A knowledge representation based on the Belnap's four-valued logic

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1 Introduction

Much recent work in artificial intelligence has required formal techniques for working with incomplete knowledge. The new approach clearly necessitated that a distinction be made between logic and the action (i.e. two components of intelligence: the epistemological and the heuristic; cf. [MCC 69]). However, certain new inference procedures associated with the new approach (for example the familiar example of default logic) have, like classical logic, been based on an underlying consistent ontology. (Cf. [GIN 87a]: "It is precisely this 'absence of information to the contrary' that makes the inference non-monotonic...")

Philosophers were the first to call attention to the new situation. (Cf. [RES 79]: "...we can live with the prospect of inconsistency — not only in epistemology, but even in ontology"; and further: "The invocation of *ontology* here is significant... Thus if $t(P)$ were to be construed not as 'P is true in itself', but as 'X (I, you, he, etc.) *maintains* that P is true', then we can clearly and unproblematically have both $t(P)$ and $t(\neg P)$...")

In 1975 N. Belnap proposed a computer-oriented ontology for contradictory knowledge (cf. [BEL 75, BEL 76]). He expressed the idea of representing possibly contradictory knowledge, in the form of epistemic states constituting in entirety an approximate lattice. On the other hand, Dana Scott introduced the notion of data type as an approximate lattice with an effective basis (cf. [SCO 71]).

In our view, Belnap's remarkable attempt to consider knowledge states as a data type has not been completed. (This is the case, for example, because the question concerning an effective basis did not even arise there.) However, although his papers appeared almost twenty years ago, we believe that their value has still not been fully recognized.

We propose a representation of knowledge (possibly with contradictions) in a propositional language, and we show how such knowledge can be maintained and how it should be transformed on receipt of new information. In this transformation, the key role is played by Scott's continuity rather than by consistency.

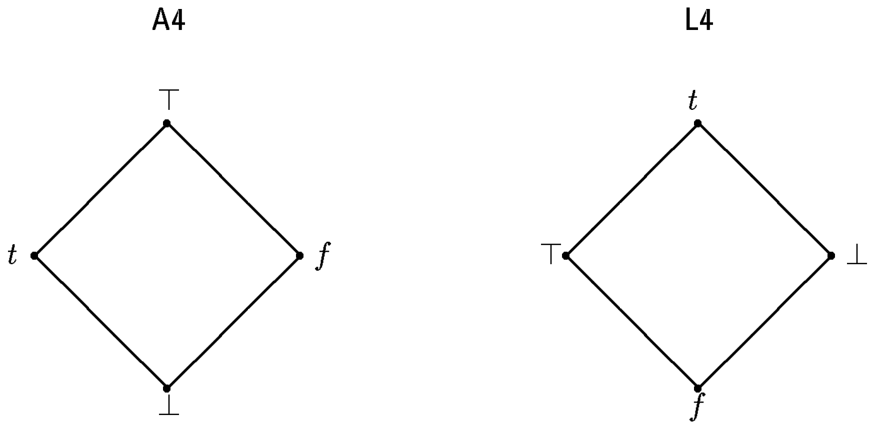
In the present paper, we are concerned only with problems of maintaining knowledge information from an ontological point of view; we leave the logical issues for later consideration. Our starting point is the concept of distinguishing feasible knowledge actually accessible to a computer from complete (i.e., ideal or thorough) knowledge and we maintain that availability of the former implies availability of the latter. We were led to consider two notions: finite epistemic state and generalized epistemic state, and to study the relationship between them. This point of view will be discussed more extensively in Section 4 on Scott Principle.

2 Preliminaries

As in [BEL 75] we suppose that a computer is able to accept information (possibly with contradiction) in the form of either formulas $\mathcal{A}$,$\mathcal{B}$,... of propositional language (containing a countable set Var $(= \{p_0, p_1, \ldots\})$ of variables and the connectives: $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation)), or in the form of conditions (or rules) $\mathcal{A} \rightarrow \mathcal{B}$. This information is represented in the computer in the form of an *epistemic state*, i.e. a nonempty set of *setups*.

A *setup* is a mapping from Var into the set $\Im$ of truth values: t (true), f (false), $\top$ (both), $\perp$ (none), which are ordered by the relation $\sqsubseteq$ in the form of the Belnap's approximate lattice $\mathsf{A4}$ $(= \langle \Im; \sqcap, \sqcup \rangle)$ (see Figure 1).

Figure 1: Lattices A4 and L4.



Two setups s_1 and s_2 are considered to be

$$s_0 \leq s_1 \text{ if and only if } s_0(\pi) \sqsubseteq s_1(\pi) \text{ for every } \pi \in \mathsf{Var}.$$

This relation is a partial order.

Proposition 1 *The partially ordered set of all the setups is a complete lattice* (AS) *with the operations:*

$$\sqcap\{s_i \mid i \in I\}(\pi) = \sqcap\{s_i(\pi) \mid i \in I\},$$

$$\sqcup\{s_i \mid i \in I\}(\pi) = \sqcup\{s_i(\pi) \mid i \in I\}$$

for every $\pi \in \mathsf{Var}$.

Proof. Let us consider $\mathsf{A4}$ as a partially ordered set. Then the Cartesian power $\mathsf{A4}^{\mathsf{Var}}$ is a complete lattice with the component-wise determination of the operations.

Following [BEL 75, BEL 76], every setup can be extended to the set of all the formulas using the "logical lattice" $\mathsf{L4}$ $(= \langle \Im; \wedge, \vee \rangle)$ (see Figure 1) which is a lattice according to $\wedge$ and $\vee$, and the $\neg$ is given as

$$\neg t = f, \neg f = t, \neg \top = \top, \neg \bot = \bot.$$

Having 'combined' $\mathsf{A4}$ and $\mathsf{L4}$ we get the so-called simplest nontrivial bilattice in the sense of [GIN 87b].

Because the logical connectives are monotonic with respect to the order on $\mathsf{A4}$, we prove the following

Proposition 2 *Let $\mathcal{A}$ be a formula, s_0 and s_1 be setups and $s_0 \leq s_1$. Then $s_0(\mathcal{A}) \sqsubseteq s_1(\mathcal{A})$.*

Remark. Consider $s(\mathcal{A})$ as a result of the application of the formula $\mathcal{A}$ to the setup s. Then one can say that the formulas are monotonic functions on AS.

By an epistemic value of a formula $\mathcal{A}$ in a state E, (symbolically $E(\mathcal{A})$), we shall mean the element $\sqcap\{s(\mathcal{A}) \,|\, s \in E\}$ of the lattice $\mathsf{A4}$ (cf. [BEL 75]). Two states E_0 and E_1 are called *formula indistinguishable* iff $E_0(\mathcal{A}) = E_1(\mathcal{A})$ for every formula $\mathcal{A}$.

We denote $V(s) \stackrel{\text{def}}{=} \{\pi \,|\, \pi \in \mathsf{Var}, s(\pi) \neq \bot\}$ for any setup s. A setup s is called *finite* if $V(s)$ is a finite set. A finite collection of finite setups is called a *finite epistemic state*. The finite epistemic states are properly accessible to a computer. In contrast to them infinite epistemic states represent elements of ideal (thorough) knowledge which admits a finite approximation as we shall show later. We denote $V(E) \stackrel{\text{def}}{=} \bigcup\{V(s) \,|\, s \in E\}$ for any state E. Thus a state E is finite if and only if $V(E)$ is a finite set.

Let V be a set of variables, i.e. $V \subseteq \mathsf{Var}$.

By the *down-restriction* of a setup s over V we call the setup $s^{V\bot}$:

$$s^{V\bot}(\pi) \stackrel{\text{def}}{=} \begin{cases} s(\pi) & \text{for } \pi \in V \\ \bot & \text{for } \pi \notin V \end{cases}$$

By the *up-restriction* of a setup s over V we call the setup $s^{V\top}$:

$$s^{V\top} \stackrel{\text{def}}{=} \begin{cases} s(\pi) & \text{for } \pi \in V \\ \top & \text{for } \pi \notin V \end{cases}$$

By the down-restriction (up-restriction) of an epistemic state E over V we call the state

$$E^{V\perp} \stackrel{\text{def}}{=} \left\{ s^{V\perp} | s \in E \right\} \quad \left(E^{V\top} \stackrel{\text{def}}{=} \left\{ s^{V\top} | s \in E \right\} \right).$$

In case it doesn't matter up- or down-restriction is in question we shall write E^V.

It is obvious that in the case when V is a finite set the down-restriction $E^{V\perp}$ is a finite state and the up-restriction $E^{V\top}$ is a finite set of setups. Let us notice an important property: $E^{V(E)\perp} = E$. (We shall use it without mention.)

We describe now some properties of restrictions.

Proposition 3 *Let V_0,V_1 be two sets of variables and E be an epistemic state. Then the following equalities hold*

$$E^{V_0\perp^{V_1\perp}} = E^{V_0 \bigcap V_1 \perp}, \quad E^{V_0\top^{V_1\top}} = E^{V_0 \bigcap V_1 \top}.$$

Proof. We shall prove the first equality. If $s \in E^{V_0\perp^{V_1\perp}}$ then there is a setup $s_0 \in E$ such that $s = {s_0}^{V_0\perp^{V_1\perp}}$. If $\pi \in V_0 \bigcap V_1$ then $s(\pi) = s_0(\pi)$. Assume that $\pi \notin V_0 \bigcap V_1$. If $\pi \notin V_0$ then ${s_0}^{V\perp}(\pi) = \perp$ and, hence, ${s_0}^{V_0\perp^{V_1\perp}}(\pi) = \perp$. It is so in the case $\pi \notin V_1$ as well. Thus we have $s = {s_0}^{V_0 \bigcap V_1 \perp}$.

Let now $s \in E^{V_0 \bigcap V_1 \perp}$, i.e. $s = s_1^{V_0 \bigcap V_1 \perp}$ for some setup $s_1 \in E$. If $\pi \in V_0 \bigcap V_1$ then $s(\pi) = s_1(\pi) = {s_1}^{V_0\perp}(\pi) = {s_1}^{V_0\perp^{V_1\perp}}(\pi)$. If $\pi \notin V_0$ then $s(\pi) = {s_1}^{V_0\perp}(\pi) = {s_1}^{V_0\perp^{V_1\perp}}(\pi) = \perp$. Case $\pi \notin V_1$ can be proved in the same way and we have $s = {s_1}^{V_0\perp^{V_1\perp}}$.

The second equality is proved similarly.

Proposition 4 *Let V be a set of variables containing all the variables of a formula $\mathcal{A}$. Then $E^V(\mathcal{A}) = E(\mathcal{A})$.*

Proof (by induction on the length of $\mathcal{A}$) is obvious.

Proposition 5 *Let V be a set of variables, s_0 and s_1 be setups. Then the following equalities are correct:*

$$\begin{array}{ll} {s_0}^{V\perp} \sqcup {s_1}^{V\perp} = (s_0 \sqcup s_1)^{V\perp}, & {s_0}^{V\top} \sqcup {s_1}^{V\top} = (s_0 \sqcup s_1)^{V\top}, \\ {s_0}^{V\perp} \sqcap {s_1}^{V\perp} = (s_0 \sqcap s_1)^{V\perp}, & {s_0}^{V\top} \sqcap {s_1}^{V\top} = (s_0 \sqcap s_1)^{V\top} \end{array}$$

Proof. We shall only prove the first equality using the Proposition 1:

$${s_0}^{V\perp} \sqcup {s_1}^{V\perp}(\pi) = s_0(\pi)^{V\perp} \sqcup s_1(\pi)^{V\perp} = \left\{ \begin{array}{cl} s_0(\pi) \sqcup s_1(\pi) & \text{if } \pi \in V \\ \perp \sqcup \perp & \text{if } \pi \notin V \end{array} \right\} =$$

$$= \left\{ \begin{array}{cl} (s_0 \sqcup s_1)(\pi) & \text{if } \pi \in V \\ \perp & \text{if } \pi \notin V \end{array} \right\} = (s_0 \sqcup s_1)^{V\perp}(\pi)$$

3 Finite epistemic states

By $m(E)$ we shall mean the set of the minimal setups in E according to the order on AS. For finite E the set $m(E)$ is nonempty because of the Descending Chain Condition.

Theorem 3.1 *If E is a finite epistemic state then E and $m(E)$ are formula indistinguishable.*

Proof is obvious.

A finite state E is called *minimal*, if $m(E) = E$. Notice that a finite state E is minimal if and only if it is an antichain in AS and that $m(m(E)) = m(E)$. Following N.Belnap we introduce a relation $\leq$ on the set of all the epistemic states:

$$E_0 \leq E_1 \stackrel{\text{def}}{=} (\forall s_1 \in E_1)(\exists s_0 \in E_0)(s_0 \leq s_1).$$

It is obvious that this relation is a partial order on the set of minimal states.

Theorem 3.2 *The collection of the minimal states forms a lattice (AFE) with operations as follows:*

$$E_0 \sqcap E_1 \stackrel{\text{def}}{=} m(E_0 \bigcup E_1)$$

$$E_0 \sqcup E_1 \stackrel{\text{def}}{=} m(\{s_0 \sqcup s_1 \,|\, s_0 \in E_0, s_1 \in E_1\}).$$

But the lattice AFE is not complete.

Proof. The first part of the statement is obvious. To illustrate the incompleteness of AFE consider the infinite set $\{E_i \,|i < \omega\}$, where each E_i consists of the only setup s_i such that $s_i(p_j) = \perp$ if $i \neq j$, and $s_i(p_i) = t$. This set has no upper-bound in AFE.

Basic Lemma 3.3 *Let E_0 be an epistemic state consisting of a finite set of setups and E_1 be any state. Then $E_0 \leq E_1$ if and only if $E_0\,(\mathcal{A}) \sqsubseteq E_1\,(\mathcal{A})$ for each formula $\mathcal{A}$.*

Proof. Assume that $E_0 \leq E_1$ is false. Since E_0 is finite, $E_0 = \{s_1, ..., s_n\}$. There is a setup $s \in E_1$ such that $s_i \leq s$ is false for each i $(1 \leq i \leq n)$. The later means that for every i there exists a variable $\pi_i \in V(s_i)$ such that $s_i(\pi_i) \sqsubseteq s(\pi_i)$ is false. Notice that $s(\pi_i) \neq \top$ for each i. Now we shall define formulas $\mathcal{A}_i$ $(1 \leq i \leq n)$ as follows:

$$\mathcal{A}_i \stackrel{\text{def}}{=} \begin{cases} \pi_i \vee \neg\pi_i & \text{if } s(\pi_i) = \perp \\ \pi_i & \text{if } s(\pi_i) = f \\ \neg\pi_i & \text{if } s(\pi_i) = t \end{cases}$$

Let now $\mathcal{A}$ be the formula $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n$. For it we have both $t \sqsubseteq s_i(\mathcal{A}_i)$ and $s(\mathcal{A}_i) \sqsubseteq f$. Therefore $t \sqsubseteq s_i\,(\mathcal{A})$ for each i, hence, $t \sqsubseteq E_0\,(\mathcal{A})$. On the other hand, $E_1\,(\mathcal{A}) \sqsubseteq s\,(\mathcal{A}) \sqsubseteq f$. Hence, $E_0\,(\mathcal{A}) \sqsubseteq E_1\,(\mathcal{A})$ is false.

At last, $E_0 \leq E_1$ implies $E_0\,(\mathcal{A}) \sqsubseteq E_1\,(\mathcal{A})$ for every $\mathcal{A}$ via the Proposition 2.

Remark. Analyzing the above proof one can see that if $E_0 \leq E_1$ is false, then there exists a formula $\mathcal{A}$, all the variables of which belong to $V(E_0)$, such that $E_0\,(\mathcal{A}) \sqsubseteq E_1\,(\mathcal{A})$ doesn't hold.

Theorem 3.4 *Any two minimal states are formula indistinguishable if and only if they are equal.*

Proof. It follows immediately from the Lemma 3.3.

The condition of finiteness in the Lemma 3.3 and the Theorem 3.4 is essential as it shows the following example. Let $E_1 (= \{s\})$ and $E_0 (= \{s_1, s_2, ...\})$ be states such that $s(\pi) = \perp$ for every $\pi \in \mathsf{Var}$ and $s_i(p_j) = \perp$ if $i \neq j$, and $s_i(p_i) = t$. It is easy to see that $E_0\,(\mathcal{A}) = E_1\,(\mathcal{A}) = \perp$ for any $\mathcal{A}$, although $E_0 \leq E_1$ doesn't hold.

4 Generalized epistemic states

Let us consider the following relation on the set of all the epistemic states:

$$E_0 \preceq E_1 \stackrel{\text{def}}{=} \text{ for every formula } \mathcal{A} \;\; E_0(\mathcal{A}) \sqsubseteq E_1(\mathcal{A}).$$

It is obvious that this relation is a quasi-order and $E_0 \leq E_1$ implies $E_0 \preceq E_1$.

By a *generalized epistemic state* we shall mean an equivalence class with respect to the relation:

$$E_0 \equiv E_1 \stackrel{\text{def}}{=} E_0 \preceq E_1 \text{ and } E_1 \preceq E_0,$$

i.e. a class of formula indistinguishable states. It is that element of ideal knowledge, which was in question above and for which we shall try to find a finite approximation (in the form of finite states). We shall consider the generalized states to be ordered by a relation as follows:

$$\overline{E_0} \leq \overline{E_1} \stackrel{\text{def}}{=} E_0 \preceq E_1.$$

It is evident that this relation puts the collection of the generalized states in partial order .

Theorem 4.1 *The generalized epistemic states form a complete lattice* AGE *according to the order* $\leq$, *in which*

$$\sqcap\left\{\overline{E_i}\,|i \in I\right\} = \overline{\bigcup\{E_i\,|i \in I\}}.$$

Proof. First of all we notice that the class of the states, which are formula indistinguishable from $\{s\}$, where $s(\pi) = \top$ for each $\pi \in \mathsf{Var}$, is the largest one with respect to $\leq$.

Now let the set $\left\{\overline{E_i}\,|i \in I\right\}$ be given. Denote $E \stackrel{\text{def}}{=} \bigcup\{E_i\,|i \in I\}$. It is evident that $E \preceq E_i$. Assume that for a state E' $E' \preceq E_i$ for each $i \in I$. Then we have for an arbitrary formula $\mathcal{A}$:

$$E'(\mathcal{A}) \sqsubseteq \sqcap\{E_i(\mathcal{A})\,|i \in I\} = \sqcap\{\sqcap\{s(\mathcal{A})\,|s \in E_i\}\,|i \in I\} =$$
$$\sqcap\{s(\mathcal{A})\,|s \in E\} = E(\mathcal{A}).$$

Therefore we conclude by virtue of a well-known argument (see [BIR 67]) that AGE is a complete lattice.

Lemma 4.2 *Let E_0, E_1 be epistemic states and V be a set of variables. Then the following hold:*

(1) If E_0 consists of a finite set of setups then $E_0 \preceq E_1$ implies that ${E_0}^V \preceq {E_1}^V$, where ${E_0}^V$ and ${E_1}^V$ either both are up-restrictions or both are down-restrictions;

(2) $E_0 \preceq E_1$ implies ${E_0}^{V\perp} \preceq {E_1}^{V\perp}$;

(3) If V is a finite set then ${E_0}^{V\top} \preceq E_1$ implies $E_0 \preceq {E_1}^{V\top}$.

Proof. (1) by the Lemma 3.3, $E_0 \preceq E_1$ implies $E_0 \leq E_1$. We notice that $s_0 \leq s_1$ always implies ${s_0}^V \leq {s_1}^V$. Hence, again by the Lemma 3.3, ${E_0}^V \preceq {E_1}^V$.

(2) For a while assume that V is a finite set. Then ${E_0}^{V\perp}$, as it was noted above, is a finite epistemic state. Using the fact that $s^{V\perp} \leq s$ and the Lemma 3.3 we conclude that ${E_0}^{V\perp} \preceq E_0 \preceq E_1$. Applying (1) proved above and the Proposition 2 we receive: ${E_0}^{V\perp} = {{E_0}^{V\perp}}^{V\perp} \preceq {E_1}^{V\perp}$.

Now let V be an arbitrary set of variables. *By $V(\mathcal{A})$ we denote the set of variables of the formula $\mathcal{A}$.* Then we have by virtue of the Propositions 2 and 3:

$$
\begin{gathered}
{E_0}^{V\perp}(\mathcal{A}) = {{E_0}^{V\perp}}^{V(\mathcal{A})\perp}(\mathcal{A}) = {{E_0}^{V(\mathcal{A})\perp}}^{V\perp}(\mathcal{A}), \text{ and} \\
{E_1}^{V\perp}(\mathcal{A}) = {{E_1}^{V\perp}}^{V(\mathcal{A})\perp}(\mathcal{A}) = {{E_1}^{V(\mathcal{A})\perp}}^{V\perp}(\mathcal{A}).
\end{gathered}
$$

Since $V(\mathcal{A})$ is a finite set, ${E_0}^{V(\mathcal{A})\perp} \preceq {E_1}^{V(\mathcal{A})\perp}$ follows from the above case. However, ${E_0}^{V(\mathcal{A})\perp}$ is a finite state. Applying (1) we receive that ${{E_0}^{V(\mathcal{A})\perp}}^{V\perp} \preceq {{E_1}^{V(\mathcal{A})\perp}}^{V\perp}$. Hence, by the Proposition 2

$$
{E_0}^{V\perp}(\mathcal{A}) \sqsubseteq {E_1}^{V\perp}(\mathcal{A}).
$$

(3) It is evident that $s \leq s^{V\top}$ and ${s^{V\perp}}^{V\top} = s^{V\top}$ for arbitrary setup s. The former implies $E_0 \leq {E_0}^{V\top}$ and, hence, $E_0 \preceq {E_0}^{V\top}$. The latter gives the equality ${{E_0}^{V\perp}}^{V\top} = {E_0}^{V\top}$. Now assume that ${E_0}^{V\top} \preceq E_1$. Since E_0 is a finite state, then, by (1), ${{E_0}^{V\perp}}^{V\top} \preceq {E_1}^{V\top}$ and, hence, $E_0 \preceq {E_1}^{V\top}$.

Theorem 4.3 *The lattice* AFE *is a sublattice of the lattice* AGE.

Proof. We shall show that the mapping $E \longmapsto \overline{E}$ is an embedding of AFE in AGE. First, by virtue of the Lemma 3.3, this mapping is an embedding of AFE into AGE as a partially ordered sets. Now we shall prove that the image of AFE is a sublattice of AGE. Consider two minimal states E_1 and E_2 and denote $V \stackrel{\text{def}}{=} V(E_1) \bigcup V(E_2)$.

It is evident that V is a finite set and $E_1 = {E_1}^{V\top}, E_2 = {E_2}^{V\top}$.

Let $\overline{E} \stackrel{\text{def}}{=} \overline{E_1} \sqcup \overline{E_2}$. Then $E^{V\perp} \preceq E$ follows from the Lemma 3.3. Since $E_1 \preceq E$ and $E_2 \preceq E$ we have ${E_1}^{V\perp} \preceq E^{V\perp}$ and ${E_2}^{V\perp} \preceq E^{V\perp}$ by virtue of the Lemma 4.2. Now, applying the Theorem 3.1 and the Lemma 3.3, we receive: $E_1 = {E_1}^{V\perp} \leq m\Big(E^{V\perp}\Big)$, and $E_2 = {E_2}^{V\perp} \leq m\Big(E^{V\perp}\Big)$. Hence, $\overline{E_1 \sqcup E_2} \leq \overline{m\Big(E^{V\perp}\Big)} = \overline{E^{V\perp}} \leq \overline{E} \leq \overline{E_1 \sqcup E_2}$

It gives the equality $\overline{E_1} \sqcup \overline{E_2} = \overline{E_1 \sqcup E_2}$. The equality for the greatest lower bound is proved similarly.

5 Scott Principle

Theorem 4.3 being proved, we may digress for the sake of a short discussion.

Our aim is to represent knowledge expressed by means of epistemic states as a data type in the sense of [SCO 71], i.e. to represent it as a complete lattice equipped by an effective basis.

So far we have the lattice AFE, whose elements we want to serve as approximations to any (generalized) epistemic state. To attach a contensive meaning to the notion of approximation among elements of a complete lattice $\mathcal{L}$ we introduce a topological structure (so-called Scott topology) on it. (Consult [SCO 72],[GIE 80] about definition and motivations). Then we define an important relation between the elements of $\mathcal{L}$: $x \ll y$ means x is less or equal (i.e. x is an approach to) each point of some neighborhood of y (see [SCO 71],[SCO 72],[GIE 80]).

By a *basis* of the lattice $\mathcal{L}$ equipped with the Scott topology we shall name an up-semilattice $\mathcal{E}$ such that $x = \sqcup\{\varepsilon \,|\, \varepsilon \in \mathcal{E},\ \varepsilon \ll x\}$ for every element $x \in \mathcal{L}$. Moreover, we want the basis to be effective, i.e. its points allowing representation in a finitary way (for example, by an effective enumeration) and the relation $\ll$ being recursive on it (cf.[SCO 71]).

Further, following [SCO 71] only Scott continuous operations are admitted on a complete lattice being a data type. We feel natural to demand also that the admitted operations must be coordinated with the basis in the sense, that the result of the operation from points of basis belongs to the basis.

This demand — to consider a data type as *a complete lattice with an effective basis admitting only Scott continuous operations coordinated with the basis* — we call Scott Principle. However, Professor D.S.Scott and Professor N.D.Belnap may find this quite strict (cf.[BEL 75, SCO 71, GIE 80]). Notice that our notion of basis is stronger than the one in the book [GIE 80]. However, it seems to be natural in our case. Note also that the sets $\{x \,|\varepsilon \ll x\}$, where ε is from the basis (in our sense) of a complete lattice $\mathcal{L}$, constitute a basis of the topological space $\sum\mathcal{L}$ (cf.[GIE 80]) .

Finally, we remark that the basis in a complete lattice turns it into *continuous* one (cf.[SCO 72, GIE 80]).

6 Lattice AFE as a basis of lattice AGE

Following the Scott Principle, we shall show that the lattice AFE is an effective basis of the lattice AGE. But first here is one more property of down-restriction.

Define $\overline{E}^{V\perp} \stackrel{\text{def}}{=} \overline{E^{V\perp}}$ for any state E and a set V of variables. By virtue of the Lemma 4.2, this definition is well-founded.

Lemma 6.1 *Let V be a finite set of variables. Then the following equality holds:*

$$\sqcup\left\{\overline{E}_i^{\,V\perp} \,|i \in I\right\} = \left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right)^{V\perp}.$$

Proof. According to the Lemma 4.2, for every $i \in I$ holds the correlation $\overline{E}_i^{\,V\perp} \leq \left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right)^{V\perp}$.

Assume that an epistemic state E is such that $\overline{E}_i^{\,V\perp} \leq \overline{E}$ for each $i \in I$, i.e. $E_i^{\,V\perp} \leq E \;\; (i \in I)$. By the Lemma 4.2, we have $E_i \preceq E^{V\top}$ for each $i \in I$. Hence, $\sqcup\left\{\overline{E}_i \,|i \in I\right\} \leq \overline{E}^{V\top}$. Let $\overline{E_1}$ be $\sqcup\left\{\overline{E}_i \,|i \in I\right\}$. Using the Lemma 4.2 again and the equality $s^{V\top^{V\perp}}{=}s^{V\perp}$, we receive: $\overline{E}_1^{\,V\perp} \leq \overline{E}^{V\top^{V\perp}} = \overline{E}^{V\perp} \leq \overline{E}$.

Now we consider the Scott topology on the set of the elements of the lattice AGE.

Lemma 6.2 *Let E be a finite epistemic state. Then the set $\left\{\overline{E}' \,\middle|\, \overline{E} \leq \overline{E}'\right\}$ is open.*

Proof. Let the directed set $\left\{\overline{E}_i \,|i \in I\right\}$ be such that $\overline{E} \leq \sqcup\left\{\overline{E}_i \,|i \in I\right\}$. From the Lemmas 4.2 and 6.1 we have
$\overline{E} = \overline{E}^{V(E)\perp} \leq \left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right)^{V(E)\perp} = \sqcup\left\{\overline{E}_i^{\,V(E)\perp} \,|i \in I\right\}$.

Now we notice that the set $\left\{\overline{E}_i^{\,V(E)\perp} \,|i \in I\right\}$ remains directed and, moreover, it is finite since $V(E)$ is a finite set. Therefore, there exists i such that $\sqcup\left\{\overline{E}_i^{\,V(E)\perp} \,|i \in I\right\} = \overline{E}_i^{\,V(E)\perp}$. However, $E_i^{V(E)\perp} \leq E_i$ and, hence, $\overline{E} \leq \overline{E}_i$.

Lemma 6.3 *Let E be a finite state and E' be any one. Then $\overline{E} \ll \overline{E}'$ if and only if $\overline{E} \leq \overline{E}'$.*

Proof. It is sufficient to prove only that $\overline{E} \leq \overline{E}'$ implies $\overline{E} \ll \overline{E}'$, because the implication in the reverse order was noted in [SCO 71]. However, an immediate application of the Lemma 6.2 completes the proof.

Theorem 6.4 *The lattice* AFE *is an effective basis of the lattice* AGE.

Proof. Since AFE is a lattice, it remains to verify the second condition in the definition of basis. And even, by virtue of the Lemma 6.3, we need only to show that every element in AGE is the least upper bound of the lower elements from AFE.

So, let an arbitrary epistemic state be given. We consider the following finite sets of variables:

$$V_i \stackrel{\text{def}}{=} \{p_0, p_1, .., p_i\} \quad (i < \omega).$$

Each down-restriction $E^{V_i\perp}$ is a finite state. Moreover, it is obvious that $E^{V_i\perp} \leq E$. We denote $\overline{E'} \stackrel{\text{def}}{=} \sqcup\left\{\overline{E}^{V_i\perp} \,|i < \omega\right\}$. It is evident that $\overline{E'} \leq \overline{E}$. Now we shall show that $E \preceq E'$, i.e. $\overline{E} \leq \overline{E'}$.

Really, for any formula $\mathcal{A}$ there is a set V_i containing all the variables belonging to $\mathcal{A}$. Applying the Proposition 2 we receive $E\left(\mathcal{A}\right) = E^{V_i\perp}\left(\mathcal{A}\right) \sqsubseteq E'\left(\mathcal{A}\right)$.

Finally, we notice that the basis is effective, because the relation $\leq$ is a decidable one on the set of the minimal states.

Corollary 6.5 *The lattice* AGE *is continuous.*

Proof. It follows immediately from the Theorem 6.4

7 Some operations on AGE

We shall consider three forms of operations.The first two correspond to the situation, when a computer accepts a message about truth or falsity of the statement expressed by a formula $\mathcal{A}$. We denote them $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively and define: $\mathcal{A}^+\left(\overline{E}\right) \stackrel{\text{def}}{=} \overline{E} \sqcup \overline{\text{Tset}\,(\mathcal{A})}$ and $\mathcal{A}^-\left(\overline{E}\right) \stackrel{\text{def}}{=} \overline{E} \sqcup \overline{\text{Fset}\,(\mathcal{A})}$, where

$$\text{Tset}\mathcal{A} \stackrel{\text{def}}{=} \{s \,|s \in \mathsf{AS}, t \sqsubseteq s\,(\mathcal{A})\,, V(s) \subseteq V\,(\mathcal{A})\},$$

$$\text{Fset}\mathcal{A} \stackrel{\text{def}}{=} \{s \,|s \in \mathsf{AS}, f \sqsubseteq s\,(\mathcal{A})\,, V(s) \subseteq V\,(\mathcal{A})\}$$

and $V\,(\mathcal{A})$ is the set of variables belonging to $\mathcal{A}$ (cf. [BEL 75]).

Warning. Everywhere below $\overline{D}\,(\mathcal{A})$ means $D\,(\mathcal{A})$ for any class $\overline{D}$.

The following properties are verified easily.

$$\mathcal{A}^-\left(\overline{E}\right) = \neg\mathcal{A}^+\left(\overline{E}\right)$$

— the change of the epistemic state upon encountering the message about the falsity of $\mathcal{A}$ is equivalent to the change of the state upon accepting the message that $\neg\mathcal{A}$ is true;

$$\mathcal{A}^+\left(\mathcal{A}^+\left(\overline{E}\right)\right) = \mathcal{A}^+\left(\overline{E}\right)$$

— the repeated message about the truth of a formula does not change the epistemic state;

$$\mathcal{A}^+\left(\mathcal{B}^+\left(\overline{E}\right)\right) = \mathcal{B}^+\left(\mathcal{A}^+\left(\overline{E}\right)\right)$$

— order of entering messages does not influence on the epistemic state obtained;

$$t \sqsubseteq \mathcal{A}^+\left(\overline{E}\right)(\mathcal{A})\,, \quad f \sqsubseteq \mathcal{A}^-\left(\overline{E}\right)(\mathcal{A})\,;$$

if $\overline{E}\,(\mathcal{A}) = \bot$ and $\mathcal{A}$ is self-consistent, i.e. $(\exists s \in \mathsf{AS}\,)(s\,(\mathcal{A}) = t)$,
then $\mathcal{A}^+\left(\overline{E}\right)(\mathcal{A}) = t$.

Let us remark that the application of the operations above to the classes defined by minimal states, by virtue of the Theorems 3.1 and 4.3, leads us to the equalities:

$$\mathcal{A}^+(E) \stackrel{\text{def}}{=} E \sqcup m(\text{Tset}\,(\mathcal{A})) \;\text{ and }\; \mathcal{A}^-(E) \stackrel{\text{def}}{=} E \sqcup m(\text{Fset}\,(\mathcal{A})).$$

So, we have proved the

Theorem 7.1 *Operations $\mathcal{A}^+$ and $\mathcal{A}^-$ are coordinated with the basis* AFE.

Now we shall establish the

Theorem 7.2 *Operations $\mathcal{A}^+$ and $\mathcal{A}^-$ are continuous in Scott topology on* AGE.

Proof. It is sufficient to justify the continuity of the former operation only, i.e. to prove the equality:

$$\mathcal{A}^+\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right) = \sqcup\left\{\mathcal{A}^+\left(\overline{E}_i\right) \,|i \in I\right\}$$

for every directed set $\left\{\overline{E}_i \,|i \in I\right\}$ (cf.[SCO 72, GIE 80]). However, this equality is obvious.

At last, the third kind of operation is associated with the situation, when a computer accepts a condition $\mathcal{A} \rightarrow \mathcal{B}$. In this case a computer must analyze each setup of a current state and, having discovered the formula $\mathcal{A}$ being true in the setup considered, change the setup in order to provide truth of $\mathcal{B}$. We shall denote this operation on AGE by $[\mathcal{A} \rightarrow \mathcal{B}]$ and define it setup-wisely in two steps (cf.[BEL 75]).

At first, we shall define an operation $[\mathcal{A} \rightarrow \mathcal{B}]'$ on the set of all the epistemic states as follows:

$$[\mathcal{A} \rightarrow \mathcal{B}]'(E) \stackrel{\text{def}}{=} \bigcup\left\{[\mathcal{A} \rightarrow \mathcal{B}]'(s) \,|s \in E\right\}$$

for every E, where

$$[\mathcal{A} \rightarrow \mathcal{B}]'(s) \stackrel{\text{def}}{=} \begin{cases} \{s \sqcup s' \,|s' \in m(\text{Tset}\,(\mathcal{B}))\} & \text{if } t \sqsubseteq s\,(\mathcal{A}) \\ \{s\} & \text{otherwise} \end{cases}$$

for every $s \in \mathsf{AS}$.

Lemma 7.3 *For every setups s_0 and s_1, if $s_0 \leq s_1$ then $[\mathcal{A} \rightarrow \mathcal{B}]'(s_0) \leq [\mathcal{A} \rightarrow \mathcal{B}]'(s_1)$.*

Proof. If $t \sqsubseteq s_0\,(\mathcal{A})$ then, according to the Proposition 2, $t \sqsubseteq s_1\,(\mathcal{A})$. Hence, we have:

$$\begin{aligned}[\mathcal{A} \rightarrow \mathcal{B}]'(s_0) = \{s_0 \sqcup s' \,|s' \in m(\text{Tset}\,(\mathcal{B}))\} \leq \\ \leq \{s_1 \sqcup s' \,|s' \in m(\text{Tset}\,(\mathcal{B}))\} = [\mathcal{A} \rightarrow \mathcal{B}]'(s_1).\end{aligned}$$

If $t \sqsubseteq s_0\,(\mathcal{A})$ is false then $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s_0) = s_0 \leq s_1 \leq [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s_1)$.

It is worth-while to remark that $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E)$ is a finite state if E is such.

Lemma 7.4 *For every finite epistemic states E_0 and E_1, if $E_0 \preceq E_1$ then $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_0) \preceq [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_1)$.*

Proof. First of all, let us notice that by virtue of the Lemma 3.3 the condition $E_0 \leq E_1$ can be substituted for the condition $E_0 \preceq E_1$.

For an arbitrary $s_1 \in [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_1)$ there exists $s_1{}' \in E_1$ such that $s_1 \in [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s_1{}')$. Then, the condition $E_0 \leq E_1$, implies that there is $s_0{}' \in E_0$ such that $s_0{}' \leq s_1{}'$. By the Lemma 7.3, we receive: $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s_0{}') \leq [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s_1{}')$. Hence, there exists $s_0 \in [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s_0{}')$ such that $s_0 \leq s_1$. Since s_0 is a setup from $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_0)$ and s_1 is an arbitrary setup from $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_1)$, we have $[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_0) \leq [\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E_1)$. Now we apply the Lemma 3.3 again to substitute $\preceq$ for $\leq$.

Lemma 7.5 *Let a set V of propositional variables contain all the variables of formulas $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$. Then the following equality holds:*

$$\left([\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E)\right)^{V\perp}(\mathcal{C}) = \left([\mathcal{A}\!\rightarrow\!\mathcal{B}]'\left(E^{V\perp}\right)\right)(\mathcal{C})\,.$$

Proof. Applying the Proposition 5, we receive:

$$\left([\mathcal{A}\!\rightarrow\!\mathcal{B}]'(E)\right)^{V\perp}(\mathcal{C}) = \left(\cup\left\{[\mathcal{A}\!\rightarrow\!\mathcal{B}]'(s)^{V\perp}\,|s \in E\right\}\right)(\mathcal{C}) =$$

$$= \cup\Big\{\{s \sqcup s_1\,|s_1 \in m(\mathrm{Tset}\,(\mathcal{B}))\}^{V\perp}\,|s \in E, t \sqsubseteq s\,(\mathcal{A})\Big\}\cup$$
$$\Big\{s^{V\perp}\,|s \in E, s\,(\mathcal{A}) \sqsubseteq f\Big\}(\mathcal{C}) =$$

$$= \cup\Big\{\Big\{s^{V\perp} \sqcup s_1{}^{V\perp}\,|s_1 \in m(\mathrm{Tset}\,(\mathcal{B}))\Big\}\,|s \in E, t \sqsubseteq s\,(\mathcal{A})\Big\}\cup$$
$$\Big\{s^{V\perp}\,|s \in E, s\,(\mathcal{A}) \sqsubseteq f\Big\}(\mathcal{C}) =$$

$$= \cup\Big\{\Big\{s^{V\perp} \sqcup s_1\,|s_1 \in m(\mathrm{Tset}\,(\mathcal{B}))\Big\}\,\Big|s \in E, t \sqsubseteq s^{V\perp}\,(\mathcal{A})\Big\}\cup$$
$$\Big\{s^{V\perp}\,\Big|s \in E, s^{V\perp}\,(\mathcal{A}) \sqsubseteq f\Big\}(\mathcal{C}) =$$

$$= \cup\Big\{\{s \sqcup s_1\,|s_1 \in m(\mathrm{Tset}\,(\mathcal{B}))\}\,\Big|s \in E^{V\perp}, t \sqsubseteq s\,(\mathcal{A})\Big\}\cup$$

$$\left\{s \,\middle|\, s \in E^{V\perp}, s(\mathcal{A}) \sqsubseteq f\right\}(\mathcal{C}) =$$

$$\left([\mathcal{A}\to\mathcal{B}]'\left(E^{V\perp}\right)\right)(\mathcal{C}).$$

Lemma 7.6 *For every epistemic states E_0 and E_1 and for every formulas $\mathcal{A}$ and $\mathcal{B}$, if $E_0 \preceq E_1$ then $[\mathcal{A}\to\mathcal{B}]'(E_0) \preceq [\mathcal{A}\to\mathcal{B}]'(E_1)$*

Proof. Assume that $E_0 \preceq E_1$ and consider an arbitrary formula $\mathcal{C}$. Denote $V \overset{\text{def}}{=} V(\mathcal{A}) \cup V(\mathcal{B}) \cup V(\mathcal{C})$.

By the Lemma 4.2, we have ${E_0}^{V\perp} \preceq {E_1}^{V\perp}$. Using the Proposition 4 and Lemmas 7.5 and 7.4 we receive:

$$\left([\mathcal{A}\to\mathcal{B}]'(E_0)\right)(\mathcal{C}) = \left([\mathcal{A}\to\mathcal{B}]'(E_0)\right)^{V\perp}(\mathcal{C}) = \left([\mathcal{A}\to\mathcal{B}]'\left({E_0}^{V\perp}\right)\right)(\mathcal{C}) \sqsubseteq$$
$$\left([\mathcal{A}\to\mathcal{B}]'\left({E_1}^{V\perp}\right)\right)(\mathcal{C}) = \left([\mathcal{A}\to\mathcal{B}]'(E_1)\right)^{V\perp}(\mathcal{C}) = \left([\mathcal{A}\to\mathcal{B}]'(E_1)\right)(\mathcal{C}).$$

Summarizing, we can prove the

Theorem 7.7 *The operation $[\mathcal{A}\to\mathcal{B}]$ on* AGE *defined as*

$$[\mathcal{A}\to\mathcal{B}]\left(\overline{E}\right) \overset{\text{def}}{=} \overline{[\mathcal{A}\to\mathcal{B}]'(E)}$$

is correct and coordinated with the basis AFE.

Proof. The first part of the theorem follows from the Lemma 7.6. The second part is a consequence from the remark that $[\mathcal{A}\to\mathcal{B}]'(E)$ is a finite state provided that E is finite and from the Theorem 3.1.

Theorem 7.8 *The operation $[\mathcal{A}\to\mathcal{B}]$ is continuous on* AGE *in Scott topology.*

Proof. By virtue of [GIE 80, SCO 72] it is sufficient to prove the following equality for every directed set $\left\{\overline{E}_i \,|i \in I\right\}$:

$$[\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right) = \sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i\right) |i \in I\right\}$$

i.e. for an arbitrary formula $\mathcal{C}$

$$\left([\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right)\right)(\mathcal{C}) = \left(\sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i\right) |i \in I\right\}\right)(\mathcal{C})$$

Let us fix a formula $\mathcal{C}$ and denote: $V \stackrel{\text{def}}{=} V(\mathcal{A}) \cup V(\mathcal{B}) \cup V(\mathcal{C})$.

Using the Proposition 4 and Lemmas 7.5 and 6.1 we receive:

$$\left([\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right)\right)(\mathcal{C}) = \left([\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right)\right)^{V\perp}(\mathcal{C}) =$$

$$\left([\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}^{V\perp}\right)\right)(\mathcal{C}) = \left([\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i^{\,V\perp} \,|i \in I\right\}\right)\right)(\mathcal{C}),$$

and

$$\left(\sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i\right) \,|i \in I\right\}\right)(\mathcal{C}) = \left(\sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i\right) \,|i \in I\right\}\right)^{V\perp}(\mathcal{C}) =$$

$$\left(\sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i\right)^{V\perp} \,|i \in I\right\}\right)(\mathcal{C}) = \left(\sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i^{\,V\perp}\right) \,|i \in I\right\}\right)(\mathcal{C}).$$

Now we notice that, by virtue of the Lemma 4.2, the set $\left\{\overline{E}_i^{\,V\perp} \,|i \in I\right\}$ remains directed and, by virtue of the Lemma 7.6, $\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i^{\,V\perp}\right) \,|i \in I\right\}$ also is directed. Since the set $\left\{\overline{E}_i^{\,V\perp} \,|i \in I\right\}$ is finite and directed, there has to be $E_{i'}^{\,V\perp} = \sqcup\left\{\overline{E}_i^{\,V\perp} \,|i \in I\right\}$ for some $i' \in I$. It is evident that the equality $[\mathcal{A}\to\mathcal{B}]\left(\overline{E_{i'}}^{V\perp}\right) = \sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i^{\,V\perp}\right) \,|i \in I\right\}$ is correct too. Hence, we conclude: $[\mathcal{A}\to\mathcal{B}]\left(\sqcup\left\{\overline{E}_i \,|i \in I\right\}\right) = \sqcup\left\{[\mathcal{A}\to\mathcal{B}]\left(\overline{E}_i\right) \,|i \in I\right\}$.

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