Dirichlet problem for the Schrödinger operator in a half-space with boundary data of arbitrary growth at infinity

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Abstract

We consider the Dirichlet problem for the Schrödinger operator in a half-space with boundary data having an arbitrary growth at infinity. A solution is constructed as the generalized Poisson integral. Uniqueness of the solution is investigated too.

1 Introduction. Statement of results

Denote $X = (x, x_{n+1}), Y = (y, y_{n+1}), \ldots$, generic points of a half- space R^{n+1}_+, $n \geq 1$, where $x,\ y \in R^n = \partial R^{n+1}_+$, and $x_{n+1}, y_{n+1} > 0$. Also let $r = |X|, \rho = |Y|$ $, \theta = X/r,\ \psi = Y/\rho$, where $|\cdot|$ is the euclidean metric. This article is devoted to the Dirichlet problem

$$\begin{cases} L_c u(X) \equiv -\Delta u(X) + c(X)u(X) = 0 & for \quad X \in R^{n+1}_+, \\ u(x) = f(x) & for\ e.a. \quad x \in R^n, \end{cases} \tag{1}$$

where Δ is the Laplace operator; assumptions on the function *(potential)* $c(X)$ will be formulated later.

Firstly let us consider the classical case $c = 0$. If the integral $\int_{R^n} |f(y)|\,(1+|y|)^{-(n+1)}\,dy$ converges, the solution of the problem (1) can be written as (absolutely convergent) Poisson's integral

$$\int_{R^n} f(y) P(|x - y|, x_{n+1})\,dy, \tag{2}$$

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where

$$P(t, x_{n+1}) = (2/\sigma_{n+1}) x_{n+1} (x_{n+1}^2 + t^2)^{-(n+1)/2}$$

is the harmonic Poisson's kernel for the half-space and σ_{n+1} is the area of the unit sphere in R^{n+1}.

If the integral (2) diverges, a solution to the problem (1) can be given as some regularization of this integral. In particular, M. Finkelstein and D. Sheinberg [3] have constructed a solution to the problem (1) with an arbitrary continuous function f. This solution is the integral with a modified Poisson's kernel derived by subtracting of some special harmonic polynomials from $P(t, x_{n+1})$. This methoda, ascending to the Wejerstrass' theorem about canonical representations of entire functions, has been used by several authors - see e.g., [3, 4, 6, 7, 15, 16, 18]. If $c = 0$, D. Siegel [18] has studied the uniqueness of this solution to the problem (1) under the following condition: the integral $\int_{R^n} | f(y) | (1+ | y |)^{-N} dy$ converges with certain $N > 0$.

We will construct the solution to the problem (1) as a *Poisson's c -integral* corresponding to the operator L_c. We will modify this kernel and get the solution for a boundary data f having an arbitrary growth at infinity. The modified Poisson's c -kernel was introduced in [9].

Under some additional conditions this solution is unique; note that the uniqueness proof uses heavily the Phragmen-Lindelöf principle for the operator L_c [11].

Now we state the results of our article. The potential c is always supposed to be non-negative and locally-integrable, namely, $c(X) \geq 0$ for $X \in R_+^{n+1}$, and $c \in L_{loc}^p(R_+^{n+1})$ with some $p \geq (n+1)/2$ as $n \geq 3$, and $p = 2$ as $n = 1, 2$. Under these assumptions the operator L_c can be extended in the usual way from the space $C_0^\infty(R_+^{n+1})$ to an essentially self-adjoint operator on $L^2(R_+^{n+1})$; we will denote it L_c as well. This last one has a Green's function $G(X, Y)$ possesing all the necessary in sequel analytic properties. We norm it as follows: $G(X, Y) \sim \gamma_{n+1} \psi_{n+1}(| X - Y |)$ as $| X - Y | \to 0$, where $\gamma_2 = 1/(2\pi)$, $\psi_2(r) = \ln(1/r)$, and $\gamma_n = 1/[(n-2)\sigma_n]$, $\psi_n = r^{2-n}$ for $n \geq 3$. Hence G is positive on R_+^{n+1} and its inner normal derivative $\partial G(X, y)/\partial n(y) \geq 0$ - we denote this derivative $P_c(X, y)$; it is called the *Poisson's c -kernel* for the half-space R_+^{n+1}.

The potential theory for the operator L_c, including existence of the Green's function G, was developed by M. Cranston, E. Fabes, Z. Zhao [2] (see excellent survey by M. Bramanti [1]) for more general potentials. Under the assumptions above the theory with all the needed details was independently developed by B. Ya. Levin and the author [11, 9].

Let $B^+(x_0, t) = \{X \in R_+^{n+1} : | X - x_0 | < t, x_0 \in R^n, x_{n+1} > 0\}$ be an (n+1)-dimensional half-ball in R_+^{n+1}. Further let us suppose that every boundary point $x_0 \in R^n = \partial R_+^{n+1}$ has a neighbourhood $B^+(x_0, t)$ such that $c \in L^{p_0}(B^+(x_0, t))$ with some $p_0 > n + 1$. These assumptions imply the regularity of the half-space for the Dirichlet problem for the operator L_c (F.-U. Maeda [13]) and also continuity of the Poisson c-kernel $P_c(X, y)$ for $X \in R_+^{n+1}$, $y \in R^n$

[1, 9].

We denote $\mathcal{C}$ the class of the potentials c satisfying all the above-mentioned conditions. Let $\mathcal{C}_r$ stands for the subclass of the radial potentials, i.e., $c(X) = c(|X|)$. We will also consider the class $\mathcal{A}$, consisting of the potentials $c \in \mathcal{C}_r$ such that there exists the finite limit $\lim_{r\to\infty} r^2 c(r) = q < \infty$, and moreover, $r^{-1} \mid r^2 c(r) - q \mid \in L(1,\infty)$, i.e., the last quantity is integrable over the interval $[1,\infty)$. This class was introduced in [11]. At last let

$$\Gamma_\alpha(\xi) = \{X = (x, x_{n+1}) \in R_+^{n+1} : \mid x - \xi \mid < \alpha x_{n+1}\},\ \alpha > 0$$

be a non-tangential cone in R_+^{n+1} with vertex $\xi \in R^n$.

Theorem 1 *Let $c \in \mathcal{C}$ and f be a locally-summable function on R^n such that the integral*

$$\int_{R^n} \mid f(y) \mid P_c(X,y)\, dy \tag{3}$$

converges at some point $X \in R_+^{n+1}$ (and consequently everywhere in R_+^{n+1}). Then the Dirichlet problem (1) has the solution

$$u_0(X) = \int_{R^n} f(y) P_c(X,y)\, dy. \tag{4}$$

For this solution there exists the limit $\lim_{\Gamma_\alpha(\xi)\ni X\to\xi} u_0(X) = f(\xi)$ at almost every point $\xi \in R^n$. If f is a continuous function, then u_0 tends to the non-tangential boundary values $f(\xi)$ at every point $\xi \in R^n$. Evidently $u_0 \geq 0$ in the case $f \geq 0$.

It should be noticed that the integral (3) may be convergent for an essentially extensive function class then (2), because the Green's function $G(X,Y)$ of the Schrödinger operator L_c can decrease at infinity faster then the harmonic Green's function $g(X,Y)$. For example let be $c(X) = q = const$ in R_+^3. Straightforward calculation shows that

$$P_q(X,y) = \frac{1 + q^{1/2} \mid X - y \mid}{2\pi \mid X - y \mid^3}\, x_3 \exp\{-q^{1/2} \mid X - y \mid\},\ X = (x_1, x_2, x_3),$$

– i.e., the Poisson q -kernel decreases exponentially as $|X - y| \to \infty$.

Under the conditions of this theorem the solution to the problem (1) is not unique in general. Even if the integral (3) is convergent, the function f can grow very quickly at infinity (on a small set) and consequently the function u_0 (the integral (4)) can have an arbitrary fast global growth in the closed half-space $\overline{R_+^{n+1}}$ [8]. Hence to guarantee the uniqueness of the solution to the problem (1) one has to restrict *apriori* growth of the solution at infinity. For this estimate of growth it is naturally to make use of the limit growth occuring in the above-mentioned Phragmen-Lindelöf theorem [11]. To give an exact formulation we need some more notions.

We let $S^+(x_0,t) = \{X \in R_+^{n+1} : | X - x_0 |= t,\ x_0 \in R^+,\ x_{n+1} > 0\}$ denote a half-sphere in R_+^{n+1}, and Δ^* be a Laplace-Beltrami operator (spherical part of the Laplacian) on the unit sphere. It is known (see, e.g., [17, p.41]) that the eigenvalue problem

$$\begin{cases} \Delta^*\varphi(\theta) + \lambda\varphi(\theta) = 0, & \theta \in S^+(0,1), \\ \varphi(\theta) = 0, & \theta \in \partial S^+(0,1) \end{cases}$$

has the eigenvalues $\lambda_k = k(k+n-1)$ with the corresponding multiplicities $\nu_k = (n+k-2)!/[(n-1)!(k-1)!], k = 1,2,\ldots$; note that $\nu_1 = 1$. Let $\varphi_{k\nu}(\theta), 1 \le \nu \le \nu_k$, stand for the corresponding eigenfunctions. We norm the eigenfunctions in $L^2(S^+(0,1))$, moreover, $\varphi_1 \equiv \varphi_{11} > 0$. Straightforward calculation gives $\varphi_1(X) = (2(n+1)/\sigma_{n+1})^{1/2}x_{n+1}$, if $| X |= 1$. But $\varphi_{k\nu}$ are the spherical harmonics "odd" with respect to x_{n+1}, namely, $\varphi_{k\nu}(x, -x_{n+1}) = -\varphi_{k\nu}(x, x_{n+1})$. Hence well-known estimates (see, e.g., [14, p. 14]) imply the inequalities

$$| \varphi_{k\nu}(\theta) |\le Qk^{(n-1)/2}, \quad | \partial\varphi_{k\nu}(\theta)/\partial n(\theta) |\le Qk^{(n+1)/2},$$

where symbols Q denote different constants depending on n only. Consequently the following inequalities are valid:

$$\begin{gathered} | \varphi_{k\nu}(\theta) | \le Qk^{(n+1)/2}\varphi_1(\theta), \\ | \sum_{\nu=1}^{\nu_k} \varphi_{k\nu}(\theta)\varphi_{k\nu}(\psi) | \le Qk^{2n-2}, \\ | \sum_{\nu=1}^{\nu_k} \varphi_{k\nu}(\theta)\partial\varphi_{k\nu}(\psi)/\partial n(\psi) | \le Qk^{2n-1}. \end{gathered} \tag{5}$$

Let $q(r) \ge 0$ be a locally summable function on the ray $0 < r < \infty$. We denote W_k and V_k respectively, the main solution and the dominant one (i.e., non-increasing and increasing solutions as $t \to +\infty$ [3, ch. 11]) of the equation

$$y''(r) + nr^{-1}y'(r) - \{\lambda_k r^{-2} + q(r)\}y(r) = 0,\ 0 < r < \infty, \tag{6}$$

normed under the condition $W_k(1) = V_k(1) = 1$. We denote the wronskian of these solutions by $\omega_k, k = 1,2,\ldots$. We will omit the index $k = 1$, i.e., $v = v_1, \varphi = \varphi_1, \ldots$.

Theorem 2 *Let $c \in \mathcal{C}$ and q be any measurable radial minorant of c: $0 \le q(| x |) \le c(x)$, $x \in R_+^{n+1}$. If f is a function such that the integral (3) converges, then the problem (1) has the unique solution in the class of functions satisfying the condition*

$$\liminf_{r\to\infty} V^{-1}(r) \int_{S^+(0,1)} | u(X) | \varphi(\theta)\, d\sigma(\theta) = 0,$$

- this solution is given by the Poisson c -integral (4).

Note that if $c = 0$, then the theorem 2 implies the theorem 2.1 [18] and its generalization to any dimensions $n = 2, 3, \ldots$.

More efficient conditions for convergence of the integrals (3) - (4) can be given if some additional information is available about the potential c. For example, let c be a radial potential of the class $\mathcal{B}$ [11], i.e., $\lim_{t\to\infty} t^2 c(t) = \infty$ and solutions of the equation (6) have JWKB-asymptotic as $r \to \infty$ [5]. Then the convergence of the integral

$$\int_1^\infty \mid f(\rho) \mid \rho^{n/2-1}(c(\rho))^{1/4} \exp\{-\int_1^\rho c^{1/2}(t)dt\}\, d\rho$$

is a sufficient condition for the integral (3) to exist.

In the case $c \in \mathcal{A}$ these results can be essentially improved and strengthened. If c is a radial potential, it is known the following expansion for the Green's function G [11, 9]:

$$G(X,Y) = \sum_{k=1}^{\infty} \omega_k^{-1} V_k(\min\{r,\rho\}) W_k(\max\{r,\rho\}) \{\sum_{\nu=1}^{\nu_k} \varphi_{k\nu}(\theta)\varphi_{k\nu}(\psi)\},\ r \neq \rho. \tag{7}$$

This series converges uniformly if either $r \le \gamma\rho$, or $\rho \le \gamma r$, $\gamma = const$, $0 < \gamma < 1$. In the case $c = 0$ this expansion coinsides with well-known result by J. Lelong-Ferrand [12]. The expansion (7) can be rewritten in the terms of the Gegenbauer polynomials. The formula (7) gives reason to introduce the kernels

$$G^{\{p\}}(X,Y) = G(X,Y) - \sum_{k=1}^{p} \omega_k^{-1} V_k(r) W_k(\rho) \{\sum_{\nu=1}^{\nu_k} \varphi_{k\nu}(\theta)\varphi_{k\nu}(\psi)\},$$
$$p = 1, 2, \ldots,\ and\ G^{\{0\}} = G.$$

Let $\hat{f}(t) = \sup\{\mid f(y) \mid : \mid y \mid \le t\}$ be the best radial nondecreasing majorant of f. Consider an integer-valued function $p(t) = 1 + [\ln(t^\varepsilon \hat{f}^{1+\varepsilon}(t))]$, where ε is any positive number and $[\cdot]$ denotes the entire part of a number. If f has a finite order λ, i.e., $\mid f(y) \mid \le K_\varepsilon \mid y \mid^{\lambda+\varepsilon}$, where $\varepsilon > 0$ is arbitrary small, we introduce the quantity

$$p_\lambda = \begin{cases} (1/2)(1 - n + \{(2\lambda + n - 1)^2 - 4q\}^{1/2})^+, & if \quad (2\lambda + n - 1)^2 > 4q, \\ 0, & if \quad (2\lambda + n - 1)^2 \le 4q. \end{cases} \tag{8}$$

Recall, that $c \in \mathcal{A}$, $q = \lim_{r\to\infty} r^2 c(r) < \infty$ and as usual, $a^+ = \max\{0; a\}$. Note that $p_\lambda = [\lambda]$ in the case $q = 0$.

For any $N = 1, 2, \ldots$, we introduce the class U_N of functions u such that

$$\liminf_{r\to\infty} V_N^{-1}(r) \int_{S^+(0,1)} \mid u(X) \mid \varphi_N(\theta) d\sigma(\theta) = 0. \tag{9}$$

Here φ_N denotes an arbitrary eigenfunction corresponding to the eigenvalue λ_N.

Theorem 3 *Suppose $c \in \mathcal{A}$ and $f(y)$ be a locally-summable real function on R^n, having an arbitrary growth as $\mid y \mid \to \infty$. Then the problem (1) has the solution*

$$u(X) = \int_{R^n} f(y) \frac{\partial G^{\{p(|y|)\}}(X,y)}{\partial n(y)} \, dy. \tag{10}$$

Moreover $\lim_{\Gamma_\alpha(\xi) \ni X \to \xi} u(X) = f(\xi)$ for e.a. $\xi \in R^n$. If f is a continuous function, then the last relation is valid for all $\xi \in R^n$.

If the function f has a finite order λ, the solution u has the same order.

If a continuous function f has a finite order λ, then the general solution to the problem (1) in the class U_N has the form

$$\int_{R^n} f(y) \frac{\partial G^{\{p_\lambda\}}(X,y)}{\partial n(y)} \, dy + P(X).$$

Here P is an arbitrary "c -harmonic polynomial" of degree not greater than $N-1$, i.e.,

$$P(X) = \sum_{k=1}^{N-1} V_k(r) \{ \sum_{\nu=1}^{\nu_k} \alpha_{k\nu} \varphi_{k\nu}(\theta) \}.$$

In the case $c = 0$ this statement implies the theorems 4.1–4.1' [18] and the results of [3].

We prove the theorems 1–3 in the section 3; the section 2 is devoted to some estimates of the kernels $G^{\{p\}}$ needed in the sequel. These estimates are similar to known ones in the case $c = 0$ [16]. In the case of a whole space, not a half-space the analogous estimates are in [9].

Remark at last that all results of the paper take place for any cone in R^{n+1} with a sufficiently smooth boundary.

Statements of the results of this article were published without proofs in [10].

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2 Estimates of the kernels $G^{\{p\}}$

Let be $c \in \mathcal{A}$; hence $\lim_{r \to \infty} r^2 c(r) = q < \infty$. Denote

$$\kappa_k = \{(1-n)^2 + 4(q + \lambda_k\}^{1/2} = \{(2k+n-1)^2 + 4q\}^{1/2}$$

and $\mu_k^\pm = (1 - n \pm \kappa_k)/2$. It is known [5] that in the case under consideration the solutions to the equation (6) have the asymptotics

$$V_k(r) \sim b_1 r^{\mu_k^+},\ W_k \sim b_2 r^{\mu_k^-},\ as\ r \to \infty,$$

where $b_{1,2}$ are some positive constants and it is possible to differentiate these relations.

Lemma 1 *Let $c \in \mathcal{A}$. For every $p = 0, 1, \ldots,$ and for any fixed $\gamma \in (0,1)$ the following inequalities are valid with the value $s \in (0,1)$ to be specified in the proof:*

$$| G^{\{p\}}(X,Y) | \le Qp^{2n-1}(1-s)^{-2n}\varphi(\theta)\varphi(\psi)V_{p+1}(r)W_{p+1}(\rho) \tag{11}$$

as $r \le \gamma\rho$.

The factor $Q = Q(n)$ here and below depends only on the dimension n.

If $\rho \le \gamma r, 1 < r = | X | < \infty$, then

$$| G^{\{p\}}(X,Y) | \le Qp^{2n-1} r^{\mu_p^+} \rho^{1-n}. \tag{12}$$

Further, in the case $r \le \gamma\rho$, $\gamma \in (0,1)$

$$| \frac{\partial G^{\{p\}}(X,y)}{\partial n(y)} | \le Qp^{2n}(1-s)^{-2n-2}\varphi(\theta)\frac{\partial \varphi(\psi)}{\partial n(\psi)} | y |^{-1} V_{p+1}(r)W_{p+1}(\rho). \tag{13}$$

At last, if $1 < \rho \le \gamma r$, then

$$| \frac{\partial G^{\{p\}}(X,y)}{\partial n(y)} | \le Qp^{2n-1} r^{\mu_p^+} \rho^{-n}. \tag{14}$$

Proof. Let be $r \le \gamma\rho$ with any fixed $0 < \gamma < 1$. It is easy to see that $\omega_k \ge \kappa_k, k = 1, 2, \ldots$. We will use the known inequalities [9]

$$| \omega_k^{-1} V_k(r) W_k(\rho) | \le | \omega_{p+1}^{-1} V_{p+1}(r) W_{p+1}(\rho) |,\ k \ge p+1,$$

and

$$| \omega_k^{-1} V_k(r) W_k(\rho) | \le \kappa_k^{-1} r^{\mu_k^+} \rho^{\mu_k^-}.$$

Hence the definition of $G^{\{p\}}$ and the inequalities (5) imply the estimates

$$| G^{\{p\}}(X,Y) | \le Q \sum_{k=p+1}^{\infty} \kappa_k^{-1} r^{\mu_k^+} \rho^{\mu_k^-} \kappa^{2n} \varphi(\theta)\varphi(\psi) \le$$

$$\le Q\varphi(\theta)\varphi(\psi) r^{\mu_{p+1}^+} \rho^{\mu_{p+1}^-} \sum_{k=p+1}^{\infty} \kappa^{2n-1} r^{\mu_k^+ - \mu_{p+1}^+} \rho^{\mu_k^- - \mu_{p+1}^-} \le$$

$$\le V_{p+1}(r)W_{p+1}(\rho)\varphi(\theta)\varphi(\psi) \sum_{k=p+1}^{\infty} k^{2n-1} \gamma^{\mu_k^+ - \mu_{p+1}^+}.$$

Moreover the relation $\mu_k^+ - \mu_{p+1}^+ = (\kappa_k - \kappa_{p+1})/2 \geq R(k-p-1)$ is valid for $p+1 \leq k < \infty$, where $R = (k+p+n)\{(2k+n-1)^2+4q\}^{-1/2}$. In the case $0 \leq p < \infty$ and $k \geq p+1$ one can check that $R \geq Q = Q(q,n)$. Consequently $\mu_k^+ - \mu_{p+1}^+ \geq Q(k-p-1)$ and

$$| G^{\{p\}} | \leq$$

$$QV_{p+1}(r)W_{p+1}(\rho)\varphi(\theta)\varphi(\psi) \sum_{k=p+1}^{\infty} k^{2n-1}\gamma^{Q(k-p-1)} =$$

$$= QV_{p+1}(r)W_{p+1}(\rho)\varphi(\theta)\varphi(\psi) \sum_{k=0}^{\infty} s^k (k+p+1)^{2n-1},$$

where $s = \gamma^Q < 1$. Making use of the following elementary inequality

$$\sum_{k=0}^{\infty} s^k (k+p+1)^{2n-1} \leq Q(p+1)^{2n-1}(1-s)^{-2n},$$

we get the estimate (11):

$$| G^{\{p\}}(X,Y) | \leq Qp^{2n-1}(1-s)^{-2n} V_{p+1}(r)W_{p+1}(\rho)\varphi(\theta)\varphi(\psi).$$

Now let be $1 < \rho \leq \gamma r$, $0 < \gamma < 1$. Then we have the inequality

$$| G^{\{p\}}(X,Y) | \leq G(X,Y) + Q\varphi(\theta)\varphi(\psi) \sum_{k=1}^{p} k^{2n-1} r^{\mu_k^+} \rho^{\mu_k^-} \leq$$

$$\leq g(X,Y) + Qp^{2n-1} r^{\mu_p^+} \sum_{k=1}^{p} \rho^{\mu_k^-},$$

where $\rho > 1$, $\mu_k^- \leq -n-k-1$ and g stands for the harmonic Green's function. Now the known relation

$$g(X,Y) \leq \gamma_{n+1} \, | X - Y |^{1-n} = o(1) \, as \, \, | X | \to \infty,$$

implies the inequality $\sum_{k=1}^{p} \rho^{\mu_k^-} < \sum_{k=1}^{\infty} \rho^{\mu_k^-} < Q\rho^{1-n}$ and the estimate (12) is proven.

The inequalities (13) and (14) follow immediately from (11) and (12) respectively. The lemma is proven completely.

3 Proofs of the theorems

Proof of the theorem 1. The kernel $P_c(X,y)$, i.e., the inner normal derivative of the Green's function G satisfies the equation $L_c u(X) = 0$ as

$X \in \Omega, y \in \partial\Omega$ - such functions are called c *-harmonic* functions. So the absolutely convergent integral (4) is a solution to the boundary problem (1). We will investigate its boundary behavior.

The situation under consideration has one essential distinction to the case $c(X) = 0$. Namely, the kernel P_c fails generally to be an approximate identity. It is known that the half-space R_+^{n+1} is a regular domain with respect to the Dirichlet problem for the operator L_c (see, e.g., [13, 9]). Hence the integral $\int_{R^n} P_c(X,y)\,dy$ is a c -harmonic function having boundary values 1. But the known relation $G(X,Y) \leq g(X,Y)$ implies the inequality

$$P_c(X,y) \leq P_0(X,y) = P(|\, x-y\,|,\, x_{n+1}), \tag{15}$$

thus

$$\int_{R^n} P_c(X,y)\,dy \leq \int_{R^n} P(|\, x-y\,|, x_{n+1})\,dy = 1,$$

and this is strickt inequality unless $c = 0$ almost everywhere. Consequently 1 is not a c -harmonic function if $c \neq 0$.

This obstacle can be overcomed due to the fact, that every boundary point of the half-space is c -regular under our suppositions.

Let $\xi \in R^n$ be a Lebesgue's point of the function f, so that, in particular, $|\, f(\xi)\,| < \infty$. Let us fix any $\epsilon > 0$ and split the difference

$$\begin{aligned} u(X) - f(\xi) = \int_{|y-\xi|<\epsilon} [f(y) - f(\xi)]\, P_c(X,y)\,dy \,+ \\ +f(\xi)\{\int_{|y-\xi|<\epsilon} P_c(X,y)\,dy - 1\} + \\ + \int_{|y-\xi|\geq\epsilon} f(y) P_c(X,y)\,dy \;\equiv\; I_1 + I_2 + I_3. \end{aligned}$$

Due to the inequality (15) and known inequalities for the harmonic functions [19, ch. 3] we have the estimate

$$\begin{aligned} \sup_{X\in\Gamma_\alpha(\xi)} |\, I_1\,| \leq \\ \leq \sup_{X\in\Gamma_\alpha(\xi)} \int_{|y-\xi|<\epsilon} |\, f(y) - f(\xi)\,|\; P_c(|\,x-y\,|,\, x_{n+1})\,dy \leq \\ \leq A_\alpha\,(Mf_\xi)(\xi), \end{aligned}$$

where $f_\xi(y) = f(y) - f(\xi)$, Mf is the maximal function:

$$(Mf)(x) = \sup_{r>0} nr^{-n}\sigma_n^{-1} \int_{|y-x|<r} |\, f(y)\,|\,dy,$$

and the constant A_α depends only on α and n. Now the equality $\lim_{\Gamma_\alpha(\xi)\ni X\to\xi} I_1 = 0$ follows from the analogous property of the usual harmonic functions.

The integral $I_4 = \int_{|y-\xi|<\epsilon} P_c(X,y)\,dy$ is a solution of the problem (1) with boundary data 1 as $| y-\xi |< \epsilon$ and 0 as $| y-\epsilon |\geq \epsilon$; hence there exists the limit $\lim_{\Gamma_\alpha(\xi)\ni X\to\xi} I_4 = 1$ due to the known estimate $0 < const < P_c(X,y) \leq P(X,y)$ [2], which is uniform in $Xandy$. It is also known [9] that the constant in this inequality is tending to zero, if the domain is shrinking to a point. But $| f(\xi) |< \infty$, thus $\lim_{\Gamma_\alpha(\xi)\ni X\to\xi} I_2 = 0$.

Now let us consider the summand I_3. If $| y-\xi |\geq \epsilon$ and $\Gamma_\alpha(\xi) \ni X \to \xi$, then the Poisson kernel $P(| x-y |, x_{n+1})$ tends uniformly to zero. The same is valid for the kernel $P_c(X,y)$, as follows from the inequality (15). Using the convergence of the integral (3) and the theorem from [2] on estimation of the ratio $P_c(X_1,\, y)/P_c(X_2,\, y)$, we apply the theorem on the dominated convergence to conclude that $\lim_{\Gamma_\alpha(\xi)\ni X\to\xi} I_3 = 0$. The theorem 1 is proved.

Further we need the Phragmen-Lindelöf principle for the operator L_c in a half-space [11]. Let u be a subfunction of the Schrödinger operator L_c, i.e., upper-semicontinuous function $u : R_+^{n+1} \to [-\infty, \infty)$, which locally satisfies the generalized mean-value inequality

$$u(X) \leq \int_{|X-Y|=t} u(Y)\, \frac{\partial G_t(X,Y)}{\partial n(Y)}\, d\sigma(Y),$$

where $G_t(X,Y)$ is the Green's function of L_c in a ball $| X-Y |< t$, vanishing at its boundary $| X-Y |= t$ - about the theory of these so-called generalized subharmonic functions see, e.g., [13, 11, 9] and the references cited there.

The theorem 2 is an immediate corollary of the following particular case of the theorem 1 [11]:

if u is a generalized subharmonic function in R_+^{n+1} such that

$$\lim_{x_{n+1}\to 0} u(X) \leq A = const$$

for every $x \in R^n$, and

$$\liminf_{r\to\infty} V^{-1}(r) \int_{S^+(0,1)} u^+(r,\theta)\, \varphi(\theta)\, d\sigma(\theta) = 0,$$

where V and φ are as above, then $u(X) \leq A^+$ everywhere in R_+^{n+1}.

Proof of the theorem 3. The kernel $\partial G^{\{p(|y|)\}}(X,y)/\partial n(y)$ is a c -harmonic function of $X \in R_+^{n+1}$ for every fixed $y \in R^n$, hence if the integral (10) converges, it is the c -harmonic function too. To prove the convergence of this integral we fix $X \in R_+^{n+1}$ and estimate the remainder

$$I_5 = \int_{e|X|<|y|} f(y)\, \frac{\partial G^{\{p(|y|)\}}(X,y)}{\partial n(y)}\, dy.$$

The inequality (13) with $\gamma = 1/e$ implies the estimate

$$\left| \frac{\partial G^{\{p(|y|)\}}(X,y)}{\partial n(y)} \right| \leq$$

$$\leq Q(p(\mid y \mid))^{2n} V_{p(|y|)+1}(r) W_{p(|y|)+1}(\rho) \leq$$

$$\leq Q(p(\rho))^{2n} r^{\mu^+_{p(\rho+1}-1} \rho^{\mu^-_{p(\rho)+1}} \leq Q(p(\rho))^{2n} \rho^n \exp\{\mu^+_{p(\rho)+1}\}.$$

Therefore the following inequality is valid

$$\mid I_5 \mid \leq \int_{er<|y|} \hat{f}(\mid y \mid)\, Q p^{2n}(\rho)\, \rho^n \exp\{\mu^+_{p(\rho)+1}\}\, dy =$$

$$= Q \int_{er}^{\infty} \hat{f}(t)\, t^{-1} p^{2n}(\rho) \exp\{-\mu^+_{p(\rho)+1}\}\, dt.$$

But $\mu^+_{p+1} \geq p+1$, so that we can choose r as large as we need, and $\rho \geq er$, such that $2n \ln p(\rho) \leq p(\rho)/2$. Consequently the last integral is dominated by the convergent integral $Q \int^{\infty} t^{-1-\epsilon}\, dt, \epsilon > 0$, if $p(t) \geq 2 \ln\{t^{\epsilon} \hat{f}^{1+\epsilon}(t)\}$. Obviously, if f is a bounded function, then the theorem 1 is applicable and we can let $p = 0$. If f has a finite order $\lambda > 0$, and p_λ has the form (8), then the integrand can be dominated by the quantity

$$Q p^{2n} r^{\mu^+_{p+1}-1} \rho^{\lambda+\epsilon+\mu^-_{p+1}} = O(\rho^{-1-\epsilon_1})\ as\ \rho \to \infty.$$

Now we study a boundary behavior of the integral (10). Fix a point $\xi \in R^n$ from the Lebesgue set of the function f and a point $X = (x, x_{n+1}) \in \Gamma_\alpha(\xi)$. We represent the function u as $u(X) = I_6 + I_7$, where

$$I_6 = \int_{|y|<2|X|} f(y)\, \frac{\partial G^{\{p(|y|)\}}(X,y)}{\partial n(y)}\, dy$$

and $I_7 = u - I_6$. The definition of the kernels $G^{\{p\}}$ implies the equality

$$I_6 = \int_{|y|<2|X|} f(y)\, \frac{\partial G(X,y)}{\partial n(y)}\, dy -$$

$$- \sum_{k=1}^{k(2|y|)} \int_{t_k \leq |y| \leq 2|X|} f(y)\, \omega_k^{-1}\, V_k(\mid X \mid)\, W_k(\mid y \mid)\, \varphi_k(\theta)\, \frac{\partial \varphi_k(\psi)}{\partial n(\psi)}\, dy,$$

where t_k are jump points of the function $p(t)$.

The following relations

$$\lim_{X \to \xi} \int_{t_k \leq |y| \leq 2|X|} f(y)\, V_k(\mid X \mid)\, W_k(\mid y \mid)\, \varphi_k(\theta)\, \frac{\partial \varphi_k(\psi)}{\partial n(\psi)}\, dy = 0$$

are obviously fulfilled for every of the last integrals, $1 \leq k \leq k(2 \mid y \mid)$. As for the first integral, it was proved in the theorem 1, that it tends to $f(\xi)$, as $X \to \xi$.

The proven estimate of the integral I_5 implies the uniform convergence of the integral I_7 on any compact set in $\overline{R^{n+1}_+}$, and consequently the relation $\lim_{\Gamma_\alpha(\xi) \ni X \to \xi} I_7 = 0$ follows.

Now we estimate the growth of the solution $u(X)$ for a function f having a finite order λ. In this case $p = p_\lambda = const$ and

$$| u(X) | \le | \int_{|y|\le 2|X|} | + | \int_{2|X|<|y|} | \le$$

$$\le \int_{|y|\le 2|X|} \hat{f}(\rho) \{ P(| x-y |, x_{n+1}) + Q \sum_{k=1}^{p} V_k(r) W_k(\rho) \rho^{-1} \} dy +$$

$$+ Q \int_{2|X|<|y|} r^{\mu_{p+1}^{+}} | y |^{\mu_{p+1}^{-}} \rho^{\lambda+\epsilon-1} dy.$$

Making use of the equality $\int_{R^n} P(| x-y |, x_{n+1}) dy \equiv 1$ for the harmonic Poisson kernel P, we have

$$| u(X) | \le Q | X |^{\lambda+\epsilon} + Q \sum_{k=1}^{p} V_k(r) \int_0^{2r} \rho^{\lambda+\epsilon+n-2+\mu_k^-} d\rho +$$

$$+ Q r^{\mu_{p+1}^{+}} \int_{2r}^{\infty} \rho^{\lambda+\epsilon+n-2+\mu_{p+1}^{-}} d\rho.$$

The first k integrals converge as $\rho \to 0$ due to the definition of the index $p = p_\lambda$, and the last integral converges as $\rho \to \infty$ (because the inequality $\mu_k^- \ge \mu_p^-$ is valid as $k \le p_\lambda$). The calculation of the integrals above implies the required estimate of the solution.

At last we consider the solution to the problem (1) under the condition (9). In our case ($c \in \mathcal{A}$, $p = p_\lambda = const$) the difference $v(X)$ of any two solutions is continuous in $\overline{R_+^{n+1}}$ and $v |_{R^n} = 0$. Hence v can be extended to a c -harmonic function in the whole space R^{n+1} as an odd function: $v(x, -x_{n+1}) = -v(x, x_{n+1})$, $x_{n+1} \ge 0$. Extended function v satisfirs the condition (9) everywhere in R^{n+1}, whereas

$$v(x, 0) = 0, \ \forall x \in R^n. \tag{16}$$

Now we introduce the functions

$$y_k(r) = \int_{S^+(0,1)} v(r, \theta) \varphi_k(\theta) d\sigma(\theta), \ k = 1, 2, \ldots,$$

where $\varphi_k = \varphi_{k\nu}$ being any spherical harmonics corresponding to the eigenvalue λ_k, odd with respect to x_{n+1}. Making use of the equality (16) and self-adjointness of the Laplace-Beltrami operator Δ^*, one can check directly (by differentiating under the integral sign) that the functions $y_k, k = 1, 2, \ldots$, satisfy the equation (6) with $q = c$. This equation has a general solution $y(r) = AV_k(r) + BW_k(r), A, B$ are some constants. Since $y_k(r)$ is bounded as $r \to 0$, we have $B = 0$, and $y_k(r) = A_k V_k(r), A_k = const$. Now the condition (9) implies $A_k = 0$ as $k \ge N$, so that $y_k(r) = 0, k = N, N+1, \ldots$.

To this end we expand the function $v(X)$, $X \in R^{n+1}$, in the Fourier- Laplace series on the spherical harmonics. This expansion contains only odd harmonics

$\varphi_{k\nu}$, because $v(X)$ is an odd function with respect to x_{n+1}. Thus

$$v(X) = \sum_{k=1}^{N-1} \{\sum_{\nu=1}^{\nu_k} \alpha_{k\nu}\, \varphi_{k\nu}(\theta)\}\, V_k(r).$$

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