ITERATIVE SUBSTRUCTURING METHODS FOR SPECTRAL ELEMENTS: PROBLEMS IN THREE DIMENSIONS BASED ON NUMERICAL QUADRATURE

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May 24, 1994

Abstract. Iterative substructuring methods form an important family of domain decomposition algorithms for elliptic finite element problems. Two preconditioners for p-version finite element methods based on continuous, piecewise Q_p functions are considered for second order elliptic problems in three dimensions; these special methods can also be viewed as spectral element methods. The first iterative method is designed for the Galerkin formulation of the problem. The second applies to linear systems for a discrete model derived by using Gauss-Lobatto-Legendre quadrature. For both methods, it is established that the condition number of the relevant operator grows only in proportion to $(1 + \log p)^2$. These bounds are independent of the number of elements, into which the given region has been divided, their diameters, as well as the jumps in the coefficients of the elliptic equation between elements. Results of numerical computations are also given, which provide upper bounds on the condition numbers as functions of p and which confirms the correctness of our theory.

Key Words. preconditioned conjugate gradient methods, spectral finite element approximation, Gauss-Lobatto-Legendre quadrature, domain decomposition, iterative substructuring

AMS(MOS) subject classifications. 41A10, 65N30, 65N35, 65N55

1. Introduction. It seems most fitting that a paper, in this special issue dedicated to Cornelius Lanczos, should address recent developments of preconditioned conjugate gradient methods for spectral elements for elliptic partial differential equations. The close connection between conjugate gradient and Lanczos' algorithm, see Lanczos [25], is of course well known, and in their well-known book [6], Canuto, Hussaini, Quarteroni, and Zang credit Lanczos with fundamental discoveries that have provided a point of departure for the development of the very active field of spectral methods for differential equations; cf. Lanczos [24].

Over the last decade, preconditioners, in particular those based on domain decomposition, have attracted increasing interest among numerical analysts; seven annual, international symposia have been held on domain decomposition methods for partial differential equations since 1987; cf. [17, 7, 8, 18, 9, 23, 22].

The iterative substructuring methods form one of two main families of domain decomposition methods for elliptic problems. They are based on a decomposition of the given region into nonoverlapping subregions. Data are only exchanged between neighboring local problems through their boundary values. The preconditioners are constructed from solvers for local problems and, in addition, a solver of a coarse problem similar to that used in a multi-grid algorithm. However, the global, coarse

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problem can be quite exotic, cf. Dryja, Smith, and Widlund [12] and Widlund [36]. See also Dryja, Sarkis, and Widlund [11] for a discussion of the use of such coarse problems in the multigrid context. The other major family of domain decomposition methods uses overlapping subregions; see Dryja and Widlund [14, 15] for a discussion of recent work. For further comments and an overview of the literature, we refer to our recent papers [12, 13, 30]. We note that the principal goal of domain decomposition theory is to provide a good upper bound on the condition number κ of the preconditioned operator. It is well known that the number of conjugate gradient iterations grows in proportion to $\sqrt{\kappa}$; see, e.g., Golub and Van Loan [19].

All the powerful domain decomposition methods are thus two-level methods. The second level certainly complicates the design and implementation of the algorithms, but it is by now well established that these methods are quite well suited for the large, relatively loosely coupled computing systems that are becoming increasingly common; cf. Gropp [20]. For experimental evidence that the best of these algorithms work well for large and very ill-conditioned problems; see, e.g., Cai, Gropp, and Keyes [3, 4], Cowsar, Mandel, and Wheeler [10], Gropp and Smith [21], and Smith [35].

The best of the results on iterative substructuring methods show that the condition number of the relevant preconditioned operator grows only polylogarithmically in the number of degrees of freedom of an individual subregion. It is important to note that these bounds are independent of the number of subproblems and that they are independent of jumps in the coefficients across subregion boundaries. Since the number of degrees of freedom per element increases rapidly with p, it is natural to use individual elements as subregions to be assigned to individual processors of a parallel computing system.

The development of iterative methods for higher order and spectral methods poses a special challenge since the stiffness matrices can be much more ill conditioned than for lower order methods. In this paper, we continue our recent work on spectral elements, described in detail in Pavarino and Widlund [30] and also announced in [29],[28]. Just as our previous algorithm, the new methods are close relatives of a method developed by Smith [33],[34] for $h-$version finite elements. In this paper, we use tools and algorithmic ideas developed in our earlier paper to derive and analyze two new, closely related methods.

The first provides a solver for the same Galerkin approximation considered in our previous work. Our second preconditioner is specially designed for the numerical quadrature based, collocation-type finite element methods that have been studied by Bernardi and Maday [2], Fischer and Rønquist [16], Maday, Meiron, Patera, and Rønquist [26], and Rønquist [31, 32]. We also note that a recent master's thesis, Pahl [27], contains a detailed description and discussion of several domain decomposition algorithms for two dimensional spectral problems as well as results of a number of interesting and systematic numerical experiments. As in our previous work, we obtain bounds on the condition numbers of our new methods that are quadratic in $\log p$, where p is the degree of the polynomial space, and independent of the number of elements, their diameters, and jumps in the coefficients across element interfaces.

In this paper, we also report on numerical experiments, which provide strict (and often quite tight) upper bounds for the condition numbers of our methods for any region formed as a union of cubic elements; see Dryja, Smith, and Widlund [12] for a discussion of the underlying theory. These experiments confirm that our theoretical bounds are correct and also show that the condition numbers of the Galerkin finite element method and the method based on numerical quadrature are quite close.

In the near future, we plan to study alternative methods among them algorithms where the performance is enhanced by adding additional (or enlarging existing) subspaces; so far we have only considered algorithms that are based on a direct sum decomposition of the spectral finite element space.

2. The continuous and discrete elliptic problems. We consider a linear, elliptic problem on a bounded domain $\Omega \subset R^3$ formulated variationally as:
Find $u \in V$, such that

$$a(u, v) \;=\; \int_{\Omega} k(x) \nabla u \cdot \nabla v \, dx \;\;=\;\; f(v) \quad \forall \; v \in \; V.$$

Here V is an appropriate subspace of $H^1(\Omega)$ chosen so as to accommodate the boundary conditions of the elliptic problem. The coefficient $k(x) > 0$ can be discontinuous, with very different values for different subregions, but we allow it to vary only moderately within each subregion. We will in fact assume that the region is the union of elements Ω_i that are cubes or images of a reference cube under reasonably smooth mappings; no element can be "too distorted". As in our previous work, almost all our technical work can in fact be carried out on a single reference cube; cf. Dryja, Smith, and Widlund [12] and Pavarino and Widlund [30]. Without decreasing the generality of our results, we will only consider the piecewise constant case of $k(x) = k_i \;\; \forall x \in \Omega_i$.

2.1. Galerkin approximation. The discrete space $V^p \subset V$ is the space of continuous, piecewise Q_p elements, constructed from a tensor product of degree p polynomials of one variable:

$$V^p = \{v \in C^0(\Omega) : v|_{\Omega_i} \in Q_p(\Omega_i), i = 1, 2, \cdots N\}.$$

This gives rise to a conforming Galerkin method; the finite element problem is obtained by restricting u and the test functions to the space V^p. The finite element solution is the projection, orthogonal with respect to the bilinear form $a(\cdot, \cdot)$, of the exact solution onto the finite element space V^p.

There are a number of good choices of basis functions; cf. Babuška, Griebel, and Pitkäranta [1]. In this paper, there is no strong reason to be very specific concerning the choice of basis in discussing the Galerkin method.

The finite element variational problem is turned into a linear system of algebraic equations, $Kx = b$, in the usual way. Here K is the stiffness matrix, and b the load vector. $K^T = K > 0$, a property inherited from the bilinear form $a(\cdot, \cdot)$.

When we need to distinguish between this standard Galerkin method and the one which is quadrature based, we will use the letters G and Q, respectively.

2.2. Quadrature based approximation. When we now turn to the study of our second variational problem, a choice of a specific, nodal basis in appropriate. A variational problem is obtained by using Gauss-Lobatto-Legendre numerical quadrature replacing the original bilinear form $a(\cdot, \cdot)$, element by element. We obtain

$$a_Q(u, v) = (k \nabla u, \nabla v)_Q = \sum_i k_i (\nabla u, \nabla v)_{Q, \Omega_i} \qquad \forall u, v \in V^p(\Omega).$$

Here the new inner product is defined as follows: Let $\Xi = \{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^{p}$ be the set of Gauss-Lobatto-Legendre points on the reference cube $\overline{\Omega}_{ref} = [-1, 1]^3$ and let ρ_i be the weight associated with ξ_i; see Bernardi and Maday [2], pp. 31-34. The contribution of an individual element to the inner product $(u,v)_Q$ is given in terms of a numerical quadrature formula defined on Ω_{ref} by

$$(1) \qquad (u,v)_{Q,\Omega_{ref}} = \sum_{i=0}^{p}\sum_{j=0}^{p}\sum_{k=0}^{p} u(\xi_i,\xi_j,\xi_k)v(\xi_i,\xi_j,\xi_k)\rho_i\rho_j\rho_k.$$

This inner product is uniformly equivalent to the standard L_2-inner product on $Q_p(\Omega_{ref})$; i.e.,

$$(2) \qquad \|u\|^2_{L_2(\Omega_{ref})} \le (u,u)_{Q,\Omega_{ref}} \le 27\|u\|^2_{L_2(\Omega_{ref})} \qquad \forall u \in Q_p(\Omega_{ref}).$$

The numerical quadrature rule is exact if uv is a polynomial in $Q_{2p-1}(\Omega_{ref})$, see Bernardi and Maday [2], p. 75, and it can also be shown that

$$(3) \qquad a(u,u) \le a_Q(u,u) \le 9a(u,u) \qquad \forall u \in V^p(\Omega);$$

see Bernardi and Maday [2], p. 85. The new bilinear form is therefore continuous and coercive. A detailed analysis of this method, including a discussion of existence, uniqueness, and error estimates for an individual element, is given in Bernardi and Maday [2], pp. 85-95. They also establish that, in case of Dirichlet boundary conditions, this numerical quadrature-based method is identical to a collocation method on the grid Ξ.

2.3. Further details on the quadrature based method. A basis for V^p is constructed, locally on the reference element, by introducing tensor products of Lagrange interpolating polynomials $l_i(x)$ defined by $l_i(\xi_j) = \delta_{ij}, 0 \le i,j \le p$. The resulting set of basis functions

$$l_i(x)l_j(y)l_k(z), \qquad 0 \le i,j,k \le p,$$

can be divided into interior (all indices differ from 0 and p), face (one of the indices is 0 or p), edge (two of the indices are 0 and/or p) and vertex basis functions (all indices are 0 and/or p). This provides a nodal basis associated with the Gauss-Lobatto-Legendre nodes Ξ in the sense that, on the reference element,

$$u(x,y,z) = \sum_{\Xi} u(\xi_i,\xi_j,\xi_k)l_i(x)l_j(y)l_k(z).$$

The stiffness matrix K is less sparse than for a lower order finite elements, but still quite well structured; see Figure 4 in Section 5. For the reference element, the matrix element corresponding to the basis functions $l_i(x)l_j(y)l_k(z)$ and $l_q(x)l_r(y)l_s(z)$ is of the form

$$\alpha_{iq}\delta_{jr}\rho_j\delta_{ks}\rho_k + \alpha_{jr}\delta_{iq}\rho_i\delta_{ks}\rho_k + \alpha_{ks}\delta_{iq}\rho_i\delta_{jr}\rho_j,$$

where for $1 \le j,r \le p-1$, the interior points,

$$\alpha_{jr} = \begin{cases} \frac{4}{p(p+1)L_p(\xi_j)L_p(\xi_r)(\xi_r-\xi_j)^2} & \text{if } \; j \ne r \\ \\ \frac{2}{3(1-\xi_j^2)L_p^2(\xi_j)} & \text{if } \; j = r, \end{cases}$$

while

$$\alpha_{j0} = \alpha_{0j} = \frac{4(-1)^p}{p(p+1)L_p(\xi_j)(1+\xi_j)^2} \quad (= \alpha_{p-j,p}), \qquad \text{for} \quad j > 0,$$

$$\alpha_{jp} = \alpha_{pj} = \frac{4}{p(p+1)L_p(\xi_j)(1-\xi_j)^2} \quad (= \alpha_{p-j,0}), \qquad \text{for} \quad j < p,$$

$$\alpha_{00} = \alpha_{pp} = \frac{p(p+1)-1}{6};$$

see Lemma 5.3, Chapter 3, of Bernardi and Maday [2].

3. Iterative Substructuring Methods. The possibly easiest way of describing an iterative substructuring method is in terms of a block-Jacobi/conjugate gradient method; cf. Dryja and Widlund [14],[15]. The stiffness matrix K is preconditioned by a matrix K_J that is the direct sum of diagonal blocks of K. Some or all of these blocks can be replaced by spectrally equivalent, or almost spectrally equivalent, block matrices in an attempt to decrease the cost of the computation. The bilinear form that corresponds to such a block matrix is denoted by $\tilde{a}_i(u,v)$, and it can be viewed as representing an approximate energy functional for a subspace. A local subspace, V_i, often corresponds to a set of adjacent degrees of freedom of a finite element method. However, for a successful method, we must first carry out a suitable change of basis and then select the blocks carefully. This change of basis is intimately connected to the choice of a coarse, global space V_0; cf. Dryja, Smith, and Widlund [12].

The eigenvalues of $K^{-1}K_J$ are given by the stationary values of the Rayleigh quotient

$$\frac{\sum_{i=0}^{N} \tilde{a}_i(u_i,u_i)}{a(u,u)}, \quad u = \sum_{i=0}^{N} u_i, \quad u_i \in V_i. \tag{4}$$

To understand this formula, we just have to realize that the quadratic form corresponding to $i'th$ diagonal block of K_J can be written as $\tilde{a}_i(u_i,u_i)$. The most challenging part of our work is to provide an upper bound of this Rayleigh quotient. Success is tied to estimating the approximate energies $\tilde{a}_i(u_i,u_i)$ uniformly, or almost uniformly, in terms of the strain energy $a(u,u)$. If inexact solvers are used for some or all of the subspaces, upper bounds on $a(u_i,u_i)/\tilde{a}_i(u_i,u_i)$, $u_i \in V_i$, also enters the bound on $\kappa(K_J^{-1}K)$.

In this study, we use the block-Jacobi framework but there is also a more general theory; cf. Dryja and Widlund [13]. Any block-Jacobi method can be viewed as an *additive Schwarz* method based on a direct sum of subspaces. There are also Gauss-Seidel-like, *multiplicative*, as well as *hybrid Schwarz* algorithms; see Dryja, Smith, and Widlund [12] for a general discussion. It follows from this general theory that bounds on the convergence of the multiplicative, block-Gauss-Seidel version of our algorithms can be obtained in a completely routine fashion, once bounds for the additive, block-Jacobi case have been established.

A crucial observation is that, for problems in three dimensions, we cannot obtain a good bound if $V_0 = Q_1$ and, at the same time, all the elements of the other subspaces vanish at the vertices of the elements. For the higher order methods considered here, several standard bases have that property; cf. Babuška, Griebel, and Pitkäranta [1].

Then $u_0 \in V_0$, in the decomposition $u = \sum u_i$ associated with the Rayleigh quotient (4), must be chosen as the interpolant onto V_0. In three dimensions, the norm of this interpolant can be much larger than the norm of u itself and any upper bound for the Rayleigh quotient must, for this coarse space, grow in proportion to p^2; cf. Pavarino and Widlund [30]. This matter is also discussed in detail in Dryja, Smith, and Widlund [12], in the $h-$method context. We note that the vertex basis functions of the nodal basis have small energy. However, by themselves, they do not suffice for building a global subspaces of an iterative method with a rate of convergence independent of the number of elements; see further discussion below.

As in the case of h-finite elements, we consider several important geometric objects: *interiors, faces, edges*, and *vertices.* Our subspaces are directly related to them. The edges and vertices of an element Ω_i are merged creating *wire baskets* W_i, and related *wire basket based* spaces V_0 will be used as our global coarse spaces.

Our methods are based on the following subspaces; we note that the choice of bases for these subspaces do not affect the spectrum of the preconditioner but can make a significant difference as far as the implementation and costs are concerned.

• An interior space for each element: $Q_p \cap H_0^1(\Omega_i)$. This is the same choice as in our previous work.

• A space for each face. These functions vanish on and outside the boundary of $\Omega_{ij} = (\Omega_i \cup \Gamma_{ij} \cup \Omega_j)$. Here Ω_i and Ω_j are two elements that share a common face Γ_{ij}; thus $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. Since it is crucial to have a good recipe for extending the values on the designated face to the interior of the two relevant elements, we use the minimal energy (discrete harmonic) extension making the face spaces orthogonal, in the energy norm, to the interior spaces. These are the same face spaces as in our previous work. If numerical quadrature is used, the functions are made discrete harmonic with respect to the bilinear form $a_Q(\cdot,\cdot)$ instead of $a(\cdot,\cdot)$.

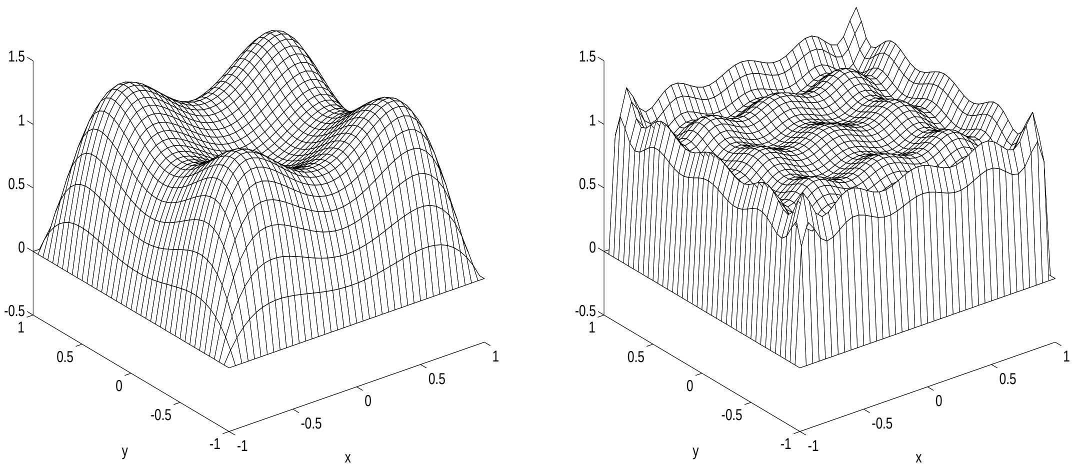
• A coarse, global space, V_0, of piecewise discrete harmonic functions, is associated with the wire baskets of the elements. Its elements are defined solely by their values on the wire baskets. A central issue is how to define the values on the faces. Here we will provide two recipes, which are different from that of our previous paper.

We remark, that the use of interior spaces, such as these, complemented with discrete harmonic subspaces, effectively decouples the problem into two. The matrix representing the restriction of the stiffness matrix K to the subspace of discrete harmonic functions is known as the Schur complement and will be denoted by S.

3.1. Wire basket space for Method G. As we have noted, the design and analysis of any iterative substructuring method involves the decomposition of an arbitrary, given function into subspace components. Typically, we first determine the component of the coarse, global space, then those of the face spaces, and we are finally left with the components of the interior, completely local, spaces.

In our previous paper, [30], the face values of the wire basket component, corresponding to a given function u, are obtained by first subtracting off components related to the vertices, and then expanding the resulting values on each edge of the wire basket in series of special polynomials which are approximate sine-functions. The extension to the faces and interior is then carried out by using polynomials which resemble hyperbolic sine functions and which provide a harmonic extension of the boundary data. This is a procedure based on separation of variables quite similar to the one used when solving a continuous Poisson problem.

FIG. 1. *$\mathcal{F}_{G,j}$ on a face of Ω_{ref}, for $p = 5$, and $p = 10$*



In this paper, we use much simpler recipes. The given value on an edge is multiplied by a special polynomial φ_0, which has been studied in detail in our previous paper. φ_0 is the degree p polynomial which satisfies

$$\min_{\varphi} \|\varphi\|_{L_2(-1,1)}, \qquad \varphi(1) = 1,\ \varphi(-1) = 0.$$

The wire basket space is given in terms of edge and vertex basis functions. As in [30], the elements of the subspace, spanned by these functions, must later be "corrected" so that they also contain certain components from the face spaces.

• The *vertex part* $\tilde{u}_V$ of a function $u \in Q_p(\Omega_{ref})$ is given, preliminarily, as the sum of eight terms. The one associated with the vertex $(1, 1, 1)$ is given by

$$\tilde{u}_V^{(1)}(x, y, z) = u(1, 1, 1)\varphi_0(x)\varphi_0(y)\varphi_0(z). \tag{5}$$

• To construct the *edge part* $\tilde{u}_E$, we first consider the restriction of $u - \tilde{u}_V$ to each of the edges. These values are then extended to the two adjacent faces and then to the interior by using the φ_0 function. One of the contributions is thus given, preliminary, by

$$\tilde{u}_E^{(1)}(x, y, z) = \varphi_0(x)\varphi_0(y)(u(1, 1, z) - \tilde{u}_V(1, 1, z)). \tag{6}$$

The preliminary edge component $\tilde{u}_E$ is given by the sum of twelve such functions.

The proof of the following lemma follows from a direct computation and observing that the L_2−inner product of φ_0 and any polynomial, which vanishes at 1 and −1, equals 0.

LEMMA 3.1. *The edge and vertex functions are discrete harmonic.* As in our previous paper, a preliminary interpolation operator $\tilde{I}_G^W : V^p \to V_0$, is defined by $\tilde{I}_G^W u = \tilde{u}_V + \tilde{u}_E$, where the individual contributions to $\tilde{u}_V$ and $\tilde{u}_E$ have already been described. This operator will not reproduce constants. We therefore construct a function $\mathcal{F}_G = 1 - \tilde{I}_G^W 1$, which vanishes on the wire basket, and which naturally can be split into six discrete harmonic components, each of which vanishes on five of the six faces:

$$\mathcal{F}_G = \sum_{j=1}^{6} \mathcal{F}_{G,j}.$$

See Figure 1 for a plot of $\mathcal{F}_{G,j}$ on a face of Ω_{ref}. It is easy to show that on one of the faces

$$\mathcal{F}_{G,1}(x,y) = (1 - \varphi_0(x) - \varphi_0(-x))(1 - \varphi_0(y) - \varphi_0(-y)).$$

These special functions are extended to the interior of the reference element as discrete harmonic functions. We define the wire basket component as the image of u under a new interpolation operator

$$u_W = I_G^W u = \tilde{I}_G^W u + \sum_{j=1}^{6} \overline{u}_{\partial F_j} \mathcal{F}_{G,j},$$

where $\overline{u}_{\partial F_j} = \frac{1}{8} \int_{\partial F_j} u$. With this definition, the wire basket space

$$V_{0,G} = Range(\tilde{I}_G^W),$$

will contain the constants, since $u_W \equiv 1$ on $\partial\Omega_{ref}$ if $u \equiv 1$ on W. This interpolation operator also defines a basis of the subspace and a change of basis in the entire space.

For the subspace $V_{0,G}$, we use a simple bilinear form defined by,

$$\tilde{a}_{0,G}(u,u) \;=\; C(1+\log p) \sum_i k_i \inf_{c_i} \|u - c_i\|^2_{L_2(W_i)}. \tag{7}$$

In the case when the basis elements are L_2-orthogonal on each edge, we obtain a a coarse problem with only one essentially global degree of freedom, c_i, per element. These values are found by solving a linear system of finite difference type. In addition, a larger linear system with a convenient diagonal matrix is solved to find all the values on the wire basket; see Dryja, Smith, and Widlund [12]. We note that it is easy to modify the algorithm if we have a basis which does not satisfy the orthogonality conditions.

Our main result for Method G is

THEOREM 1. *For the iterative substructuring method, Method G, just introduced by the subspaces and the bilinear form $\tilde{a}_{0,G}(\cdot,\cdot)$, we have*

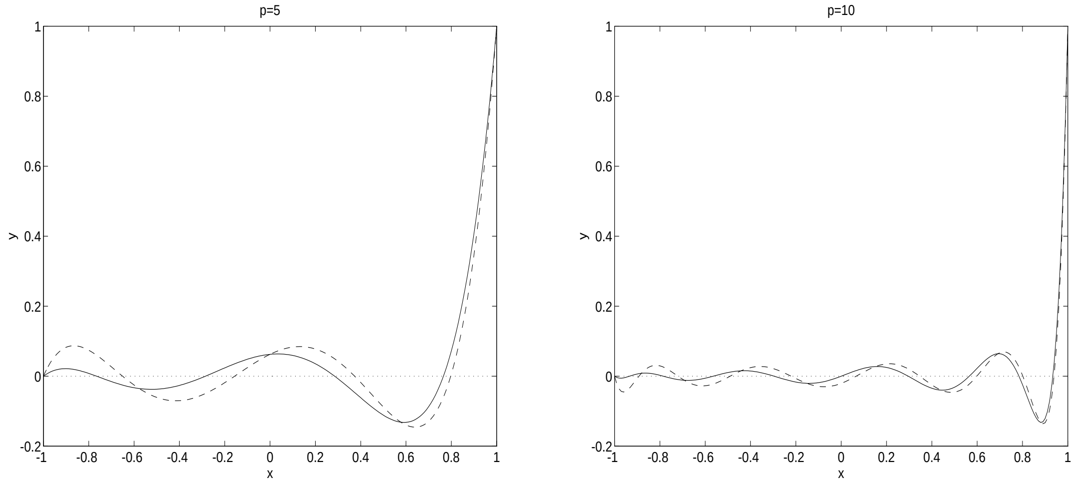
$$\kappa(K_{J,G}^{-1} K) \leq const.(1 + \log p)^2.$$

Here the constant is independent of the number of elements, their diameters, the degree p, and the size of the jumps of the coefficient $k(x)$ across element boundaries.

3.2. Wire basket space for Method Q. In deriving our second method, we first replace φ_0 by l_0, the degree p polynomial that vanishes at all the Gauss-Lobatto-Legendre points ξ_i except at 1. See Figure 2 for a plot comparing l_0 and φ_o. We then proceed very much as in the previous subsection.

It is easy to see that l_0 provides the minimum norm, defined by the one-dimensional quadrature formula underlying formula (1), among all polynomials that satisfy the

FIG. 2. l_0 *(solid line) and* φ_0 *(dashed line) for* $p = 5$ *and* $p = 10$



same boundary conditions as φ_0. We also work with harmonic extensions defined in terms of the bilinear form $a_Q(\cdot,\cdot)$ rather than $a(\cdot,\cdot)$.

A preliminary interpolation operator $\tilde{I}_Q^W : V^p \rightarrow V_0$, is defined by $\tilde{I}_Q^W u = \tilde{u}_V + \tilde{u}_E$, where the different component are defined as in the previous subsection except that l_0 has replaced φ_0.

The follwing result is as easy to prove as Lemma 3.1.

LEMMA 3.2. *The edge and vertex functions are discrete harmonic.*

The operator $\tilde{I}_Q^W$ will not reproduce constants. We therefore construct a function $\mathcal{F}_Q = 1 - \tilde{I}_Q^W 1$, which vanishes on the wire basket, and which naturally can be split into six discrete harmonic components, each of which vanishes on five of the six faces:

$$\mathcal{F}_Q = \sum_{j=1}^{6} \mathcal{F}_{Q,j}.$$

See Figure 3 for a plot of $\mathcal{F}_{Q,j}$ on a face of Ω_{ref}. It is easy to show that on one of the faces

$$\mathcal{F}_{Q,1}(x,y) = (1 - l_0(x) - l_p(x))(1 - l_0(y) - l_p(y)).$$

This function can be expanded into the nodal basis restricted to the face and we find that

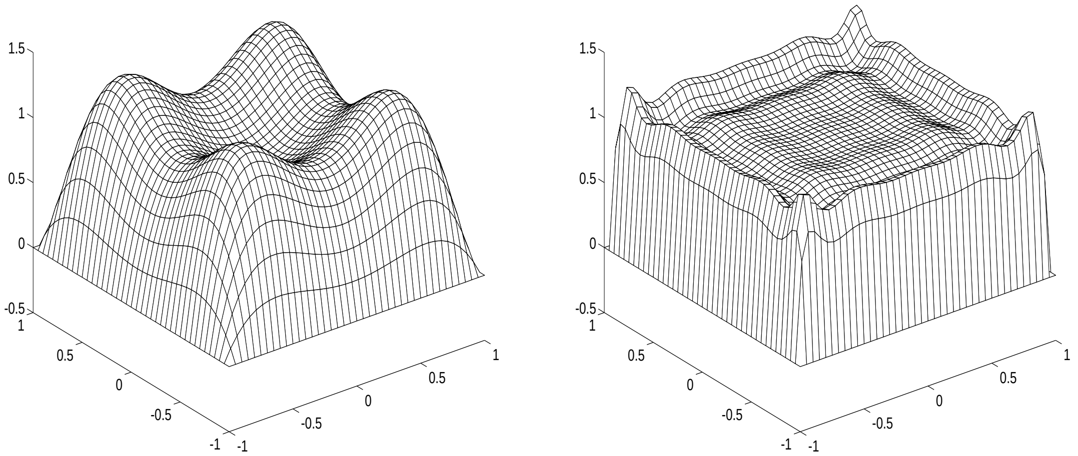
$$\mathcal{F}_{Q,1}(x,y) = \sum_{i,j=1}^{p-1} l_i(x) l_j(y).$$

This means that $\mathcal{F}_{Q,j}$ is represented in the basis of one of the face spaces by the vector $[1, 1, \cdots, 1]$, just as the corresponding function in the h-version case; cf. Dryja, Smith, and Widlund [12]. These special functions are extended to the interior of the reference element as discrete harmonic functions.

We define the wire basket component in terms of a new interpolation operator

$$u_{W,Q} = I_Q^W u = \tilde{I}_Q^W u + \sum_{j=1}^{6} \overline{u}_{\partial F_j} \mathcal{F}_{Q,j},$$

FIG. 3. $\mathcal{F}_{Q,j}$ *on a face of* Ω_{ref}*, for* $p = 5$*, and* $p = 10$



where $\overline{u}_{\partial F_j} = \frac{1}{8}(u,1)_{Q,\partial F_j}$. With this definition, the wire basket space

$$V_{0,Q} = Range(\tilde{I}_Q^W),$$

will contain the constants, since if $u \equiv 1$ on W, then $u_{W,Q} \equiv 1$ on $\partial\Omega_{ref}$. This interpolation operator also defines a basis in the wire basket space and a change of basis in the entire space. In our Algorithm Q, the face and interior spaces are defined as before and the bilinear form for $V_{0,Q}$ is given by

$$\tilde{a}_{0,Q}(u,u) \;=\; C(1+\log p)\sum_i k_i \inf_{c_i} \|u - c_i\|^2_{Q,W_i}. \tag{8}$$

We note that the restriction of any two different basis functions to an edge are orthogonal in sense of the quadrature-based innner product.

Our main result for Method Q is

THEOREM 2. *For the iterative substructuring method, Method Q, just introduced, by the subspaces and the bilinear form* $\tilde{a}_{0,Q}(\cdot,\cdot)$*, we have*

$$\kappa(K_{J,Q}^{-1}K) \leq const.(1+\log p)^2.$$

Here the constant is independent of the number of elements, their diameters, the degree p*, and the size of the jumps of the coefficient* $k(x)$ *across element boundaries.*

4. Proofs of the Theorems. We can use the proofs of the corresponding results in Pavarino and Widlund [30] as models. We confine ourselves to pointing out those details for which changes are necessary.

We first need analogs of Lemma 4.4 in [30]:

LEMMA 4.1. *For all* $p \geq 1$ *and* $1 \leq i \leq p-1$*,*

$$|\varphi_0|^2_{H^1(-1,1)} + \lambda_i^{(p)}\|\varphi_0\|^2_{L_2(-1,1)} \leq Cp^2\,. \tag{9}$$

Here $\lambda_i^{(p)}$ *are eigenvalues defined in Definition 2 of [30].*

Proof. The L_2-part of the inequality follows immediately from Lemma 4.1 of [30] and the bound on the eigenvalues $\lambda_i^{(p)} \leq Cp^4$. The H^1-bound then follows from a polynomial inverse inequality, Lemma 4.3 of [30].

□

The proof of the following lemma is very similar once we note that $\|l_0\|^2_{L_2(-1,1)} \leq C/p^2$. This result follows directly by computing the Gauss-Lobatto-Legendre quadrature approximation of the polynomial, noting that the quadrature weight at the end point is sufficiently small, and using the equivalence of the L_2- and quadrature-norms; see Canuto and Funaro [5] for a similar argument.

LEMMA 4.2. *For all $p \geq 1$ and $1 \leq i \leq p-1$,*

$$|l_0|^2_{H^1(-1,1)} + \lambda_i^{(p)} \|l_0\|^2_{L_2(-1,1)} \leq Cp^2 \, . \tag{10}$$

Here $\lambda_i^{(p)}$ are eigenvalues defined in Definition 2 of [30].

As in the previous work, we can then obtain estimates of the interpolation operators $\tilde{I}^W_G$ and $\tilde{I}^W_Q$. The proofs are very similar to that of Lemma 5.6 of [30].

LEMMA 4.3. *The energy of the $\tilde{I}^W_G$ interpolant of u is bounded by*

$$|\tilde{I}^W_G u|^2_{H^1(\Omega_{ref})} \leq C \|\tilde{I}^W_G u\|^2_{L_2(W)} = C\|u\|^2_{L_2(W)}.$$

LEMMA 4.4. *The energy of the $\tilde{I}^W_Q$ interpolant of u is bounded by*

$$|\tilde{I}^W_Q u|^2_{H^1(\Omega_{ref})} \leq C \|\tilde{I}^W_Q u\|^2_{L_2(W)} = C\|u\|^2_{L_2(W)}.$$

We next turn our attention to two results that are direct counterparts of Lemma 5.7 of [30]. Their proofs require no new ideas.

LEMMA 4.5. *For each face F_k of $\Omega_{ref}, k = 1, \cdots, 6$,*

$$\|u - \tilde{I}^W_G u\|^2_{H^{1/2}_{00}(F_k)} \leq C(1+\log p)^2 \|u\|^2_{H^1(\Omega_{ref})}.$$

LEMMA 4.6. *For each face F_k of $\Omega_{ref}, k = 1, \cdots, 6$,*

$$\|u - \tilde{I}^W_Q u\|^2_{H^{1/2}_{00}(F_k)} \leq C(1+\log p)^2 \|u\|^2_{H^1(\Omega_{ref})}.$$

We also need to obtain bounds for the norms of the new special face functions $\mathcal{F}_{G,j}$ and $\mathcal{F}_{Q,j}$; cf. Lemma 5.8 of [30]. No new ideas are required.

LEMMA 4.7. *Let $\mathcal{F}_{G,j}$ be the special face function for the face F_j, defined before. Then,*

$$\|\mathcal{F}_{G,j}\|^2_{H^{1/2}_{00}(F_j)} \leq C(1+\log p).$$

LEMMA 4.8. *Let $\mathcal{F}_{Q,j}$ be the special face function for the face F_j, defined before. Then,*

$$\|\mathcal{F}_{Q,j}\|^2_{H^{1/2}_{00}(F_j)} \leq C(1+\log p).$$

With these new tools, we can repeat the proof of Theorem 3.1 in [30], almost line by line, and prove Theorem 1. In the proof of Theorem 2, we proceed in the same way, using integrals rather than the quadrature rule, in effect, establishing a result for an additional Galerkin method. Then, we use the equivalence given by formula (3) to complete the proof.

5. Numerical results. As we already have pointed out, it follows from standard theory for iterative substructuring methods that an upper bound for the condition number for the whole problem can be obtained by considering a preconditioner for the Neumann problem on the reference element. We can therefore compute such a bound from the eigenvalues of a matrix pencil defined by the contributions from an individual element to the stiffness matrix and the preconditioner, respectively. Both these matrices are singular and have the same null space; only the space orthogonal to this one dimensional space is relevant in our analysis.

We have carried out a series of MATLAB 4.1 experiments, which closely parallel similar studies in our own work [30], and in Smith [35] for the case of piecewise linear elements. In our tables, $S^{(j)}$ denotes the local Schur complement, obtained from the local stiffness matrix $K^{(j)}$ by eliminating the interior variables. We recall that this is the restriction of the stiffness matrix to the discrete harmonic part of the space V^p. The two local preconditioners $S^{(j)}_{J,G}$ and $S^{(j)}_{J,Q}$ are given by Methods G and Q. The condition number κ is the ratio of the largest eigenvalue λ_{max} and the smallest positive eigenvalue λ_{min}. They are computed using Lanczos' algorithm. Figures 4 and 5 show the sparsity structure of the local stiffness matrix $K^{(j)}$, the local Schur complement $S^{(j)}$ and its preconditioner $S^{(j)}_{J,Q}$ with Method Q. See our previous paper [30] for the sparsity structure of matrices obtained by a variant of Method G.

Method G. In Table 1, we report on the condition numbers $\kappa(S^{(j)^{-1}}_{J,G} S^{(j)})$, the extreme nonzero eigenvalues, and the optimal choice of the scaling constant C of the bilinear form (7), for $p = 2, \cdots, 15$. Since the values of $\kappa(S^{(j)^{-1}}_{J,G} S^{(j)})$ produce a zigzag graph, we have analyzed the even and odd values of p separately. The coefficients α_i of the linear ($n = 1$), quadratic ($n = 2$) or cubic ($n = 3$) least squares approximation $f_n(p) = \sum_{i=0}^{n} \alpha_i (\log p)^i$, are given in Table 2. They clearly indicate a $log^2 p$ growth of κ: the coefficient α_3 of the cubic term is negative for the even case and negligible for the odd case. A $log^2 p$ growth of κ as a function of p is also clear from log-log and semi-log plots.

Method Q. In Table 3, we report on the condition numbers $\kappa(S^{(j)^{-1}}_{Q} S^{(j)})$, the extreme nonzero eigenvalues and the optimal choice of the scaling constant C of the bilinear form (8), for $p = 2, \cdots, 15$. The results of a polylogarithmic least squares approximation are reported in Table 4. As before, they clearly indicate a $log^2 p$ growth of κ, because the coefficient α_3 of the cubic term is negative.

A graphic comparison of the condition numbers of the two methods is given by the log-log plot in Figure 6. It is interesting to note that the condition numbers of Methods G and Q are quite close. This is in spite of the relatively large constants in the formulas (2) and (3). We have also run experiments in which the vertex block of the preconditioner is weighted by the factor $\frac{1}{2}$. This seems reasonable since a vertex belongs to twice as many subdomains as an edge. The resulting condition numbers are only slightly better than those obtained without a weight. Thus for $p = 15, \kappa_Q$ drops to 69.2543.

FIG. 4. *Sparsity pattern of $K^{(j)}$, for $p = 5$, Method Q.*

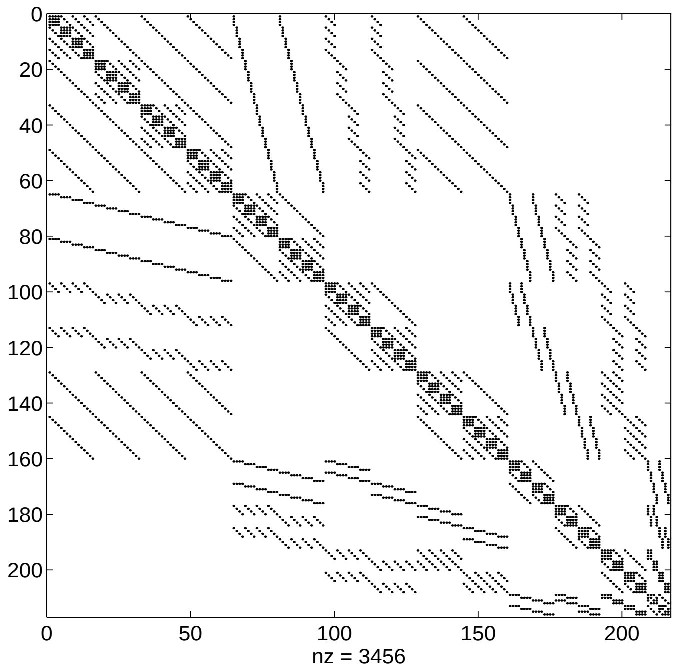


FIG. 5. *Sparsity pattern of $S^{(j)}$ (left) and of $S^{(j)}_{J,Q}$ (right), for $p = 5$, Method Q.*

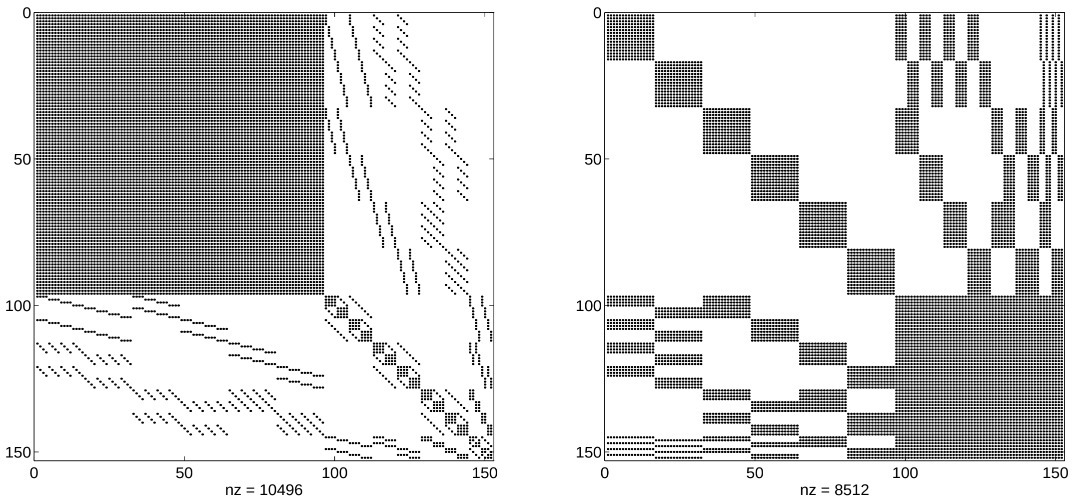


Table 1
Local condition numbers for Method G

p	$\kappa(S_{J,G}^{(j)^{-1}} S^{(j)})$	λ_{max}	λ_{min}	C
2	7.9396	1.5838	0.1995	0.5419
3	14.4306	2.5801	0.1788	0.3626
4	18.6801	2.6576	0.1423	0.3267
5	26.6551	2.8598	0.1073	0.3839
6	30.4150	2.8686	0.0943	0.3636
7	37.8139	2.9944	0.0792	0.4068
8	41.3753	2.9934	0.0723	0.3953
9	47.7892	3.0716	0.0643	0.4255
10	51.0944	3.0797	0.0603	0.4144
11	56.7553	3.1116	0.0548	0.4437
12	59.8160	3.1291	0.0523	0.4314
13	64.8876	3.1639	0.0488	0.4512
14	67.6676	3.1458	0.0465	0.4505
15	72.3285	3.1898	0.0441	0.4619

Table 2
Least squares approximation of $\kappa(S_{J,G}^{(j)^{-1}} S^{(j)})$ *for Method G*

p even	α_0	α_1	α_2	α_3	$\|error\|_{l^\infty}$	$\|error\|_{l^2}$
linear	-19.8334	31.0844			6.2269	11.7167
quadratic	8.5671	-9.3232	12.0423		0.1651	0.3046
cubic	10.5607	-13.9235	15.0669	-0.6007	0.1135	0.1509
p odd	α_0	α_1	α_2	α_3	$\|error\|_{l^\infty}$	$\|error\|_{l^2}$
linear	-29.3205	36.2140			3.9655	7.3055
quadratic	7.5792	-5.8270	10.9816		0.0092	0.0165
cubic	7.5487	-5.7732	10.9520	0.0052	0.0093	0.0164

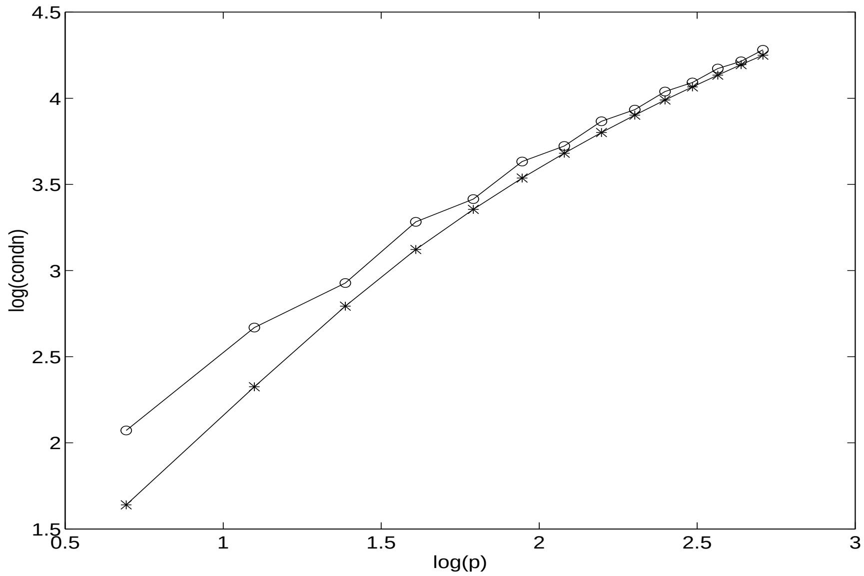
Table 3
Local condition numbers for Method Q

p	$\kappa(S_{J,Q}^{(j)^{-1}} S^{(j)})$	λ_{max}	λ_{min}	C
2	5.1548	1.4778	0.2867	0.4625
3	10.2272	2.3078	0.2257	0.2976
4	16.3348	2.5979	0.1590	0.3120
5	22.6897	2.7686	0.1220	0.3386
6	28.6569	2.8773	0.1004	0.3580
7	34.3494	2.9498	0.0859	0.3757
8	39.6878	2.9934	0.0754	0.3926
9	44.7432	3.0395	0.0679	0.4034
10	49.5128	3.0704	0.0620	0.4149
11	54.0420	3.0986	0.0573	0.4238
12	58.3412	3.1196	0.0535	0.4324
13	62.4449	3.1137	0.0499	0.4472
14	66.3533	3.1510	0.0475	0.4475
15	70.0997	3.1664	0.0452	0.4532

Table 4
Least squares approximation of $\kappa({S_{J,Q}^{(j)}}^{-1} S^{(j)})$ for Method Q

	α_0	α_1	α_2	α_3	$\|error\|_{l^\infty}$	$\|error\|_{l^2}$
linear	-27.7334	34.0835			9.2633	15.2392
quadratic	4.7275	-7.8590	11.8393		0.2509	0.4666
cubic	7.2757	-13.3428	15.3021	-0.6684	0.0709	0.0902

Fig. 6. *Log-log plot of κ for $p = 2, \cdots, 15$: o = Method G, $*$ = Method Q*



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