results are even worse for a domain partitioned into many elements, and they suggest that an unpreconditioned conjugate gradient method

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is likely to require many iterations; this is actually seen in numerical tests. Diagonal preconditioning of these full matrices has also been used, but the condition number still increases quadratically with $N$; see [5, 21].

Many domain decomposition preconditioners can be viewed as block-Jacobi preconditioners after an appropriate change of basis has been made. Each block is determined by a subspace of the discrete space, and by an exact or inexact solver; see [9]. The decomposition into subspaces corresponds to the elimination of the coupling between different sets of basis functions. We note that it has been determined experimentally that there is a very strong coupling between the interior and the standard interface basis functions [4]. A block-Jacobi preconditioner that eliminates the problem associated with this strong coupling has been proposed by Babuška, Craig, Mandel, and Pitkäranta [2] for the p-version finite element method. A change of basis is performed by computing the *Schur complement* with respect to the interior degrees of freedom; the new interface basis functions are orthogonal to the interior ones. In this new basis, the preconditioner is built from one block of relatively small dimension associated with a global problem, one block for each edge of the triangulation into elements, and one block for the interior of each element; exact solvers are used for all blocks. The condition number of this algorithm is bounded from above by $C(1+\log(N))^2$; see [2]. This result can be extended straightforwardly to the spectral element method.

However, for all the implementations that we know of, the Schur complement blocks associated with the edges are preconditioned by their diagonals; in other words, inexact solvers are used to totally decouple the edge degrees of freedom. This substantially reduces the amount of work in constructing and evaluating the action of the preconditioner, because it eliminates the need to assemble and factor the edge Schur complement blocks, or, alternatively, the need to solve, in each iteration, Dirichlet problems in the unions of pairs of subregions; see [2, 9]. The use of this diagonal preconditioner has been found not to increase the condition number of the overall iterative process appreciably, if at all; see [2, 3, 4, 14]. No theoretical result is derived in [2] to support this particular variant of the algorithm.

The goal of this paper is to prove that the blocks of the Schur complement associated with each edge, preconditioned by their diagonal, have condition numbers that grow approximately linearly with $N$, both for the p-version and for the spectral element method; see Theorems 1 and 2.

There are at least two applications of our results: the first immediate consequence is that, for the algorithm as actually implemented in [2] and [3], the condition number $\kappa$ grows faster than polylogarithmically in $N$. In fact, $\kappa$ satisfies

$$CN(1+\log(N)) \leq \kappa \leq CN(1+\log(N))^3.$$

A very similar estimate holds for the spectral element case. The numerical results presented here demonstrate that the linear growth predicted by this estimate is present for large $N$, but also that the actual condition numbers are relatively small, even for $N$ on the order of 50; see Figs. 1 and 2.

Many domain decomposition algorithms have also been developed for problems in three dimensions; see, e.g. [8, 15, 16, 18, 19, 20]. Again, the Schur complement blocks associated with the faces play a major role. We expect that diagonal preconditioners for these blocks have to be modified in order to produce reasonable condition numbers. Our results might be a starting point for deriving efficient (although possibly non-diagonal) preconditioners for the face blocks.

**2. On Polynomials and Trace Norms.** Let $\Omega = [-1,+1]^2$, with the side $[-1,+1] \times \{-1\}$ identified with $\Lambda = [-1,+1]$. Let $P^N(\Lambda)$ be the space of polynomials of degree less than or equal to $N$, and let $P_0^N(\Lambda)$ be the set of polynomials in $P^N(\Lambda)$ that vanish at $-1$ and $1$.

The space $P^N(\Omega)$ is given by tensorization of $P^N(\Lambda)$; analogously, $P_0^N(\Omega)$ is the tensor product of $P_0^N(\Lambda)$ with itself.

The Legendre polynomial basis $\{L_n\}_{n\geq 0}$ results from applying the Gram-Schmidt procedure to the set $1, x, x^2, \ldots,$ and normalizing so that $L_n(1) = 1$. The following properties are classical, and can be found in [5]:

$$((1-x^2)L_n'(x))' + n(n+1)L_n(x) = 0 \quad (n \geq 0), \tag{2}$$

$$\int_{-1}^{1} L_n^2(t)\,dt = \frac{1}{n+1/2} \quad (n \geq 0), \tag{3}$$

$$\int_{-1}^{x} L_n(t)\,dt = \frac{1}{2n+1}(L_{n+1}(x) - L_{n-1}(x)) \quad (n \geq 1). \tag{4}$$

For each $N$, the Gauss-Lobatto-Legendre quadrature of order $N$ is denoted by GLL($N$) and satisfies:

$$\forall p \in P^{2N-1}(\Lambda), \quad \int_{-1}^{1} p(x)\,dx = \sum_{j=0}^{N} p(\xi_j)\rho_j. \tag{5}$$

Here, the quadrature points $\xi_j$ are numbered in increasing order, and are the zeros of $(1-x^2)L_n'(x)$. The weights $\rho_j$ are given by:

$$\rho_j = \frac{2}{N(N+1)L_N^2(\xi_j)} \quad (0 \leq j \leq N). \tag{6}$$

The GLL($N$) quadrature has the following important property:

$$\forall p_N \in P^N(\Lambda), \quad ||p_N||_{L^2(\Lambda)}^2 \leq \sum_{j=0}^{N} p_N^2(\xi_j)\rho_j \leq 3||p_N||_{L^2(\Lambda)}^2. \tag{7}$$

We next describe the basis functions used in the two methods. Following Babuška and Szabó [22], a polynomial basis for the p-version finite element method on $P^N(\Lambda)$ is defined by $\phi_0(x) = (1-x)/2$, $\phi_1(x) = (1+x)/2$, and

$$\phi_i(x) = \frac{1}{||L_{i-1}||_{L^2(\Lambda)}} \int_{-1}^{x} L_{i-1}(t)\,dt \quad (i \geq 2). \tag{8}$$

A p-version polynomial basis for $P^N(\Omega)$ is given by tensorization of this one dimensional basis.

The basis for the spectral element method on $P^N(\Lambda)$ is given by $\{\ell_j\}_{j=0}^N$, the Lagrange interpolation basis at the GLL points, i.e. $\ell_j(\xi_i) = \delta_{ij}$. The spectral element basis in two dimensions is also given by tensorization of the one dimensional basis.

The remainder of this section describes some Schur complement and trace norm properties. They are valid for both the p-version and the spectral element method. In each case, the basis can be partitioned into two sets of functions. The first is formed by the basis functions vanishing on $\partial\Omega$; these are the interior (i) basis functions. The others are the boundary (b) basis functions. The Schur complement is defined by $S = K_{bb} - K_{ib}^t K_{ii}^{-1} K_{ib}$, where the subscripts refer to blocks of the stiffness matrix $K$, ordered appropriately.

Let $w$ be the restriction of a function of $P^N(\Omega)$ to $\partial\Omega$, let $\underline{w}_b$ be the vector of its boundary degrees of freedom, and let $||\cdot||_{H^1(\Omega)}$ and $|\cdot|_{H^1(\Omega)}$ be the standard Sobolev norm and semi-norm, respectively. We easily find that

$$\underline{w}_b^t S \underline{w}_b = \min_u |u|^2_{H^1(\Omega)} = |\mathcal{H}w|^2_{H^1(\Omega)}, \tag{9}$$

where the minimum is taken over all functions $u \in P^N(\Omega)$ such that $u|_{\partial\Omega} = w$, and $\mathcal{H}w$ is the function achieving the minimum. It is also easy to see that $(w_i, w_b)^t = \underline{\mathcal{H}w}$ satisfies:

$$K_{ii} w_i + K_{ib} w_b = 0.$$

The first expression of (9) defines a Schur complement symmetric bilinear form that only depends on the boundary values of the function, and can be estimated in terms of a trace norm. Let $H^{1/2}(\partial\Omega)$ be the trace space of $H^1(\Omega)$, which can also be defined by the K-method of interpolation as $H^{1/2}(\partial\Omega) = [L^2(\partial\Omega), H^1(\partial\Omega)]_{1/2}$; see [11]. Then, by Theorem 7.4 of [2], for any $w \in P^N(\Omega)$, there is a $u \in P^N(\Omega)$ with $u = w$ on $\partial\Omega$, such that

$$||u||_{H^1(\Omega)} \leq C||w||_{H^{1/2}(\partial\Omega)}. \tag{10}$$

Throughout the paper, we use the standard convention that $c > 0$ and $C < \infty$ are constants independent of $N$.

The space $H^{1/2}_{00}(\Lambda)$ is the space of functions $v \in H^{1/2}(\partial\Omega)$ that vanish outside $\Lambda$, endowed with the norm $||v||_{H^{1/2}(\partial\Omega)}$. This space is isomorphic to the interpolation space $[L^2(\Lambda), H^1_0(\Lambda)]_{1/2}$; see [11]. An equivalent norm for $H^{1/2}_{00}(\Lambda)$ is given by:

$$||v||_*^2 = \int_{-1}^{1}\int_{-1}^{1} \left(\frac{v(x)-v(y)}{x-y}\right)^2 dx\, dy + \int_{-1}^{1} \frac{v^2(x)}{1-x^2}\, dx; \tag{11}$$

see [17].

Let $v_\Lambda$ be the trace on $\Lambda \sim [-1,1] \times \{-1\}$ of a function of $P^N(\Omega)$ that vanishes on $\partial\Omega \setminus \Lambda$. Let $\underline{v}_\Lambda$ be the vector of degrees of freedom associated with the interior of $\Lambda$, and let $S_\Lambda$ be the Schur complement restricted to these degrees of freedom. Then, by using (9) and (10), we obtain, $\forall v_\Lambda \in P_0^N(\Lambda)$:

$$c||v_\Lambda||_*^2 \leq \underline{v}_\Lambda^t S_\Lambda \underline{v}_\Lambda \leq C||v_\Lambda||_*^2. \tag{12}$$

**3. Diagonal Preconditioning.** In what follows, we only work with $S_\Lambda$, the Schur complement restricted to $\Lambda$, as in (12), and we therefore drop the subscript $\Lambda$. Accordingly, the vectors consist of the degrees of freedom associated with the interior of $\Lambda$. The p-version and spectral element Schur complements are denoted by $S_p$ and $S_N$, respectively.

THEOREM 1. *Let $D_p$ be the diagonal of $S_p$. Then, $\forall u \in P_0^N(\Lambda)$,*

$$\lambda^N_{min}(\underline{u}^t D_p \underline{u}) \leq \underline{u}^t S_p \underline{u} \leq \lambda^N_{max}(\underline{u}^t D_p \underline{u}), \tag{13}$$

*with*

$$c \le \lambda^N_{max} \le C, \tag{14}$$

*and*

$$\frac{c}{N\log(N)} \le \lambda^N_{min} \le \frac{C\log(N)}{N}. \tag{15}$$

*Proof.* Let $u(x) = \sum_{i=2}^N a_i\phi_i(x)$. By using (12) and the Courant-Fischer characterization of the extreme eigenvalues via Rayleigh quotients, we only need to prove estimates of $||u||_*^2$ in terms of $\sum_{i=2}^N a_i^2||\phi_i||_*^2$. We start by showing that $||\phi_i||_*^2 \asymp 1/i$. Indeed, from (2), we have

$$\phi_i = -\frac{\sqrt{i-1/2}}{i(i-1)}(1-x^2)L'_{i-1}. \tag{16}$$

Then, by integrating by parts and using (2) again, the second term of (11) is easily seen to be of order $1/i^2$. To compute the first term of (11), we note that it is the square of the $L^2$-norm of a polynomial of total degree less than or equal to $i-1$. We use the GLL$(i-1)$ quadrature rule which, by (7), gives the value of the integral, to within a multiplicative constant. The use of this quadrature results in a double sum that can be reduced to

$$\sum_{j=0}^{i-1}(\phi_i'(\xi_j))^2\rho_j^2,$$

since the $\xi_j$ are zeros of the $\phi_i$, by (16). This last sum can be computed exactly by using (6) and (8) for $N = i-1$, and we find that $||\phi_i||_*^2 \asymp 1/i$.

We prove only the right inequality of (14), since the left inequality is clear by taking $u = \phi_2$. Given $u \in P_0^N(\Lambda)$, we define an extension of $u$, $E(u) \in P^N(\Omega)$ such that $E(u) = u$ on $\Lambda$, and $E(u)$ vanishes on $\partial\Omega \setminus \Lambda$. By (9) and (12), it suffices to show that $|E(u)|^2_{H^1(\Omega)} \le C\sum_{i=2}^N(a_i^2/i)$. We choose $E(u)(x,y) = \sum_{i=2}^N a_i\phi_i(x)\psi_i(y)$, for some $\psi_i \in P^N(\Lambda)$, $\psi_i(-1) = 1$, $\psi_i(1) = 0$, that will be chosen momentarily. A simple computation shows that

$$|E(u)|^2_{H^1(\Lambda)} \;=\; \sum_{i,j=2}^N a_ia_j((\phi_i',\phi_j')(\psi_i,\psi_j) + (\phi_i,\phi_j)(\psi_i',\psi_j')).$$

Here, $(\cdot,\cdot)$ is the $L^2(-1,1)$ inner product. By using (3), (4), and (8), we find:

$$(\phi_i,\phi_i) \asymp \frac{1}{i^2}, \quad (\phi_i',\phi_i') \asymp 1. \tag{17}$$

Moreover, we also have $(\phi_i',\phi_j') = 0$ if $i \neq j$, and $(\phi_i,\phi_j) = 0$ if $|i-j| > 2$. These estimates together with the Cauchy-Schwarz inequality imply

$$|E(u)|^2_{H^1(\Lambda)} \;\le\; C\sum_{i=1}^N a_i^2(||\psi_i||^2_{L^2(\Lambda)} + (1/i^2)||\psi_i'||^2_{L^2(\Lambda)}).$$

The piecewise linear interpolant using the GLL($N$) points $I_N^h$ is defined for any $v_N \in P^N(\Lambda)$ and is given by $v_h = I_N^h(v_N)$, $v_N(\xi_j) = v_h(\xi_j)$, for $j = 0, 1, \ldots, N$. The inverse of $I_N^h$ is denoted by $I_h^N$. By using results of Canuto [7], we have:

$$||v_N||_{L^2(\Lambda)} \asymp ||v_h||_{L^2(\Lambda)} \text{ and } ||v_N'||_{L^2(\Lambda)} \asymp ||v_h'||_{L^2(\Lambda)}. \tag{18}$$

For $i = 2, \ldots, N$, let $\xi_{j(i)}$ be one of the GLL($N$) points, with a distance to $-1$ proportional to $1/i$. Let $\psi_{i,h}(x)$ be the piecewise linear function that goes from 1 to 0 linearly between $-1$ and $\xi_{j(i)}$, and is zero for $x \geq \xi_{j(i)}$, and choose $\psi_i = I_h^N(\psi_{i,h})$. By (18), we have $||\psi_i||^2_{L^2(\Lambda)} \asymp 1/i$, and $||\psi_i'||^2_{L^2(\Lambda)} \asymp i$, since this is true for $\psi_{i,h}$. Then,

$$|E(u)|^2_{H^1(\Lambda)} \leq C \sum_{i=1}^{N} a_i^2((1/i) + (1/i^2)i),$$

which implies the right inequality of (14).

We next prove the left inequality of (15). We recall that $u(x) = \sum_{i=2}^N a_i \phi_i(x)$. Since $\{\phi_i'\}$ is an orthonormal set in $L^2(\Lambda)$, we have

$$a_i = \int_{-1}^{1} u'(x)\phi_i'(x)\, dx \tag{19}$$

By integration by parts and a duality argument, we get

$$\begin{aligned} a_i &\leq |\int_{-1}^{1} u(x)\phi_i''(x)\, dx\, | \\ &\leq ||u||_* ||\phi_i''||_{(H_{00}^{1/2})'} \\ &\leq ||u||_* ||\phi_i'||_{H^{1/2}} \\ &\leq ||u||_* \sqrt{i - 1/2}\, ||L_{i-1}||_{H^{1/2}}, \end{aligned}$$

where the penultimate inequality follows from [11, Proposition 12.1]. The $H^{1/2}$-norm of $L_i$ has been computed in [1], using Gaussian quadrature rules, and is known to be bounded from above by $C(\log(i))^{1/2}$. Therefore,

$$\sum_{i=2}^{N} a_i^2 ||\phi_i||_*^2 \leq C(\sum_{i=2}^{N} \log(i-1))||u||_*^2,$$

which implies the left inequality of (15).

We prove the right inequality of (15) only for the case of $N$ even. For $N$ odd, the same proof applies, with $N$ replaced by $N-1$. Let $u(x) = (L_N(x) - 1) \in P_0^N(\Lambda)$. By (19) and integration by parts, we obtain, for $2 \leq i < N$:

$$\begin{aligned} a_i &= \sqrt{i - 1/2}\,(\int_{-1}^{1} L_{i-1}'(x) - \int_{-1}^{1} L_N(x) L_{i-1}'(x)\, dx\, ) \\ &= 2\sqrt{i - 1/2}\,, \end{aligned}$$

if $i$ is even, and zero, otherwise. Again by results of [1], $||L_N - 1||_*^2 \leq C\log(N)$. Therefore:

$$\frac{\sum_{i=2}^N a_i^2 ||\phi_i||_*^2}{||u||_*^2} \geq \frac{C}{\log(N)} \sum_{i\, even,\, i \geq 2}^{N} 1,$$

which implies the right inequality of (15). □

THEOREM 2. *Let $D_N$ be the diagonal of $S_N$. Then, $\forall u \in P_0^N(\Lambda)$,*

$$\lambda_{min}^N(\underline{u}^t D_N \underline{u}) \leq \underline{u}^t S_N \underline{u} \leq \lambda_{max}^N(\underline{u}^t D_N \underline{u}), \tag{20}$$

*with*

$$c \leq \lambda_{max}^N \leq C, \tag{21}$$

*and*

$$\frac{c}{N\log(N)} \leq \lambda_{min}^N \leq \frac{C}{N}. \tag{22}$$

*Proof.* Let $u(x) = \sum_{i=1}^{N-1} u(\xi_i)\ell_{i,N}(x)$, where $\{\ell_{i,N}\}$ is the Lagrange interpolation basis related to the GLL($N$) points. By (12), we only need to estimate $||u||_*^2$ in terms of $\sum_{i=2}^{N} u^2(\xi_i)||\ell_{i,N}||_*^2$.

Let $v^N \in P_0^N(\Lambda)$. A consequence of (18) is that not only are the $L^2$- and $H^1$-norms of $v_N$ and $v_h = I_N^h(v_N)$ equivalent, but also

$$||v_N||_* \asymp ||v_h||_*; \tag{23}$$

a detailed argument can be found in [8].

We start by showing that $||\ell_{i,N}||_* \asymp 1$. Let $\ell_{i,h} = I_N^h(\ell_{i,N})$, $h_i = \xi_{i+1} - \xi_i$, and $\eta_i = \arccos(\xi_i)$. Then, for $1 \leq i \leq N-1$,

$$\frac{(N-i-(1/2))\pi}{N} \leq \eta_i \leq \frac{(N-i+1)\pi}{N}; \tag{24}$$

see [5, p. 76]. This relation implies that $h_{i+1} \asymp h_i$, for $0 \leq i \leq N-1$. A simple computation shows that for $1 \leq i \leq N-1$, $||\ell_{i,h}||_{L^2(\Lambda)}^2 \leq Ch_i$ and $||\ell_{i,h}||_{H^1(\Lambda)}^2 \leq C/h_i$. By interpolating between these two spaces and using (18), we obtain $||\ell_{i,h}||_* \leq C$. A rather tedious, yet elementary, computation using (11) shows that one of the positive terms which form $||\ell_{i,h}||_*^2$ is greater than a positive constant, and this shows that $||\ell_{i,h}||_* \asymp 1$. Then, by (23), we find that $||\ell_{i,N}||_* \asymp 1$.

The left inequality of (21) follows by taking $u = \ell_{2,N}$, and using that $||u||_* \asymp 1$. To prove the right inequality, we use (23) and restrict ourselves to piecewise linear functions. Let $E(u_h)(x,y) = \sum_{i=2}^{N} u_h(\xi_i)\ell_{i,h}(x)\psi_{i,h}(y)$, for some $\psi_{i,h}$ with $\psi_{i,h}(-1) = 1$ and $\psi_{i,h}(1) = 0$. We go through the same steps as in the previous proof, and since the mass and stiffness matrices corresponding to the $\ell_{i,h}$ are tridiagonal, we obtain, as before:

$$|E(u_h)|_{H^1(\Lambda)}^2 \leq C\sum_{i=1}^{N-1}(u_h(\xi_i))^2((1/h_i)||\psi_{i,h}||_{L^2(\Lambda)}^2 + (h_i)||(\psi_{i,h})'||_{L^2(\Lambda)}^2).$$

By (18), we can choose the $\psi_{i,h}$ so that the coefficients of $(u_h(\xi_i))^2$ can be bounded above by a constant, thus proving (21).

Our task is now to prove (22), and we start with the left inequality. For $u \in P_0^N(\Lambda)$, it is well known that

$$||u||_{L^\infty(\Lambda)}^2 \leq C(1+\log(N))||u||_*^2; \tag{25}$$

see Theorem 6.2 of [2]. Then,

$$\sum_{i=1}^{N-1} |u(\xi_i)|^2 \leq C \sum_{i=1}^{N-1} (1+\log(N))||u||_*^2,$$

and this, in turn, implies the left inequality of (22).

For the right inequality, let $u_h(x) = 1 - |x|$, and let $u_N = I_h^N(u_h)$. A standard argument of interpolation between $L^2(\Lambda)$ and $H_0^1(\Lambda)$ and a simple computation shows that $||u||_* \asymp 1$. We also have

$$\begin{aligned}
\sum_{i=1}^{N-1} (u(\xi_i))^2 &\geq \sum_{i=\lceil N/2 \rceil}^{N-1} (u(\xi_i))^2 \\
&= \sum_{i=\lceil N/2 \rceil}^{N-1} (1-\xi_i)^2 \\
&= N \sum_{i=\lceil N/2 \rceil}^{N-1} \frac{1}{N}(1-\cos(\eta_i))^2,
\end{aligned}$$

where, by (24), the $\eta_i$ are asymptotically equidistant on the interval $[0, \pi/2]$. This last sum is a Riemann sum for $(1/\pi)\int_0^{\pi/2}(1-\cos(\eta))^2\, d\eta$, and therefore,

$$\sum_{i=1}^{N-1} (u(\xi_i))^2 \geq CN,$$

completing the proof of (22). □

REMARK 1. *The Schur complement associated with an edge for a finite element space based on a quasi-uniform triangulation with a parameter $h$ has a condition number on the order of $1/h$; see [6]. The techniques used to prove Theorem 2 can be used to establish that this condition number is between $1/h$ and $|\log(h)|/h$. Although this is a slightly weaker result, our methods can be used in a context more general than for quasi-uniform meshes, e.g. for the GLL mesh of Theorem 2.*

We have performed numerical experiments to determine the actual values of the eigenvalues of Theorems 1 and 2. The results for $4 \leq N \leq 50$ are given in Figures 1 and 2. They agree, in a clear cut way, with the theoretical results developed here. We remark that for these values of $N$, the approximate linear growth of the inverse of the smallest eigenvalue is clear, and that the graph of the largest eigenvalue appears to approach a horizontal asymptote. We note that the relatively small values of the resulting condition numbers help explain the good convergence rates experienced with the algorithm implemented in [2].

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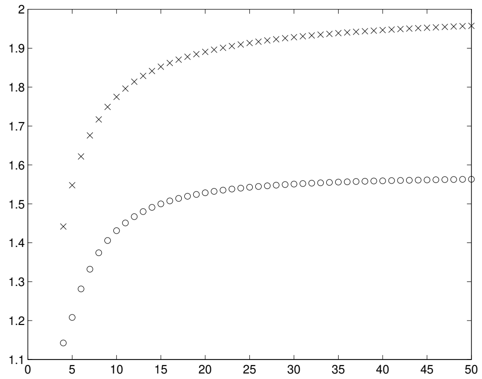


FIG. 1. $\lambda_{max}(D_p^{-1}S_p) = \circ$, $\lambda_{max}(D_N^{-1}S_N) = \times$

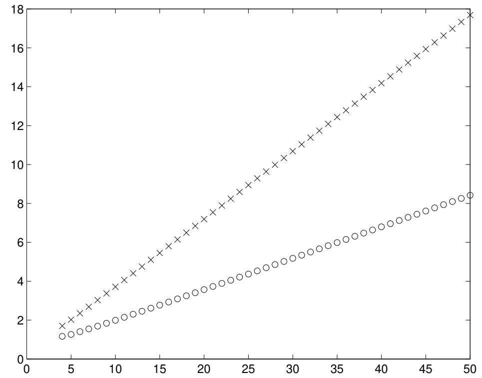


FIG. 2. $1/\lambda_{min}(D_p^{-1}S_p) = \circ$, $1/\lambda_{min}(D_N^{-1}S_N) = \times$


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