On the First Degree Entailment of Two 3-Valued Logics

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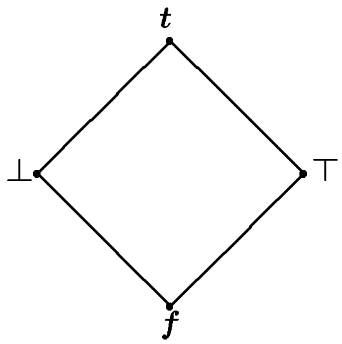
Abstract

We note first that the first degree entailment of Łukasiewicz's 3-valued logic and a 3-valued logic that is extracted of Belnap's 4-valued logic is the same. Then, we give an axiomatization of that entailment as the calculus $E_{fde} + A \wedge \neg A \rightarrow B \vee \neg B$, where E_{fde} is the first degree entailment of Anderson-Belnap's logic E of relevance and necessity.

We consider propositional language $\mathbf{L}$ based on an infinite set Var of propositional variables and connectives: $\wedge$, $\vee$, $\rightarrow$ and $\neg$, denoting arbitrary formulas via A, B,... (probably with subscripts). Following [AB 75], we call the formulas of the form $A \rightarrow B$, where both A and B do not contain any occurrances of $\rightarrow$, *first degree entailments*. Thus, from now on, we will refer to formulas as not containing the connective $\rightarrow$ and to the first degree entailments as simply entailments.

Interest to the logics of the first degree entailment arises in connection with an attempt to present the computer-represented knowledge in the form of domain structure, finding further for the last a suitable informative system in the sense of [Sco 82], using for those purposes that or another calculus of first degree entailment (cf. [Mur 94, Mur 95a, Mur 95b]).

As it was established in [AB 75], the first degree entailment fragment of the logic E (of relevance and necessity) is axiomatized in the form of the calculus E_{fde} and coincides with the first degree entailment fragment of the 4-valued logic that arises in considering 4-valued matrix $\{\boldsymbol{t}, \boldsymbol{f}, \bot, \top\}$ with the single designated value $\boldsymbol{t}$ and connectives defined as follows: $\wedge$ and $\vee$ are *infimum* and *supremum* on the following 4-valued distributive lattice called further B4 (after Nuel Belnap; cf. [Bel 75]), respectively, and other connectives are defined with respect to the following tables:



x	$\neg x$
$\boldsymbol{t}$	$\boldsymbol{f}$
$\boldsymbol{f}$	$\boldsymbol{t}$
$\bot$	$\bot$
$\top$	$\top$

$x \to y$	$\boldsymbol{t}$	$\bot$	$\top$	$\boldsymbol{f}$
$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{f}$	$\boldsymbol{f}$	$\boldsymbol{f}$
$\bot$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{f}$	$\boldsymbol{f}$
$\top$	$\boldsymbol{t}$	$\boldsymbol{f}$	$\boldsymbol{t}$	$\boldsymbol{f}$
$\boldsymbol{f}$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{t}$

Considering B4 as a universal algebra of the signature $< \wedge, \vee, \to, \neg >$, we notice that $< \{\boldsymbol{t}, \bot, \boldsymbol{f}\}, \wedge, \vee, \to, \neg >$ is one of its subalgebras. We denote it via B3. It is easy to see that $\wedge$, $\vee$ and $\neg$ are defined in it as in well-known 3-valued logics of Łukasiewicz and Kleene (cf. [Res 69]; also [Łuk 20] and [Kle 52], respectively). However, the implication $\to$ seems to be new. Recall that implication $\to$ in Łukasiewicz's logic, Ł3, and B3 are defined as it is pictured in the following tables:

Ł3

$x \to y$	$\boldsymbol{t}$	$\bot$	$\boldsymbol{f}$
$\boldsymbol{t}$	$\boldsymbol{t}$	$\bot$	$\boldsymbol{f}$
$\bot$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\bot$
$\boldsymbol{f}$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{t}$

B3

$x \to y$	$\boldsymbol{t}$	$\bot$	$\boldsymbol{f}$
$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{f}$	$\boldsymbol{f}$
$\bot$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{f}$
$\boldsymbol{f}$	$\boldsymbol{t}$	$\boldsymbol{t}$	$\boldsymbol{t}$

Let $\leq$ mean the relation of order on $\mathsf{B3}$, defined as usual:

$$x \leq y \iff x \wedge y = x, \text{ or } x \vee y = y.$$

The following proposition follows immediately from definitions.

Proposition 1 *For every $x, y \in \mathsf{B3}$, the following conditions are equivalent:*

$$\begin{array}{rl} i) & x \leq y; \\ ii) & x \rightarrow y = \boldsymbol{t} \text{ in } \mathsf{Ł3}; \\ iii) & x \rightarrow y = \boldsymbol{t} \text{ in } \mathsf{B3}. \end{array}$$

The Proposition 1 shows us that the first degree entailment of Ł3 and B3 coincide. (That is why we use the "two" in the title.) We present below an axiomatization of this first degree entailment in the form of calculus $E3$. Thus, Lukaciewicz's logic is one source of our interest for that. However, more principal one is that $\{\boldsymbol{t}, \boldsymbol{f}, \perp\}$ along with the imposed order $\sqsubseteq$ defined as

$$x \sqsubseteq y \iff x = \perp$$

constitutes the simplest *epistemic structure* in the sense of [Mur 95a, Mur 95b], that generates a domain which can be considered as a knowledge carrier for the computer-represented knowledge.

Following [AB 75, Bel 75], we call *setup* (or assignment) a mapping s from Var into $\{\boldsymbol{t}, \boldsymbol{f}, \perp\}$, being extended to the set of formulas with respect to the following well-known conditions:

- $s(A \wedge B) = s(A) \wedge s(B)$;
- $s(A \vee B) = s(A) \vee s(B)$;
- $s(\neg A) = \neg s(A)$.

Thus, in virtue of the Proposition 1, an entailment $A \rightarrow B$ belongs to B3 (or is true in B3) if and only if for every setup s, $s(A) \leq s(B)$.

Now let

$$E3 \stackrel{\text{def}}{=} E_{fde} + A \wedge \neg A \rightarrow B \vee \neg B,$$

where the last is thought of as an axiom scheme.

Theorem 1 *For any formulas A,B, the following conditions are equivalent:*

$$\begin{array}{ll} i) & \vdash_{E3} A \to B; \\ ii) & s(A) \le s(B) \textit{ for every setup } s. \end{array}$$

Proof. The implication $(i) \Rightarrow (ii)$ follows from the two facts: 1) B3 is a subalgebra of B4 and, hence, all the entailments derived in E_{fde} are valid on B3; and 2) the entailment $A \wedge \neg A \to B \vee \neg B$ is valid on B3, because for every setup s, $s(A \wedge \neg A) \in \{\boldsymbol{f}, \bot\}$ and $s(B \vee \neg B) \in \{\bot, \boldsymbol{t}\}$, independently of which formulas A and B are.

Now prove the implication $(ii) \Rightarrow (i)$. Assume an entailment $A \to B$ is such that for every setup s, $s(A) \le s(B)$. We have to show that $\vdash_{E3} A \to B$.

First of all, notice that $A \to B$ can be reduced by means of E_{fde} to a *normal form*,

$$A_1 \vee \ldots \vee A_m \to B_1 \wedge \ldots \wedge B_n,$$

where each A_i and B_j is a *primitive conjunction* and a *primitive disjunction*, *i.e.* a conjunction of literals[1] and a disjunction of literals, respectively. A pair of literals p and $\neg p$ is called *contrary*. Thus, our premise is: for every setup s,

$$s(A_1 \vee \ldots \vee A_m) \le s(B_1 \wedge \ldots \wedge B_n). \tag{1}$$

Consider any pair A_i and B_j. Assume the entailment $A_i \to B_j$ is *explicitly tautological*, [AB 75] that is, A_i and B_j have a common literal. Then $\vdash_{E_{fde}}$ $A_i \to B_j$ and, hence, $\vdash_{E3} A_i \to B_j$.

Suppose $A_i \to B_j$ is not explicitly tautological. Then A_i and B_j have no common literal. Rewrite the entailment $A_i \to B_j$ in the form:

$$a_1 \wedge \ldots \wedge a_k \to b_1 \vee \ldots \vee b_l.$$

Thus, we have $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_l\} = \emptyset$. Denote the sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$ via Π and Σ, respectively. Consider the following cases.

Case 1: there is no contrary pair in Π. Define a setup s_1 as follows:

$$s_1(p) = \begin{cases} \boldsymbol{t} & \text{if } p \in \Pi \\ \boldsymbol{f} & \text{if } \neg p \in \Pi \\ \bot & \text{otherwise.} \end{cases}$$

[1] We call a *literal* a propositional variable from Var or its negation. The authors of [AB 75] prefer the term *atom* in the same sense.

Then we see that $s_1(a_1 \wedge \ldots \wedge a_k) = \boldsymbol{t}$ and $s_1(b_1 \vee \ldots \vee b_l) \in \{\bot, \boldsymbol{f}\}$ and, hence, $s_1(A_i) \not\leq s_1(B_j)$.

Case 2: there is no contrary pair in Σ. Define a setup s_2 as follows:

$$s_2(p) = \begin{cases} \boldsymbol{t} & \text{if } \neg p \in \Sigma \\ \boldsymbol{f} & \text{if } p \in \Sigma \\ \bot & \text{otherwise.} \end{cases}$$

Then we find that $s_2(a_1 \wedge \ldots \wedge a_k) \in \{\boldsymbol{t}, \bot\}$ and $s_2(b_1 \vee \ldots \vee b_l) = \boldsymbol{f}$. So we have $s_2(A_i) \not\leq s_2(B_j)$.

However, in both cases, we must have according to our premise (1):

$$s_{1,2}(A_i) \leq s_{1,2}(A_1 \vee \ldots \vee A_m) \leq s_{1,2}(B_1 \wedge \ldots \wedge B_n) \leq s_{1,2}(B_j).$$

A contradiction.

Thus, both Π and Σ have contrary pairs, for instance, $p, \neg p \in \Pi$ and $q, \neg q \in \Sigma$. In that case, $\vdash_{E3} p \wedge \neg p \rightarrow q \vee \neg q$ and, hence, $\vdash_{E3} A_i \rightarrow B_j$.

Now by means of E_{fde}, we conclude that $\vdash_{E3} A \rightarrow B$.

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