A POLYLOGARITHMIC BOUND FOR AN ITERATIVE SUBSTRUCTURING METHOD FOR SPECTRAL ELEMENTS IN THREE DIMENSIONS

LUCA F. PAVARINO* AND OLOF B. WIDLUND†

Abstract. Iterative substructuring methods form an important family of domain decomposition algorithms for elliptic finite element problems. A p-version finite element method based on continuous, piecewise Q_p functions is considered for second order elliptic problems in three dimensions; this special method can also be viewed as a conforming spectral element method. An iterative method is designed for which the condition number of the relevant operator grows only in proportion to $(1+\log p)^2$. This bound is independent of jumps in the coefficient of the elliptic problem across the interfaces between the subregions. Numerical results are also reported which support the theory.

Key Words. p-version finite elements, spectral approximation, domain decomposition, iterative substructuring

AMS(MOS) subject classifications. 41A10, 65N30, 65N35, 65N55

1. Introduction. Over the last decade, domain decomposition has developed into an active research area; see, e.g., [26, 15, 16, 27, 17, 32, 31].

The iterative substructuring methods form an important family of domain decomposition methods for elliptic problems. They are based on a decomposition of the given region into nonoverlapping subregions and, as all other domain decomposition methods, provide preconditioners for conjugate gradient type methods. The preconditioners are constructed from solvers for local problems and, in addition, a solver of a coarse problem similar to that used in a multi-grid algorithm. However, the global, coarse problem can be quite exotic; see, e.g., Dryja, Smith, and Widlund [20] and Widlund [52].

When an iterative substructuring method is used, data is only exchanged between neighboring local problems through their boundary values. In this they differ from the Schwarz methods that use overlapping subregions; see, e.g., Dryja and Widlund [23, 24] for a discussion of recent work on this other major family of methods. We also note that similar results, for higher order methods and both two and three dimensions, are given in Pavarino [43, 42].

All these iterative methods are thus two-level methods and convincing arguments have been put forward supporting the opinion that they are particularly well suited for the large, relatively loosely coupled computing systems that are becoming increasingly common; cf. Gropp [29]. The best of these algorithms have proven quite powerful and very large and very ill-conditioned systems of linear algebraic equations, arising when elliptic problems are discretized by finite elements and finite differences, have

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been solved quite economically; cf., e.g., Bjørstad et al. [8, 9], Cai, Gropp, and Keyes [12, 13], Cowsar, Mandel, and Wheeler [18], Gropp and Smith [30], Mandel [39, 41, 40], and Smith [51].

A well-known bound on the decrease of the energy norm of the error, after k steps, of the standard preconditioned conjugate gradient method is given by the formula

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \qquad \text{where} \qquad \kappa = \frac{\lambda_{max}(B^{-1}A)}{\lambda_{min}(B^{-1}A)}.$$

Here A is the coefficient matrix of the original system, and B that of the preconditioner. Therefore, the principal goal of this paper, and domain decomposition theory in general, is to provide good upper bounds on the condition number of the preconditioned operator.

Early work on iterative substructuring methods focused on the h-version finite element methods; see, e.g., Bramble, Pasciak, and Schatz [10], Dryja [19], Dryja and Widlund [21], and Smith [49, 50, 51] for work on three-dimensional elliptic problems. A recent paper by Dryja, Smith, and Widlund [20] summarizes our knowledge of the h-version case. The best of these results show that the condition number of the relevant preconditioned operator grows only linearly with the logarithm of the number of degrees of freedom of an individual subregion. It is important to note that these bounds are independent of the number of subproblems and that they are independent of jumps in the coefficients across subregion interfaces. We also note that there are considerable differences between good iterative substructuring algorithms for two and three dimensional problems; some algorithms that are successful for problems in two dimensions are quite mediocre in three.

The development of iterative methods for higher order and spectral methods poses a special challenge since the stiffness matrices can be much more ill conditioned than those of lower order methods. The domain decomposition methods that have been proposed have also been less well understood. Since the number of degrees of freedom per element increases rapidly with p, it is natural to use individual elements as subregions to be assigned to individual processors of a parallel computing system. In this paper, we design and analyze an algorithm with a polylogarithmic bound in the degree p of the spectral elements. While doing so, we also develop a number of technical tools, which are likely to be useful in future studies of other domain decomposition algorithms. We note that the method considered in this paper is directly inspired by a method developed by Smith [50, 51] for the h-version. Our basic result has previously been announced in Pavarino [44] and Pavarino and Widlund [45].

Important progress has previously been reported, for problems in two dimensions, by Babuška, Craig, Mandel, and Pitkäranta [1], in which polylogarithmic bounds for some methods are proved; see also Pavarino [42] for results in two dimensions, which are similar to those of this paper. Early experimental work is reported in Babuška and Elman [2]. In three dimensions, pioneering work has been carried out by Mandel [38, 37, 36, 41]. Some of his algorithms, which use global spaces which differ from ours, are used in daily industrial practice. A number of domain decomposition methods for spectral elements have been considered by Fischer and Rønquist [25] and Rønquist [47, 48]; for a general introduction to spectral element methods, we refer to Maday and Patera [35] and Bernardi and Maday [7]. However, we know of no previous theoretical results that show only polynomial growth in $\log p$ for problems in three dimensions.

In our analysis, we use a special basis for our finite element space and we have also used this basis in our numerical experiments; our analysis relies heavily on separation of variables. After introducing the elliptic problems and the basic iterative method in Section 2, we introduce the different subspaces and their bases, in some detail, in Sections 3 and 4. A number of technical tools and a proof of our main result are given in Section 5. Section 6 provides a useful alternative description of our problem and iterative method using matrix notations. We also show that there is a great deal of flexibility in choosing the bases of our subspaces. The paper is concluded by a report on some numerical experiments, which support the theoretical results, and an appendix which contains the long proof of one of our auxiliary results.

We note that it is known that the use of Gauss-Lobatto-Legendre quadrature results in a coefficient matrix that is uniformly spectrally equivalent to the stiffness matrix derived from the Galerkin procedure considered here; see Bernardi and Maday [7]. It therefore appears likely that our algorithm could also be of use for such methods.

2. The elliptic problem and block-Jacobi methods. We consider a linear, selfadjoint, elliptic problem on a bounded domain $\Omega \subset R^3$ formulated variationally as

$$a(u,v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx \quad = \quad f_\Omega(v) \qquad \forall \, v \in \; V.$$

V is an appropriate subspace of $H^1(\Omega)$, which incorporates the boundary conditions of the problem. We always assume that the boundary conditions do not change type except at the boundary between two subregions. $\rho(x) > 0$ can be discontinuous, with very different values for different subregions, but we allow this coefficient to vary only moderately within each subregion Ω_j. In fact, without decreasing the generality of our results, we will only consider the piecewise constant case of $\rho(x) = \rho_j, x \in \Omega_j$.

We focus on the case where the subregions Ω_j form a finite element decomposition of the region Ω. The elements are all cubes, or images of a reference cube, under reasonably smooth mappings; no element can be "too distorted". Almost all our technical work can in fact be carried out on a single reference cube $\Omega_{ref} = (-1,1)^3$.

The discrete space $V^p \subset V$ is the space of continuous, piecewise Q_p elements, i.e. the tensor product of three copies of the space of degree p polynomials of one variable. This results in a conforming Galerkin method; the finite element problem is obtained by restricting u and the test function v to the space V^p. The finite element solution is a projection of the exact solution onto the finite element space; this projection is orthogonal with respect to the bilinear form $a(\cdot,\cdot)$.

The finite element variational problem is turned into a linear system of algebraic equations, $Kx = b$, in the usual way. Here K is the stiffness matrix, and b the load vector. $K^T = K > 0$, a property inherited from the bilinear form $a(\cdot,\cdot)$.

Here we view our iterative substructuring method as a block-Jacobi/conjugate gradient method; see Dryja and Widlund [24]. The stiffness matrix K is preconditioned by a matrix K_J, which is the direct sum of diagonal blocks of K. We can also replace some of these blocks by spectrally equivalent (or almost spectrally equivalent) block matrices in order to decrease the cost of the computation. Each block of the Jacobi splitting corresponds to a set of degrees of freedom that define a subspace V_i. In the case considered in this paper, the space V^p is the direct sum of these subspaces. However, to arrive at a successful method, we must first carry out a suitable change of basis and then select the blocks carefully.

Block-Jacobi methods such as these can also be viewed differently. For each subspace V_i, we introduce an orthogonal projection $P_i : V^p \rightarrow V_i$, given by

$$a(P_i u, v) = a(u, v) \qquad \forall v \in V_i, \quad u \in V^p,$$

or an approximation thereof, $T_i : V^p \rightarrow V_i$, defined by a different inner product,

$$\tilde{a}_i(T_i u, v) = a(u, v) \qquad \forall v \in V_i, \quad u \in V^p.$$

We note that the choice of $\tilde{a}_i(\cdot, \cdot)$ determines the operator T_i and vice versa. In a simple case, when a subspace corresponds to a set of adjacent degrees of freedom of a finite element method, P_i simply corresponds to the inverse of the relevant diagonal block of K, padded with zero blocks, times K; the sum of these operators represents $K_J^{-1}K$. To obtain T_i, the special block of K is replaced by an approximate solver for the given operator restricted to the subregion. The resulting block diagonal matrix will be called $\tilde{K}_J$. We will see that our successful method results from selecting one of the subspaces quite differently from those of this simple example.

The spectrum relevant for this iterative method is that of the operator

$$T = \sum_{i=0}^{N} T_i.$$

The eigenvalues of $K^{-1}\tilde{K}_J$, which are identical to those of the inverse of the operator T, are the stationary values of the Rayleigh quotient

$$\frac{\sum_{i=0}^{N} \tilde{a}_i(u_i, u_i)}{a(u, u)}, \qquad u = \sum_{i=0}^{N} u_i, \quad u_i \in V_i; \tag{1}$$

cf. Dryja and Widlund [24].

The most challenging part of our work is to provide an upper bound of this Rayleigh quotient. Success is tied to estimating the approximate energies $\tilde{a}_i(u_i, u_i)$ uniformly, or almost uniformly, from above, in terms of the strain energy $a(u, u)$. An upper bound on $a(u_i, u_i)/\tilde{a}_i(u_i, u_i)$, $u_i \in V_i$, also enters the bound on $\kappa(\tilde{K}_J^{-1}K)$ if inexact solvers are used for some or all of the subspaces.

In this study, we use the block-Jacobi framework but there is also a more general theory; see Dryja and Widlund [22]. Thus, any block-Jacobi method can be viewed as an *additive Schwarz* method based on a direct sum of subspaces. There are also Gauss-Seidel-like, *multiplicative*, as well as *hybrid Schwarz* algorithms; cf. Dryja, Smith, and Widlund [20] for a general discussion. Using the estimates of this paper, we can obtain strong results for a number of these alternative algorithms in a completely routine way.

3. A choice of subspaces. Our method is primarily defined by a set of subspaces; the mathematical description of the method is complete when, in addition, the bilinear forms $\tilde{a}_i(\cdot,\cdot)$ have been specified. In designing methods, we can learn from the h-version case. The first lesson is that we cannot obtain an asymptotically satisfactory bound if $V_0 = Q_1$ and, at the same time, all the elements of the other subspaces vanish at the vertices of the elements. Such a choice of the additional, local subspaces is in fact very natural for the finite element space considered in this paper since any basis function, except those of the vertices, typically is associated with either the interior of the region, or the interior of a face or an edge; see, e.g., Babuška, Griebel, and Pitkäranta [3]. In such a case, all elements of the local spaces vanish at the vertices. We must then choose $u_0 \in V_0$ as the Q_1-interpolant in the decomposition $u = \sum u_i$. In three dimensions, the norm of this interpolant can be larger than the norm of u by a factor p and any upper bound on the Rayleigh quotient (1) must be on the order of p^2. (An example of a function with small energy and with a Q_1-interpolant with much larger energy is given by our vertex basis functions introduced in Subsection 4.3; see further Lemmas 4.1 and 5.4 i) from which a growth on the order of p^2 can be obtained.) For piecewise linear finite elements, this point is discussed in detail in Dryja, Smith, and Widlund [20] where remedies, and their consequences, are also discussed.

As in the case of h-version finite elements, we thus consider several important geometric objects: *interiors, faces, edges*, and *vertices* and subspaces directly related to them. We merge the edges and vertices of the individual elements, creating *wire baskets* $\mathcal{W}_j$ of the elements Ω_j.

Our new method is based on the following subspaces:

• An *interior* space for each element: $Q_p \cap H_0^1(\Omega_j)$.

• A space for each *face*. These functions vanish on and outside the boundary of $\Omega_{jk} = \Omega_j \cup F_{jk} \cup \Omega_k$. Here two elements share a common face F_{jk} and $\overline{F}_{jk} = \overline{\Omega}_j \cap \overline{\Omega}_k$. Since it is crucial to have a good, low energy extension of the values given on the designated face to the interior of the two relevant elements, we use the minimal energy, *discrete harmonic* extension.

• A coarse, global space, V_0, of piecewise discrete harmonic functions associated with the wire baskets $\mathcal{W}_j$ of the elements. Its elements are defined solely by their values on the wire baskets. A central issue is how to define the values on the faces of the elements; once the face values are given, we use a discrete harmonic extension to the interiors of the elements. It is known from previous work that it is crucial to include the constants in this coarse, global space; cf. Mandel [37], or Dryja, Smith, and Widlund [20]. We must therefore make sure that an element of V_0, which is constant on the wire basket of an element, takes on the same constant value everywhere in the element; see the next section where further details are provided on all the spaces.

For all these subspaces, except the last one, we use exact solver, i.e. the bilinear form $a(\cdot,\cdot)$. For the subspace V_0, we use the bilinear form

$$\tilde{a}_0(u,u) \;=\; (1+\log p)\sum_j \rho_j \inf_{c_j} \|u-c_j\|^2_{L^2(\mathcal{W}_j)}$$

if the restriction of the basis elements of this subspace to the wire basket are L^2-

orthonormal; for a discussion of the general case, see Section 6. Such a choice of bilinear form leads to a coarse problem with only one essentially global degree of freedom, c_j, per element. These values are found by solving a linear system of finite difference type; cf. Dryja, Smith, and Widlund [20]. In addition, a larger linear system, with a convenient diagonal matrix, is solved to find all the degrees of freedom related to the wire basket.

The following is the main result of the paper.

THEOREM 3.1. *For the iterative substructuring method defined by these spaces and bilinear forms,*

$$\kappa(T) \leq C(1+\log p)^2.$$

Here the constant C is independent of the number of elements, their diameters, the degree p, and the size of the jumps of the coefficient $\rho(x)$ across element interfaces.

4. Separating variables. It is clear from the work of Babuška, Griebel, and Pitkäranta [3], Babuška, Craig, Mandel, and Pitkäranta [1], and others that the choice of bases for the different subspaces is quite crucial for p-version finite element methods and for the design of good preconditioners. Several different sets of basis functions have been suggested and some of them have been implemented in industrial codes. For our analysis, we will select a particular set of basis functions; see however Subsection 6.2 for a discussion of how our algorithm can be used more generally. Our subspaces are constructed from several sets of special polynomials on the interval $[-1,+1]$ that will be introduced in Subsection 4.2. We note that similar sets of functions have been used in the work of Babuška, Griebel, and Pitkäranta [3] and Canuto and Funaro [14]. These polynomials can be regarded as discrete analogs of the sine and hyperbolic sine functions used when solving Laplace's equation in a square or a cube by the method of separation of variables. We briefly describe this method, in order to motivate what follows.

We will use capital letters V_k, E_k, F_k to denote vertices, edges and faces of the reference cube Ω_{ref}; see Figure 1 for the ordering of these geometric objects. We will use lower case, $v^{(k)}, e_i^{(k)}, f_{ij}^{(k)}$ to denote the basis functions associated with these geometric objects; they are introduced in Subsections 4.3 and 4.4.

4.1. The continuous case. Consider the continuous problem

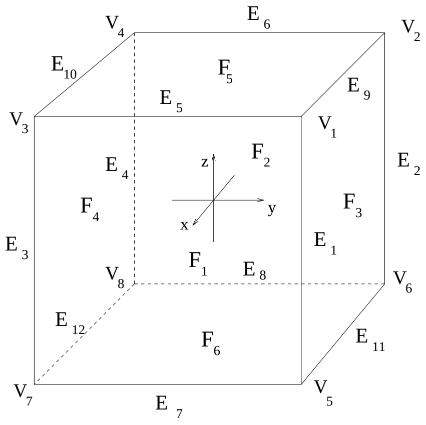
$$\left\{\begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega_{ref} = (-1,1)^3, \\ u & = & g & \text{on } \Gamma \subset \partial\Omega_{ref}\,, \\ u & = & 0 & \text{on } \partial\Omega_{ref} \setminus \Gamma\,. \end{array}\right. \tag{2}$$

We will consider the construction of i) face basis functions and ii) edge basis functions.

i) Consider the case where Γ is a face, e.g., the open set F_1 defined by $x=1$. Let $u(x,y,z) = X(x)Y(y)Z(z)$; then (2) becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = \lambda + \mu,$$

FIG. 1. *Numbering of the vertices, edges and faces of the reference cube* $\overline{\Omega}_{ref} = [-1,1]^3$.



and we obtain two Sturm-Liouville problems and a boundary value problem for a second order ODE for X:

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(-1) = Y(1) = 0 \end{cases}, \qquad \begin{cases} Z'' + \mu Z = 0 \\ Z(-1) = Z(1) = 0 \end{cases}, \tag{3}$$

$$\begin{cases} X'' - (\lambda + \mu)X = 0 \\ X(-1) = 0, X(1) = 1 \end{cases}. \tag{4}$$

The eigenvalues and eigenfunctions of the first two are the same:

$$\lambda_i = (\frac{i\pi}{2})^2, \qquad Y_i(y) = \sin(\frac{i\pi}{2}(y+1)), \qquad i \geq 1,$$

$$\mu_j = \lambda_j, \qquad Z_j(z) = Y_j(z), \qquad j \geq 1.$$

These eigenfunctions form an orthonormal basis of $L^2(-1,1)$. The solutions of (4) are:

$$X_{ij}(x) = \frac{e^{\sqrt{\lambda_i+\lambda_j}(x+1)} - e^{-\sqrt{\lambda_i+\lambda_j}(x+1)}}{e^{2\sqrt{\lambda_i+\lambda_j}} - e^{-2\sqrt{\lambda_i+\lambda_j}}} = \frac{\sinh(\sqrt{\lambda_i+\lambda_j}(x+1))}{\sinh(2\sqrt{\lambda_i+\lambda_j})}, \qquad i,j \geq 1,$$

and the solution of (2) is $u = \sum_{ij} \beta_{ij} X_{ij} Y_i Z_j$, where $\beta_{ij} = \int_{-1}^{1}\int_{-1}^{1} g Y_i Z_j dy dz$.

It is easily to seen that the resulting harmonic functions, $X_{ij}(x)Y_i(y)Z_j(z)$, are H^1-orthogonal and that

$$\|u\|^2_{H^1(\Omega_{ref})} = \sum_{ij} \beta_{ij}^2 \sqrt{\lambda_i+\lambda_j} \coth(2\sqrt{\lambda_i+\lambda_j}). \tag{5}$$

A simple computation of norms of the boundary data $g(x,y) = \sum_{ij} \beta_{ij} Y_i Z_j$ shows that

$$\|g\|^2_{L^2(F_1)} = \sum_{ij} \beta_{ij}^2 \qquad \text{and} \qquad \|g\|^2_{H^1_0(F_1)} = \sum_{ij} \beta_{ij}^2 (\lambda_i + \lambda_j) .$$

We can use Peetre's K-method, see, e.g., Lions and Magenes [33] pp. 66-69, 98-99, to compute the $H_{00}^{1/2}(F_1)-$norm of the same function by interpolating between the Hilbert spaces $L^2(F_1)$ and $H_0^1(F_1)$. We recall that the functional $K(t, g)$ and the norm are defined by

$$K(t,g) = \inf_{g_0}(|g_0|^2_{H_0^1(F_1)} + t^2\|g - g_0\|^2_{L^2(F_1)})^{1/2},$$

and

$$\|g\|^2_{H_{00}^{1/2}(F_1)} = \int_0^\infty t^{-2}K(t,g)^2 dt.$$

By separating the variables and solving a variational problem, we can find an explicit formula for $K(t, g)$. We find, by a straightforward computation, that

$$\|g\|^2_{H_{00}^{1/2}(F_1)} = \sum_{ij} \beta_{ij}^2 \int_0^\infty \frac{\lambda_i + \lambda_j}{\lambda_i + \lambda_j + t^2} dt = \frac{\pi}{2} \sum_{ij} \beta_{ij}^2 \sqrt{\lambda_i + \lambda_j}. \tag{6}$$

We note that the formulas (5) and (6) for the energy- and trace-norms, provides proofs, in a special instance, of both a trace and an extension theorem.

We will also find it convenient to use alternative formulas for the $H_{00}^{1/2}$-norm. We note that the space $H_{00}^{1/2}(F_1)$ is the completion of $C_0^\infty(F_1)$ with respect to the norm

$$\|g\|^2_{H^{1/2}(\partial\Omega_{ref})} = |g|^2_{H^{1/2}(\partial\Omega_{ref})} + \|g\|^2_{L^2(\partial\Omega_{ref})},$$

where

$$|g|^2_{H^{1/2}(\partial\Omega_{ref})} = \int_{\partial\Omega_{ref}} \int_{\partial\Omega_{ref}} \frac{|g(y) - g(z)|^2}{|y - z|^3} ds(y) ds(z).$$

As demonstrated in Grisvard [28] and Lions and Magenes [33], a norm equivalent to (6) is given by

$$|g|^2_{H^{1/2}(F_1)} + \int_{F_1} \frac{|g(z)|^2}{dist(z, \partial F_1)} dz.$$

Since $F_1 = \{1\} \times (-1, 1)^2$, this norm can be replaced by

$$|g|^2_{H^{1/2}(F_1)} + \int_{-1}^1 \int_{-1}^1 \frac{|g|^2}{1 - y^2} dy dz + \int_{-1}^1 \int_{-1}^1 \frac{|g|^2}{1 - z^2} dy dz. \tag{7}$$

ii) In the second case, let Γ be the union of an edge E_1 defined by $x = y = 1$ and the two faces sharing this edge. We are looking for a function that coincides with $g(z), z \in [-1, 1], g(-1) = g(1) = 0$, on E_1. There are of course many ways of extending this function to the two faces which share this edge. Here we use the same Ansatz as before and obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} = \lambda.$$

We satisfy these equations by solving a Sturm-Liouville problem for Z and a boundary value problem for each of X and Y:

$$\left\{ \begin{array}{c} Z'' + \lambda Z = 0 \\ Z(-1) = Z(1) = 0 \end{array} \right. , \tag{8}$$

$$\left\{ \begin{array}{c} X'' - \frac{\lambda}{2} X = 0 \\ X(-1) = 0, X(1) = 1 \end{array} \right. , \qquad \left\{ \begin{array}{c} Y'' - \frac{\lambda}{2} Y = 0 \\ Y(-1) = 0, Y(1) = 1 \end{array} \right. . \tag{9}$$

The eigenvalues and eigenfunctions of (8) are:

$$\lambda_i = (\frac{i\pi}{2})^2, \qquad Z_i(z) = \sin(\frac{i\pi}{2}(z+1)), \qquad i \geq 1,$$

while the two equations (9) have the same solutions

$$X_i(x) = \frac{\sinh(\sqrt{\lambda_i/2}(x+1))}{\sinh(2\sqrt{\lambda_i/2})}, \quad Y_i(y) = X_i(y), \qquad i \geq 1.$$

Then $u = \sum_i \alpha_i X_i Y_i Z_i$, with $\alpha_i = \int_{-1}^{1} g Z_i dz$.

We note that we can obtain the same formulas by first solving two two-dimensional elliptic problems

$$2u_{xx} + u_{zz} = 0 \qquad \text{and} \qquad 2u_{yy} + u_{zz} = 0 \tag{10}$$

on the two adjacent faces with a Dirichlet boundary condition given by $g(z)$ continued by zero onto the other edges. We then extend the boundary values, thus obtained, by zero onto the four additional faces and extend harmonically to the interior of the region. It is clear that we could also define the values on the faces in terms of solutions of $u_{xx} + u_{zz} = 0$ and $u_{yy} + u_{zz} = 0$. We will not consider the resulting, alternative algorithm in this paper.

As in case i), several equivalent characterizations can be given of the $H_{00}^{1/2}(E_1)$-norm. One of them is

$$\|g\|^2_{H_{00}^{1/2}(E_1)} = |g|^2_{H^{1/2}(E_1)} + \int_{-1}^{1} \frac{|g|^2}{1-z^2} dz \ , \tag{11}$$

where now

$$|g|^2_{H^{1/2}(E_1)} = \int_{E_1} \int_{E_1} \frac{|g(z_1) - g(z_2)|^2}{|z_1 - z_2|^2} dz_1 dz_2 \ .$$

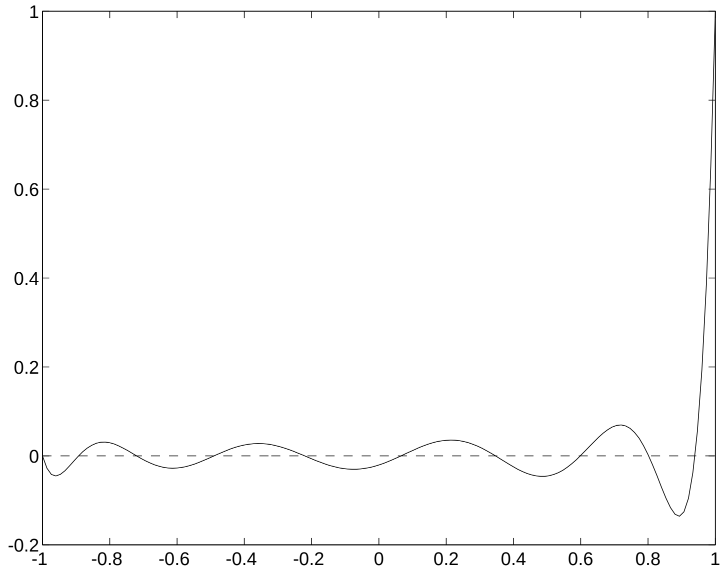
Another follows by using interpolation in Hilbert spaces. Since $g = \sum_i \alpha_i Z_i$ on E_1, we obtain

$$\|g\|^2_{L^2(E_1)} = \sum_i \alpha_i^2 \qquad \text{and} \qquad \|g\|^2_{H_0^1(E_1)} = \sum_i \alpha_i^2 \lambda_i.$$

As before, we also obtain

$$\|g\|^2_{H_{00}^{1/2}(E_1)} = \frac{\pi}{2} \sum_i \alpha_i^2 \sqrt{\lambda_i} \ . \tag{12}$$

FIG. 2. *The polynomial φ_0 for $p = 10$*



4.2. Some special sets of polynomials. Let P^p be the space of degree p polynomials on $[-1, 1]$ and let P_0^p be the subspace of polynomials that vanish at the endpoints of the interval.

DEFINITION 1. *Let φ_0 be the degree p polynomial satisfying*

$$\min_{\varphi} \|\varphi\|_{L^2(-1,1)}, \qquad \varphi(1) = 1, \ \varphi(-1) = 0.$$

We also define $\varphi_0^-(x) \equiv \varphi_0(-x)$ and will also use the notation $\varphi_0^+(x) \equiv \varphi_0(x)$.

See Figure 2 for the case of $p = 10$.

It is possible to compute the Legendre expansion of φ_0 explicitly.

LEMMA 4.1.

$$\varphi_0 = \sum_{n=0}^{p} \frac{r_n}{2} L_n, \qquad \textit{where} \qquad r_n = \begin{cases} (2n+1)r_0 & \textit{if } n \textit{ is even} \\ (2n+1)r_1/3 & \textit{if } n \textit{ is odd} \end{cases} \qquad \textit{and}$$

$$r_0 = \frac{2}{(p+1)(p+2)}, \qquad \frac{r_1}{3} = \frac{2}{p(p+1)},$$

when p is even. When p is odd, then r_0 and $r_1/3$ must be interchanged. Moreover,

$$\|\varphi_0\|^2_{L^2(-1,1)} = \frac{2}{p(p+2)} \qquad \textit{and} \qquad \|\varphi_0\|^2_{L^\infty(-1,1)} \le 1.$$

See the Appendix for a proof. The next result shows that φ_0^+ and φ_0^- are almost orthogonal.

LEMMA 4.2.

$$(\varphi_0^+, \varphi_0^-)_{L^2(-1,1)} = \frac{(-1)^{p+1}}{p+1} \|\varphi_0\|^2_{L^2(-1,1)}.$$

Proof. The Legendre polynomials satisfy the relation $L_n(-x) = (-1)^n L_n(x)$. Therefore,

$$\varphi_0^- = \sum_{n=0}^{p} (-1)^n \frac{r_n}{2} L_n, \quad \text{and} \quad (\varphi_0^+, \varphi_0^-)_{L^2(-1,1)} = \sum_{n=0}^{p} (-1)^n \frac{r_n^2}{4} \frac{2}{2n+1}.$$

From the proof of the previous lemma

$$(\varphi_0^+, \varphi_0^-)_{L^2(-1,1)} = \frac{1}{2} \sum_{even\ n} r_0^2 (2n+1) - \frac{1}{2} \sum_{odd\ n} (\frac{r_1}{3})^2 (2n+1)$$

$$= \begin{cases} \frac{1}{2}(r_0 - \frac{r_1}{3}) = \frac{-2}{p(p+1)(p+2)} = \frac{-1}{p+1} \|\varphi_0\|^2_{L^2(-1,1)} & \text{if } p \text{ is even,} \\ \frac{1}{2}(\frac{r_1}{3} - r_0) = \frac{2}{p(p+1)(p+2)} = \frac{1}{p+1} \|\varphi_0\|^2_{L^2(-1,1)} & \text{if } p \text{ is odd.} \end{cases}$$

□

DEFINITION 2. *Let $\Phi_i \in P_0^p$ and $\lambda_i^{(p)}, i = 1, \cdots, p-1$, be the eigenfunctions and eigenvalues defined by*

$$\int_{-1}^{1} \frac{d\Phi_i(x)}{dx} \frac{dv(x)}{dx} \, dx = \lambda_i^{(p)} \int_{-1}^{1} \Phi_i(x) v(x) \, dx \qquad \forall v \in P_0^p.$$

We normalize these functions to have unit H^1-norm.

We note that if we replace P_0^p by H_0^1, then we would obtain the sine functions discussed on the previous subsection. Similarly, the next two definitions provide sets of polynomials that replace the hyperbolic sine functions.

DEFINITION 3. *Let $\{\lambda_i^{(p)}\}_{i=1}^{p-1}$ be the eigenvalues of Definition 2. Define a set $\{\varphi_i\}_{i=1}^{p-1}$ of degree p polynomials by*

$$\int_{-1}^{1} \frac{d\varphi_i(x)}{dx} \frac{dv(x)}{dx} \, dx + \frac{\lambda_i^{(p)}}{2} \int_{-1}^{1} \varphi_i(x) v(x) \, dx = 0 \qquad \forall v \in P_0^p$$

and $\varphi_i(-1) = 0$, $\varphi_i(1) = 1$.

See Figure 3 for the case of $p = 10$. We also note that we can use the Φ_i and φ_i to solve a finite element approximation of (10) by separation of variables.

DEFINITION 4. *Let $\{\lambda_i^{(p)}\}_{i=1}^{p-1}$ be the eigenvalues of Definition 2. Define a set $\{\varphi_{ij}\}_{i,j=1}^{p-1}$ of degree p polynomials by*

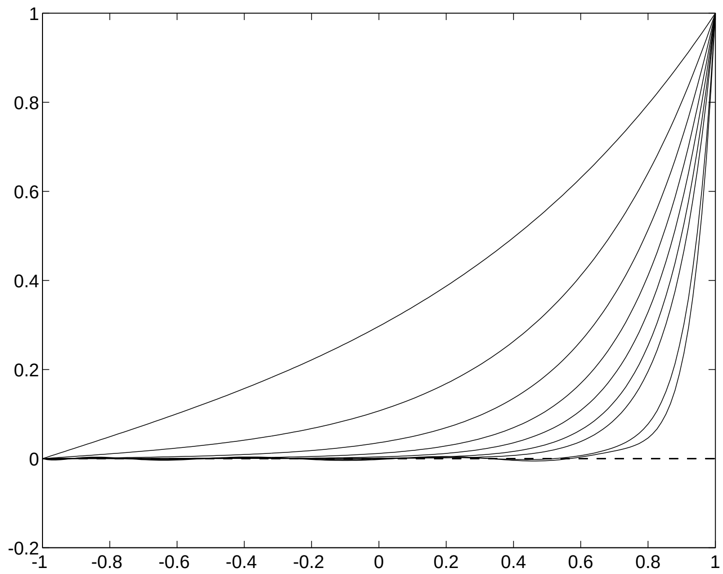
$$\int_{-1}^{1} \frac{d\varphi_{i,j}(x)}{dx} \frac{dv(x)}{dx} \, dx + (\lambda_i^{(p)} + \lambda_j^{(p)}) \int_{-1}^{1} \varphi_{i,j}(x) v(x) \, dx = 0 \qquad \forall v \in P_0^p$$

and $\varphi_{ij}(-1) = 0$, $\varphi_{ij}(1) = 1$.

We will also need the polynomials that satisfy the same boundary conditions at the opposite end points. They are obtained by changing x into $-x$ and we denote them by $\{\varphi_i^-\}$ and $\{\varphi_{ij}^-\}$. Sometimes, we also will use the notation $\varphi_i^+ \equiv \varphi_i$ and $\varphi_{ij}^+ \equiv \varphi_{ij}$.

The polynomials introduced in Definitions 2, 3, and 4, are different from the sine and hyperbolic sine functions used in the continuous case in several respect; e.g., $\lambda_{p-1}^{(p)}$

FIG. 3. *The family of polynomials* $\{\varphi_i\}_{i=1}^{p-1}$ *for* $p = 10$



grows approximately as $Cp^{3.5}$, not as Cp^2. A bound proportional to p^4 follows from a classical polynomial inverse inequality attributed to E. Schmidt, and to Hille, Szego, and Tamarkin, by Bellman [4].

LEMMA 4.3. *Let f be a polynomial of degree p on the interval $[-1, 1]$. Then*

$$|f|_{H^1(-1,1)} \leq \frac{(p+1)^2}{\sqrt{2}} \|f\|_{L^2(-1,1)}.$$

A short proof, based on an expansion in Legendre polynomials, is given in Bellman's paper.

A more general form of the following lemma has already appeared in Bernardi and Maday [6]; see also Canuto and Funaro [14]. For completeness, we include a proof. We note that quite similar inequalities, for the hyperbolic sine functions, were used to arrive at formula (5).

LEMMA 4.4. *For all $p \geq 1$ and $1 \leq i, j \leq p - 1$,*

$$|\varphi_i|^2_{H^1(-1,1)} + \frac{\lambda_i^{(p)}}{2} \|\varphi_i\|^2_{L^2(-1,1)} \leq C\sqrt{\lambda_i^{(p)}}\,, \tag{13}$$

$$|\varphi_{ij}|^2_{H^1(-1,1)} + (\lambda_i^{(p)} + \lambda_j^{(p)}) \|\varphi_{ij}\|^2_{L^2(-1,1)} \leq C\sqrt{\lambda_i^{(p)} + \lambda_j^{(p)}}\,. \tag{14}$$

Proof. From the variational characterization of φ_i, we know that for any polynomial σ of degree $\leq p$ satisfying $\sigma(-1) = 0$, $\sigma(1) = 1$, we have

$$|\varphi_i|^2_{H^1(-1,1)} + \frac{\lambda_i^{(p)}}{2} \|\varphi_i\|^2_{L^2(-1,1)} \leq |\sigma|^2_{H^1(-1,1)} + \frac{\lambda_i^{(p)}}{2} \|\sigma\|^2_{L^2(-1,1)}. \tag{15}$$

By Lemma 4.3 and Definition 2,

$$\lambda_i^{(p)} \le Cp^4, \qquad 1 \le i \le p-1.$$

Therefore, there exists an integer $q \le p$ such that

$$C(q-1)^4 \le \lambda_i^{(p)} \le Cq^4. \tag{16}$$

We select $\sigma = \varphi_0^{(q)}$, the polynomial of degree q given by Definition 1. By Lemmas 4.1 and 4.3, this polynomial satisfies

$$|\varphi_0^{(q)}|^2_{H^1(-1,1)} + q^4 \|\varphi_0^{(q)}\|^2_{L^2(-1,1)} \le Cq^2. \tag{17}$$

We obtain, by using (16), (17),

$$|\varphi_i|^2_{H^1(-1,1)} + \frac{\lambda_i^{(p)}}{2}\|\varphi_i\|^2_{L^2(-1,1)} \le Cq^2 \le 2C(q-1)^2 \le C\sqrt{\lambda_i^{(p)}}.$$

The proof of the other inequality is very similar.

□

We note that by using Lemma 4.4, it is easy to prove a constant bound for the maximum of φ_i and φ_{ij} over $[-1,1]$ since

$$\varphi_i^2(x) = \int_{-1}^{x} \frac{d\varphi_i^2(s)}{ds} ds \le 2\int_{-1}^{1} |\frac{d\varphi_i}{ds}||\varphi_i| ds = 2|\varphi_i|_{H^1(-1,1)}\|\varphi_i\|_{L^2(-1,1)} \le C. \tag{18}$$

A more precise estimate, given in Canuto and Funaro [14], shows that $\varphi_i^2(x) \le 1$ and that, in fact, $\varphi_i(x)$ decays monotonically with decreasing x; see also Figure 3.

4.3. The basis functions. We are now ready to describe our basis on the reference cube $\Omega_{ref} = (-1,1)^3$.

- The *interior basis functions* are defined by

$$\Phi_i(x)\Phi_j(y)\Phi_k(z), \quad i,j,k = 1,\cdots,p-1. \tag{19}$$

They are $a(\cdot,\cdot)-$ and $L^2(\Omega_{ref})-$orthogonal.

- One of the sets of *face basis functions*, for the face $F_1 = \{x = 1\}$, is given by

$$f_{ij}^{(1)}(x,y,z) = \varphi_{i,j}(x)\Phi_i(y)\Phi_j(z), \quad i,j = 1,\cdots,p-1. \tag{20}$$

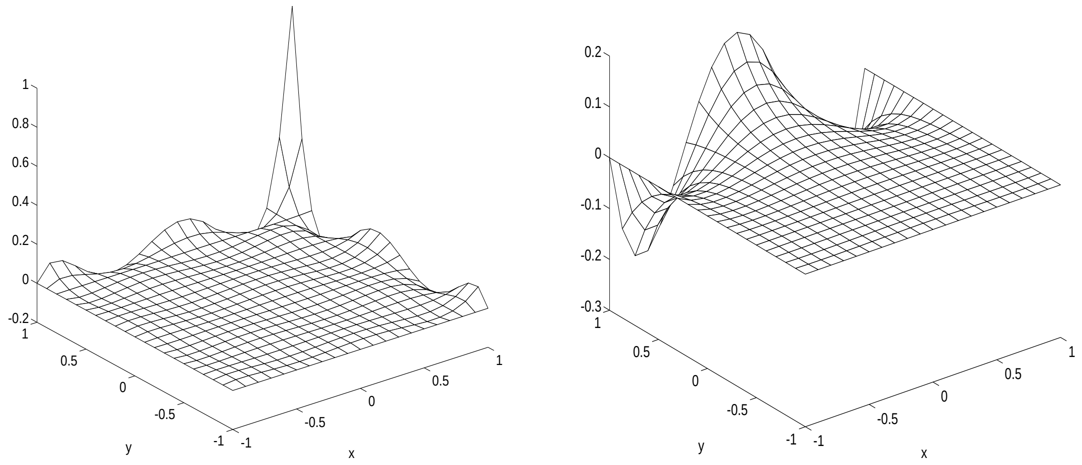
It is easy to show that any two face basis functions, associated with the same face, are $a(\cdot,\cdot)-$ and $L^2(\Omega_{ref})-$orthogonal; see the proof of Lemma 4.5 for a similar argument.

The wire basket space is given in terms of edge and vertex basis functions. As we will see, the elements of the subspace spanned by these functions are later "corrected" so that they also contain certain components from the face spaces.

- One of the sets of *edge basis functions*, for the edge $E_1 = \{x = 1, y = 1\}$, is given preliminarily by

$$\tilde{e}_i^{(1)}(x,y,z) = \varphi_i(x)\varphi_i(y)\Phi_i(z), \quad i = 1,\cdots,p-1. \tag{21}$$

FIG. 4. *Vertex function $\varphi_0(x)\varphi_0(y)\varphi_0(1)$ and edge function $\Phi_3(x)\varphi_3(y)\varphi_3(1)$, for $p=5$*



It is easy to show that any two edge basis functions, associated with the same edge, are $a(\cdot,\cdot)-$ and $L^2(\Omega_{ref})-$orthogonal.

• One of the eight *vertex basis functions*, the one for the vertex $V_1=(1,1,1)$, is given preliminarily by

$$\tilde{v}^{(1)}(x,y,z)=\varphi_0(x)\varphi_0(y)\varphi_0(z). \tag{22}$$

See Figure 4 for examples of a vertex and an edge basis functions on a face.

An easy computation shows that

LEMMA 4.5. *The face, edge, and vertex basis functions are discrete harmonic.*

Proof. i) Consider, without loss of generality, the face $F_1=\{x=1\}$:

$$\int_{\Omega_{ref}} \nabla(\varphi_{ij}(x)\Phi_i(y)\Phi_j(z))\cdot\nabla(\Phi_l(x)\Phi_m(y)\Phi_n(z))dxdydz$$

$$=\int_{-1}^{1}\frac{d\varphi_{ij}}{dx}\frac{d\Phi_l}{dx}dx\int_{-1}^{1}\Phi_i\Phi_m dy\int_{-1}^{1}\Phi_j\Phi_n dz+\int_{-1}^{1}\varphi_{ij}\Phi_l dx\int_{-1}^{1}\frac{d\Phi_i}{dy}\frac{d\Phi_m}{dy}dy\int_{-1}^{1}\Phi_j\Phi_n dz$$

$$+\int_{-1}^{1}\varphi_{ij}\Phi_l dx\int_{-1}^{1}\Phi_i\Phi_m dy\int_{-1}^{1}\frac{d\Phi_j}{dz}\frac{d\Phi_n}{dz}dz$$

$$=\delta_{im}\delta_{jn}\left(-\frac{\lambda_i^{(p)}+\lambda_j^{(p)}}{\lambda_i^{(p)}\lambda_j^{(p)}}+\frac{1}{\lambda_j^{(p)}}+\frac{1}{\lambda_i^{(p)}}\right)\int_{-1}^{1}\varphi_{ij}\Phi_l dx=0.$$

ii) Consider the edge $E_1=\{x=y=1\}$:

$$\int_{\Omega_{ref}} \nabla(\varphi_i(x)\varphi_i(y)\Phi_i(z))\cdot\nabla(\Phi_l(x)\Phi_m(y)\Phi_n(z))dxdydz$$

$$= \int_{-1}^{1} \frac{d\varphi_i}{dx}\frac{d\Phi_l}{dx} dx \int_{-1}^{1} \varphi_i \Phi_m dy \int_{-1}^{1} \Phi_i \Phi_n dz + \int_{-1}^{1} \varphi_i \Phi_l dx \int_{-1}^{1} \frac{d\varphi_i}{dy}\frac{d\Phi_m}{dy} dy \int_{-1}^{1} \Phi_i \Phi_n dz$$

$$+ \int_{-1}^{1} \varphi_i \Phi_l dx \int_{-1}^{1} \varphi_i \Phi_m dy \int_{-1}^{1} \frac{d\Phi_i}{dz}\frac{d\Phi_n}{dz} dz$$

$$= \delta_{in}\left(-\frac{1}{\lambda_i^{(p)}}\frac{\lambda_i^{(p)}}{2} - \frac{1}{\lambda_i^{(p)}}\frac{\lambda_i^{(p)}}{2} + 1\right) \int_{-1}^{1} \varphi_i \Phi_l dx \int_{-1}^{1} \varphi_i \Phi_m dy = 0.$$

iii) Consider the vertex $V_1 = (1,1,1)$:

$$\int_{\Omega_{ref}} \nabla(\varphi_0(x)\varphi_0(y)\varphi_0(z)) \cdot \nabla(\Phi_l(x)\Phi_m(y)\Phi_n(z)) dx dy dz$$

vanishes for all $1 \le l, m, n \le p-1$, because

$$\int_{-1}^{1} \varphi_0 \Phi_k dx = 0, \qquad 1 \le k \le p-1.$$

This follows from the definition of φ_0 and a simple variational argument.

$\square$

Let F be a face of Ω_{ref}. If u is a polynomial of degree p which vanishes on ∂F, then we can try to compute $\|u\|_{H_{00}^{1/2}(F)}$ using the basis just introduced and the interpolation argument of Subsection 4.1; cf. (6). For $u = \sum_{ij} \beta_{ij} \Phi_i \Phi_j$ on F, it follows, with the different normalization of the discrete eigenfunctions, that

$$\|u\|^2_{L^2(F)} = \sum_{i,j=1}^{p-1} \left(\frac{\beta_{ij}}{\sqrt{\lambda_i^{(p)}\lambda_j^{(p)}}}\right)^2 \qquad \text{and} \qquad \|u\|^2_{H_0^1(F)} = \sum_{i,j=1}^{p-1} \left(\frac{\beta_{ij}}{\sqrt{\lambda_i^{(p)}\lambda_j^{(p)}}}\right)^2 (\lambda_i^{(p)} + \lambda_j^{(p)}) \ .$$

By using the K-method, we can conclude that

$$\|u\|^2_{H_{00}^{1/2}(F)} \le \frac{\pi}{2} \sum_{i,j=1}^{p-1} \left(\frac{\beta_{ij}}{\sqrt{\lambda_i^{(p)}\lambda_j^{(p)}}}\right)^2 \sqrt{\lambda_i^{(p)} + \lambda_j^{(p)}} \ . \tag{23}$$

The quite subtle question if the expression on the right hand side is in fact equivalent to the left hand side, has been settled affirmatively by Maday [34] and Ben Belgacem [5].

Similarly, let E be an edge of Ω_{ref}. Let $u = \sum_i \alpha_i \Phi_i$ be a polynomial defined on E. Then,

$$\|u\|^2_{L^2(E)} = \sum_{i=1}^{p-1} \frac{\alpha_i^2}{\lambda_i^{(p)}} \qquad \text{and} \qquad \|u\|^2_{H_0^1(E)} = \sum_{i=1}^{p-1} \alpha_i^2,$$

and $\|u\|^2_{H_{00}^{1/2}(E)}$ is equivalent to

$$\sum_{i=1}^{p-1} \frac{\alpha_i^2}{\sqrt{\lambda_i^{(p)}}}. \tag{24}$$

4.4. Extension from the wire basket. As we have previously noted, the key part of the proof of our main result involves a decomposition of an arbitrary polynomial on Ω_{ref} into components in the different subspaces and estimates of the energy norms of these components. In this subsection, we will define these components and also give a full description of the coarse, wire basket based space.

Let u be a polynomial in $Q_p(\Omega_{ref})$. We will use the decomposition

$$u = u_W + \sum_{k=1}^{6} u_{F_k} + u_I,$$

which is a sum of a wire basket component (vertex and edges), six face components each with nonzero values only on one face, and an interior component which vanishes on the boundary of Ω_{ref}. In order to define this decomposition, we need to extend the values of a function, given at a vertex, on an edge, or on a face, to the whole of Ω_{ref} in an appropriate way.

Vertex components: For the vertices V_k of Ω_{ref}, let

$$\begin{array}{rcl} u_{V_1}(x,y,z) & = & u(1,1,1)\tilde{v}^{(1)}(x,y,z) \\ u_{V_2}(x,y,z) & = & u(-1,1,1)\tilde{v}^{(2)}(x,y,z) \\ \vdots & & \vdots \\ u_{V_8}(x,y,z) & = & u(-1,-1,-1)\tilde{v}^{(8)}(x,y,z). \end{array}$$

The vertex component of u is then

$$u_V = \sum_{k=1}^{8} u_{V_k}.$$

Edge components: On each edge E_k, $u - u_V$ can be expanded in the $\{\Phi_i\}$ basis: $u - u_V = \sum_{i=1}^{p-1} \alpha_i^{(k)} \Phi_i$, with $\alpha_i^{(k)} = \lambda_i^{(p)} \int_{E_k} (u - u_V)\Phi_i ds$. Let

$$\begin{array}{rcll} u_{E_1}(x,y,z) & = & \sum_{i=1}^{p-1} \alpha_i^{(1)} \tilde{e}_i^{(1)}(x,y,z), & \text{for } E_1 = \{x = 1, y = 1\} \\ u_{E_2}(x,y,z) & = & \sum_{i=1}^{p-1} \alpha_i^{(2)} \tilde{e}_i^{(2)}(x,y,z), & \text{for } E_2 = \{x = -1, y = 1\} \\ \vdots & & \vdots & \vdots \\ u_{E_{12}}(x,y,z) & = & \sum_{i=1}^{p-1} \alpha_i^{(12)} \tilde{e}_i^{(12)}(x,y,z), & \text{for } E_{12} = \{y = -1, z = -1\}. \end{array}$$

The edge component of u is then

$$u_E = \sum_{k=1}^{12} u_{E_k}.$$

Wire basket component: We first introduce a preliminary interpolation operator $\tilde{I}^W : V^p \to V_0$, defined by $\tilde{I}^W u = u_V + u_E$. However, this operator will not reproduce constants; see Figures 5 and 6. It is known that the bound for the condition number

FIG. 5. *Vertex and edge components of* $\tilde{I}^W 1$ *on a face of* Ω_{ref}, *for* $p = 5$

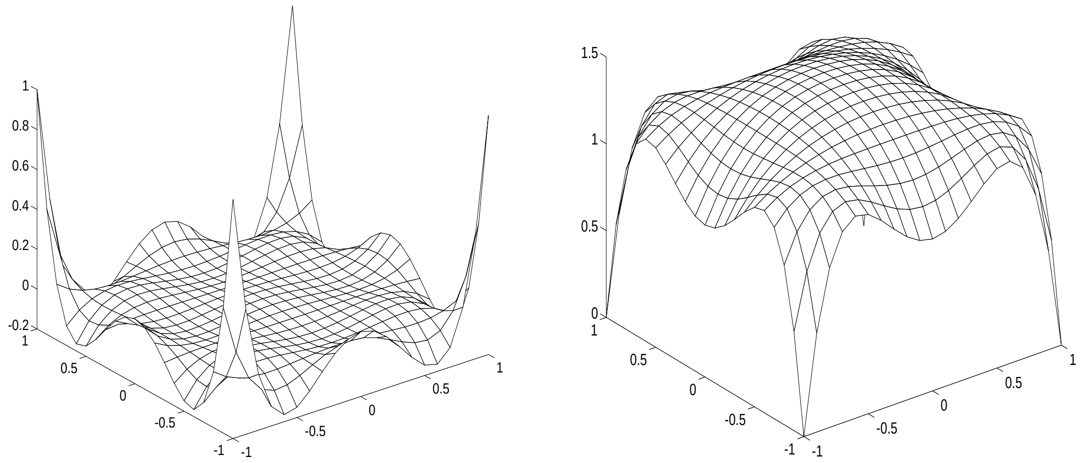
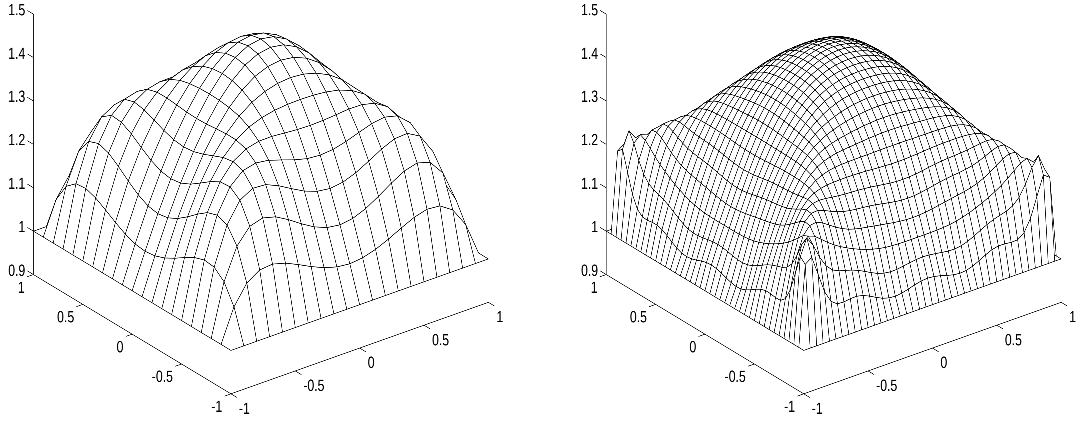


FIG. 6. $\tilde{I}^W 1 = 1 - \mathcal{F}$ *on a face of* Ω_{ref}, *for* $p = 5$, *and* $p = 10$



then must depend on the number of subregions; cf. Mandel [37], or Dryja, Smith, and Widlund [20]. In order to overcome this difficulty, we construct the function $\mathcal{F} = 1 - \tilde{I}^W 1$, which vanishes on the wire basket, and which naturally can be split into six discrete harmonic components, each with nonzero values on only one face:

$$\mathcal{F} = \sum_{j=1}^{6} \mathcal{F}_j.$$

We then define the wire basket component as the image of u under a new interpolation operator

$$u_W = I^W u = \tilde{I}^W u + \sum_{j=1}^{6} \overline{u}_{\partial F_j} \mathcal{F}_j,$$

where $\overline{u}_{\partial F_j} = \frac{1}{8}\int_{\partial F_j} u$. With this definition, the wire basket space will contain the constants, since if $u \equiv 1$ on W, then $u_W \equiv 1$ on $\partial\Omega_{ref}$. This interpolation operator defines a change of basis in the space; the preliminary vertex and edge basis functions, given above by (22) and (21), are mapped into:

$$\begin{array}{rclclcl} v^{(k)} & = & I^W \tilde{v}^{(k)} & = & \tilde{v}^{(k)} & + & \sum_j \overline{\tilde{v}}^{(k)}_{\partial F_j} \mathcal{F}_j \\ e_i^{(k)} & = & I^W \tilde{e}_i^{(k)} & = & \tilde{e}_i^{(k)} & + & \sum_j \overline{\tilde{e}}^{(k)}_{i,\partial F_j} \mathcal{F}_j \end{array} \tag{25}$$

We note that for a vertex basis function, the sum is only over the three faces sharing the vertex and that the weights are

$$\overline{\tilde{v}}^{(k)}_{\partial F_j} = \frac{1}{8}\int_{\partial F_j} \tilde{v}^{(k)} ds = \frac{1}{4}\int_{-1}^{1} \varphi_0(s) ds.$$

For an edge basis function, the sum is only over the two faces sharing the edge and

$$\overline{\tilde{e}}^{(k)}_{i,\partial F_j} = \frac{1}{8}\int_{\partial F_j} \tilde{e}_i^{(k)} ds = \frac{1}{8}\int_{-1}^{1} \Phi_i(s) ds.$$

Face components: $u - u_W$ vanishes on the wire basket. On each face F_k it can be expanded in the $\{\Phi_i\Phi_j\}$ basis: $u - u_W = \sum_{i,j=2}^{p} \beta_{ij}^{(k)} \Phi_i \Phi_j$, with

$$\beta_{ij}^{(k)} = \lambda_i^{(p)} \lambda_j^{(p)} \int_{F_k} (u - u_W) \Phi_i(s_1) \Phi_j(s_2) ds_1 ds_2.$$

The six face components are defined by

$$\begin{array}{rcll} u_{F_1}(x,y,z) & = & \sum_{i,j=2}^{p} \beta_{ij}^{(1)} f_{ij}^{(1)}(x,y,z), & \text{for } F_1 = \{x = 1\} \\ u_{F_2}(x,y,z) & = & \sum_{i,j=2}^{p} \beta_{ij}^{(2)} f_{ij}^{(2)}(x,y,z), & \text{for } F_2 = \{x = -1\} \\ \vdots & & \vdots & \vdots \\ u_{F_6}(x,y,z) & = & \sum_{i,j=2}^{p} \beta_{ij}^{(6)} f_{ij}^{(6)}(x,y,z), & \text{for } F_6 = \{z = -1\}. \end{array}$$

Interior component: $u - u_W - \sum_{k=1}^{6} u_{F_k}$ vanishes on $\partial\Omega_{ref}$. The final component of the decomposition is given by

$$u_I = u - u_W - \sum_{k=1}^{6} u_{F_k}.$$

5. Technical tools and a proof of the main result. The proof of the main result, Theorem 3.1, is based on local arguments concerning polynomials on the reference cube Ω_{ref}. We note that our final bounds are independent of the diameter H of the elements. We have chosen not to show how the constants of our auxiliary estimates depend on H.

5.1. Technical tools. We will now give a series of lemmas that are needed in the proof of Theorem 3.1. We begin with the classical Markov inequality; cf. Rivlin [46].

LEMMA 5.1. *Let f be a polynomial of degree p defined on $[-1,1]$. Then*

$$\max_{[-1,1]} |f'(x)| \leq p^2 \max_{[-1,1]} |f(x)|.$$

The following result is a discrete Sobolev inequality for polynomials; see Theorem 6.2 in Babuška, Craig, Mandel, and Pitkäranta [1].

LEMMA 5.2. *Let $F = (-1,1)^2$ and let $u \in Q_p(F)$. Then*

$$\|u\|^2_{L^\infty(F)} \leq C(1+\log p)\|u\|^2_{H^1(F)}. \tag{26}$$

Moreover, if $x_0 \in F$, then

$$\|u - u(x_0)\|^2_{L^\infty(F)} \leq C(1+\log p)|u|^2_{H^1(F)}. \tag{27}$$

Proof. We apply Lemma 2.2 in Bramble and Xu [11]: If D is a bounded Lipschitz domain in R^2, then

$$\|u\|_{L^\infty(D)} \leq C(|\log\epsilon|^{1/2}\|u\|_{H^1(D)} + \epsilon\|u\|_{W^{1,\infty}(D)}) \quad \forall u \in W^{1,\infty}(D), \quad \epsilon \in (0,1). \tag{28}$$

By Lemma 5.1

$$\|u\|_{W^{1,\infty}(F)} \leq (1+2p^2)\|u\|_{L^\infty(F)}.$$

Choosing $\epsilon = \frac{1}{6Cp^2}$ in (28), we obtain:

$$\|u\|_{L^\infty(F)} \leq C|\log 6C + 2\log p|^{1/2}\|u\|_{H^1(F)} + \frac{1+2p^2}{6p^2}\|u\|_{L^\infty(F)},$$

and finally, since $\frac{1+2p^2}{6p^2} \leq \frac{1}{2}$,

$$\|u\|^2_{L^\infty(F)} \leq C(1+\log p)\|u\|^2_{H^1(F)}.$$

To prove (27), we use the fact that if u vanishes at some point in F, then

$$\|u\|_{L^\infty(F)} \leq \|u+\alpha\|_{L^\infty(F)} + \|\alpha\|_{L^\infty(F)} \leq 2\|u+\alpha\|_{L^\infty(F)}$$

for any constant α. Then, by (26),

$$\|u\|^2_{L^\infty(F)} \leq C(1+\log p)\|u+\alpha\|^2_{H^1(F)}.$$

Minimizing over α and using Poincaré's inequality, we obtain

$$\|u\|^2_{L^\infty(F)} \leq C(1+\log p)|u|^2_{H^1(F)}.$$

To prove (27), we apply this estimate to $u - u(x_0)$.

□

Lemma 5.3. *Let I be a line segment in $\overline{\Omega}_{ref}$, which is parallel to a coordinate axis. Then*

$$\|u\|^2_{L^2(I)} \leq C(1+\log p)\|u\|^2_{H^1(\Omega_{ref})}.$$

Moreover, if $\overline{u}_W$ is the average of u over the wire basket W, then

$$\|u-\overline{u}_W\|^2_{L^2(W)} \leq C(1+\log p)|u|^2_{H^1(\Omega_{ref})}.$$

Proof. Let I be parallel to the x-axis. We apply Lemma 5.2 to a two dimensional slice F of $\overline{\Omega}_{ref}$, orthogonal to the x-axis. Then,

$$\|u\|^2_{L^2(I)} = \int_{-1}^{1} |u(x,y,z)|^2 dx \leq \int_{-1}^{1} \|u(x,\cdot,\cdot)\|^2_{L^\infty(F)} dx$$

$$\leq C(1+\log p)\int_{-1}^{1} \|u(x,\cdot,\cdot)\|^2_{H^1(F)} \leq C(1+\log p)\|u\|^2_{H^1(\Omega_{ref})}.$$

The second estimate for $u-\overline{u}_W$, is obtained by a quotient space argument.

□

It follows immediately from Lemma 5.3 that

$$(\overline{u}_W)^2 \leq C(1+\log p)\|u\|^2_{H^1(\Omega_{ref})}. \tag{29}$$

Lemma 5.4.
i) The energy of a vertex basis function $\tilde{v}^{(k)}(x,y,z)$ satisfies

$$|\tilde{v}^{(k)}|^2_{H^1(\Omega_{ref})} \leq C|\tilde{v}^{(k)}|^2_{L^2(W)} = 3C\|\varphi_0\|^2_{L^2(-1,1)}.$$

ii) The energy of an edge basis function $\tilde{e}_i^{(k)}(x,y,z)$ satisfies

$$|\tilde{e}_i^{(k)}|^2_{H^1(\Omega_{ref})} \leq C|\tilde{e}_i^{(k)}|^2_{L^2(E_k)} = C\|\Phi_i\|^2_{L^2(-1,1)}.$$

iii) The energy of a face basis function, $f_{ij}^{(k)}(x,y,z)$ satisfies

$$|f_{ij}^{(k)}|^2_{H^1(\Omega_{ref})} \leq C\|f_{ij}^{(k)}\|^2_{H^{1/2}_{00}(F_k)}.$$

Proof. i) By a direct computation

$$|\tilde{v}^{(k)}|^2_{H^1(\Omega_{ref})} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |\nabla\varphi_0^{\pm}(x)\varphi_0^{\pm}(y)\varphi_0^{\pm}(z)|^2 dxdydz$$

$$= 3\|\varphi_0\|^4_{L^2(-1,1)}\|\frac{d\varphi_0}{dx}\|^2_{L^2(-1,1)} \leq 3C\|\varphi_0\|^2_{L^2(-1,1)}.$$

The last bound follows from Lemma 4.3 and the explicit formula $\|\varphi_0\|^2_{L^2(-1,1)} = \frac{2}{p(p+2)}$.
ii) By a direct computation

$$|\tilde{e}_i^{(k)}|^2_{H^1(\Omega_{ref})} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |\nabla \underbrace{\varphi_i^{\pm}(x)\varphi_i^{\pm}(y)\Phi_i(z)}_{permutations}|^2 \, dxdydz$$

$$= \|\varphi_i\|^2_{L^2(-1,1)}(\|\frac{d\varphi_i}{dy}\|^2_{L^2(-1,1)} + \frac{\lambda_i^{(p)}}{2}\|\varphi_i\|^2_{L^2(-1,1)})2\|\Phi_i\|^2_{L^2(-1,1)}.$$

From Lemma 4.4

$$\|\frac{d\varphi_i}{dy}\|^2_{L^2(-1,1)} + \frac{\lambda_i^{(p)}}{2}\|\varphi_i\|^2_{L^2(-1,1)} \le C\lambda_i^{(p)^{1/2}}.$$

It follows that $\|\varphi_i\|^2_{L^2(-1,1)} \le C\lambda_i^{(p)^{-1/2}}$. Therefore

$$\|\varphi_i\|^2_{L^2(-1,1)}(\|\frac{d\varphi_i}{dy}\|^2_{L^2(-1,1)} + \frac{\lambda_i^{(p)}}{2}\|\varphi_i\|^2_{L^2(-1,1)}) \le C.$$

iii) By a direct computation and Lemma 4.4

$$|f_{ij}^{(k)}|^2_{H^1(\Omega_{ref})} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |\nabla \underbrace{\varphi_{ij}^{\pm}(x)\Phi_i(y)\Phi_j(z)}_{permutations}|^2 \, dxdydz$$

$$= \frac{1}{\lambda_i^{(p)}\lambda_j^{(p)}}(\|\frac{d\varphi_{ij}}{dx}\|^2_{L^2(-1,1)} + (\lambda_i^{(p)} + \lambda_j^{(p)})\|\varphi_{ij}\|^2_{L^2(-1,1)}) \le C\frac{\sqrt{\lambda_i^{(p)} + \lambda_j^{(p)}}}{\lambda_i^{(p)}\lambda_j^{(p)}}.$$

From the characterization of the $H^{1/2}_{00}(F_k)-$norm given by (23), we conclude

$$|f_{ij}^{(k)}|^2_{H^1(\Omega_{ref})} \le C\frac{\sqrt{\lambda_i^{(p)} + \lambda_j^{(p)}}}{\lambda_i^{(p)}\lambda_j^{(p)}} \le C\|f_{ij}^{(k)}\|^2_{H^{1/2}_{00}(F_k)}.$$

□

Since edge basis functions associated with the same edge and face basis functions associated with the same face are $a(\cdot,\cdot)-$ and $L^2(\Omega_{ref})-$orthogonal, we can generalize Lemma 5.4 to provide bounds for any edge and face components of a function $u \in Q_p(\Omega_{ref})$:

COROLLARY 5.5.

$$|u_{E_k}|^2_{H^1(\Omega_{ref})} \le C|u_{E_k}|^2_{L^2(E_k)} \ ,$$

$$|u_{F_k}|^2_{H^1(\Omega_{ref})} \le C\|u_{F_k}\|^2_{H^{1/2}_{00}(F_k)} \ .$$

Lemma 5.6. *The energy of the $\tilde{I}^W$ interpolant is bounded by*

$$|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} \leq C\|\tilde{I}^W u\|^2_{L^2(W)} = C\|u\|^2_{L^2(W)}.$$

Proof. By definition

$$\tilde{I}^W u = \sum_{i=1}^{8} u(V_i)\tilde{v}^{(i)}(x,y,z) + \sum_{k=1}^{12}\sum_{j=1}^{p-1} \alpha_j^{(k)} \tilde{e}_j^{(k)}(x,y,z).$$

Hence, by the orthogonality of the edge functions on a given edge and by Lemma 5.4,

$$|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} \leq C(\sum_{i=1}^{8} |u(V_i)|^2 |\varphi_0^{\pm}\varphi_0^{\pm}\varphi_0^{\pm}|^2_{H^1(\Omega_{ref})} + \sum_{k=1}^{12}\sum_{j=1}^{p-1} (\alpha_j^{(k)})^2 |\underbrace{\Phi_j \varphi_j^{\pm}\varphi_j^{\pm}}_{permutations}|^2_{H^1(\Omega_{ref})})$$

$$\leq C(\sum_{i=1}^{8} |u(V_i)|^2 \|\varphi_0\|^2_{L^2(-1,1)} + \sum_{k=1}^{12}\sum_{j=1}^{p-1} (\alpha_j^{(k)})^2 \|\Phi_j\|^2_{L^2(-1,1)}). \tag{30}$$

On the other hand

$$\|\tilde{I}^W u\|^2_{L^2(W)} = \|\sum_{i=1}^{8} u(V_i)\tilde{v}^{(i)}\|^2_{L^2(W)} + \|\sum_{k=1}^{12}\sum_{j=1}^{p-1} \alpha_j^{(k)}\tilde{e}_j^{(k)}\|^2_{L^2(W)},$$

because $(\Phi_j, \varphi_0)_{L^2(-1,1)} = 0$. Since the wire basket is the union of the closure of the edges, $W = \cup_{k=1}^{12} \overline{E}_k$, we have

$$\|\sum_{k=1}^{12}\sum_{j=1}^{p-1} \alpha_j^{(k)}\tilde{e}_j^{(k)}\|^2_{L^2(W)} = \sum_{k=1}^{12} \int_{E_k} (\sum_j \alpha_j^{(k)}\Phi_j)^2 ds = \sum_{k=1}^{12}\sum_{j=1}^{p-1} (\alpha_j^{(k)})^2 \|\Phi_j\|^2_{L^2(-1,1)}. \tag{31}$$

Here, we have used that only the $k-th$edge component differs from zero on E_k. Analogously, denoting by V_{k_1} and V_{k_2} the endpoints of E_k and using Lemma 4.2, we obtain

$$\|\sum_{i=1}^{8} u(V_i)\tilde{v}^{(i)}\|^2_{L^2(W)} = \sum_{k=1}^{12} \int_{E_k} (u(V_{k_1})\varphi_0^+ + u(V_{k_2})\varphi_0^-)^2 ds$$

$$\geq \sum_{k=1}^{12} \frac{1}{2}[u(V_{k_1})^2 + u(V_{k_2})^2]\|\varphi_0\|^2_{L^2(-1,1)} = \frac{3}{2}\sum_{k=1}^{8} u(V_k)^2 \|\varphi_0\|^2_{L^2(-1,1)}. \tag{32}$$

The desired bound is obtained by combining inequalities (30), (31), and (32).

□

Lemma 5.7. *For each face F_k of $\Omega_{ref}, k = 1, \cdots, 6$,*

$$\|u - \tilde{I}^W u\|^2_{H^{1/2}_{00}(F_k)} \leq C(1+\log p)^2 \|u\|^2_{H^1(\Omega_{ref})}.$$

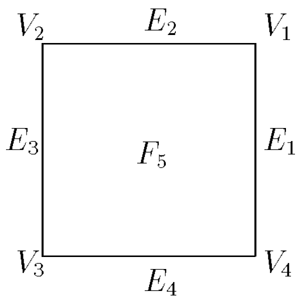


FIG. 7. *Renumbering vertices and edges of the face* $\overline{F}_5$

Proof. We consider, without loss of generality, the face $F_5 = \{z = 1\}$, and, for simplicity, we renumber its vertices and edges as in Figure 7. On this face, $w = u - \tilde{I}^W u$ can be written as

$$w = u - u(V_1)\varphi_0(x)\varphi_0(y) - u(V_2)\varphi_0^-(x)\varphi_0(y) - u(V_3)\varphi_0(x)\varphi_0^-(y) - u(V_4)\varphi_0^-(x)\varphi_0^-(y)$$
$$-\sum_j \alpha_j^{(1)}\varphi_j(x)\Phi_j(y) - \sum_j \alpha_j^{(3)}\varphi_j^-(x)\Phi_j(y) - \sum_j \alpha_j^{(2)}\Phi_j(x)\varphi_j(y) - \sum_j \alpha_j^{(4)}\Phi_j(x)\varphi_j^-(y).$$

Since w vanishes on ∂F_5, we can use the characterization of the $H_{00}^{1/2}(F_5)$—norm given by (7).

$$\|w\|^2_{H_{00}^{1/2}(F_5)} = |w|^2_{H^{1/2}(F_5)} + \int_{F_5} \frac{|w|^2}{1-x^2}dxdy + \int_{F_5} \frac{|w|^2}{1-y^2}dxdy. \tag{33}$$

The idea of the proof is to bound each of these terms by the L^2-norm of u over an interval on $\partial\Omega_{ref}$ parallel to one of the axes and then use Lemma 5.3 to complete the argument.

i) Consider the first term in (33). By a trace theorem and Lemma 5.6,

$$|\tilde{I}^W u|^2_{H^{1/2}(F_5)} \leq C|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} \leq C\|u\|^2_{L^2(W)}.$$

Again by a trace theorem and Lemma 5.3, we then obtain

$$|w|^2_{H^{1/2}(F_5)} \leq 2(|u|^2_{H^{1/2}(F_5)} + |\tilde{I}^W u|^2_{H^{1/2}(F_5)}) \leq C(1+\log p)\|u\|^2_{H^1(\Omega_{ref})}.$$

ii) In order to bound the second term of (33), we divide $w = w^I + w^{II}$ into two parts, which both vanish on the two opposite edges E_1 and E_3:

$$\begin{aligned} w^I = \quad & u - [u(V_1)\varphi_0(x)\varphi_0(y) + u(V_2)\varphi_0^-(x)\varphi_0(y) + u(V_3)\varphi_0(x)\varphi_0^-(y) + u(V_4)\varphi_0^-(x)\varphi_0^-(y))] \\ & - \textstyle\sum_j \alpha_j^{(1)}\varphi_j(x)\Phi_j(y) - \sum_j \alpha_j^{(3)}\varphi_j^-(x)\Phi_j(y) \\ w^{II} = \quad & - \textstyle\sum_j \alpha_j^{(2)}\Phi_j(x)\varphi_j(y) - \sum_j \alpha_j^{(4)}\Phi_j(x)\varphi_j^-(y). \end{aligned}$$

We now bound the first term of w^{II}, associated with the edge E_2, by using the two equivalent characterizations of the $H_{00}^{1/2}(E_2)$- norm given by (11) and (24):

$$\int_{F_5} \frac{|\sum_j \alpha_j^{(2)}\Phi_j(x)\varphi_j(y)|^2}{1-x^2}dxdy = \int_{-1}^1 (\int_{-1}^1 \frac{|\sum_j \alpha_j^{(2)}\Phi_j(x)\varphi_j(y)|^2}{1-x^2}dx)dy$$

$$\leq \int_{-1}^{1} \| \sum_j \alpha_j^{(2)} \Phi_j(\cdot)\varphi_j(y)\|^2_{H_{00}^{1/2}(-1,1)} dy \leq C \int_{-1}^{1} \sum_j (\alpha_j^{(2)})^2 \varphi_j^2(y) \frac{1}{\sqrt{\lambda_j^{(p)}}} dy$$

$$\leq C \sum_j (\alpha_j^{(2)})^2 \frac{1}{\lambda_j^{(p)}} = C \| \sum_j \alpha_j^{(2)} \Phi_j \|^2_{L^2(E_2)} = C\|u\|^2_{L^2(E_2)}.$$

In the same way, we bound the second term of w^{II}, associated with the edge E_4:

$$\int_{F_5} \frac{|\sum_j \alpha_j^{(4)} \Phi_j(x)\varphi_j^-(y)|^2}{1-x^2} dx dy \leq C\|u\|^2_{L^2(E_4)}.$$

To obtain a bound for w^I, we divide the integral with respect to x into three:

$$\int_{F_5} \frac{|w^I(x,y)|^2}{1-x^2} dx dy = \int_{-1}^{1} \frac{\|w^I(x,\cdot)\|^2_{L^2(-1,1)}}{1-x^2} dx = \int_{-1}^{-1+\epsilon} + \int_{-1+\epsilon}^{1-\epsilon} + \int_{1-\epsilon}^{1} \tag{34}$$

and choose $\epsilon = 2/p^2$. The second integral is easily bounded by

$$\int_{-1+\epsilon}^{1-\epsilon} \frac{\|w^I(x,\cdot)\|^2_{L^2(-1,1)}}{1-x^2} dx \leq \max_x \|w^I(x,\cdot)\|^2_{L^2(-1,1)} \int_{-1+\epsilon}^{1-\epsilon} \frac{dx}{1-x^2}$$

$$\leq 2\log p \max_x \|w^I(x,\cdot)\|^2_{L^2(-1,1)},$$

since $\int_{-1+\epsilon}^{1-\epsilon} \frac{dx}{1-x^2} = \log(\frac{2}{\epsilon}-1) = \log(p^2-1) \leq 2\log p$.
Moreover, $\|w^I(x,\cdot)\|^2_{L^2(-1,1)}$ can be bounded in terms of $\|u(x,\cdot)\|^2_{L^2(-1,1)}$:

$$\begin{aligned} &\|w^I(x,\cdot)\|^2_{L^2(-1,1)} \leq 3(\|u(x,\cdot)\|^2_{L^2(-1,1)} \\ &+\| \textstyle\sum_j \alpha_j^{(1)} \varphi_j(x)\Phi_j(\cdot) + u(V_1)\varphi_0(x)\varphi_0(\cdot) + u(V_4)\varphi_0^-(x)\varphi_0^-(\cdot)\|^2_{L^2(-1,1)} \\ &+\| \textstyle\sum_j \alpha_j^{(3)} \varphi_j^-(x)\Phi_j(\cdot) + u(V_2)\varphi_0^-(x)\varphi_0(\cdot) + u(V_3)\varphi_0(x)\varphi_0^-(\cdot)\|^2_{L^2(-1,1)}). \end{aligned} \tag{35}$$

Because of the L^2-orthogonality of Φ_j and φ_0, we find that

$$\int_{-1}^{1} [\sum_j \alpha_j^{(1)} \varphi_j(x)\Phi_j(y) + u(V_1)\varphi_0(x)\varphi_0(y) + u(V_4)\varphi_0^-(x)\varphi_0^-(y)]^2 dy$$

$$= \varphi_j^2(x) \int_{-1}^{1} [\sum_j \alpha_j^{(1)} \Phi_j(y)]^2 dy + \max\{\varphi_0(x), \varphi_0^-(x)\} \int_{-1}^{1} [u(V_1)\varphi_0(y) + u(V_4)\varphi_0^-(y)]^2 dy$$

$$\leq \|u(1,\cdot)\|^2_{L^2(E_1)}.$$

Here we have used the fact that $\|\varphi_j\|_{L^\infty} \leq 1$ and $\|\varphi_0\|_{L^\infty} \leq 1$; see Lemma 4.1. In the same way, we can bound the last term of (35) by $\|u(-1,\cdot)\|^2_{L^2(E_3)}$. Therefore

$$\int_{-1+\epsilon}^{1-\epsilon} \frac{\|w^I(x,\cdot)\|^2_{L^2(-1,1)}}{1-x^2} dx \leq C \log p \max_{x\in[-1,1]} \|u(x,\cdot)\|^2_{L^2(-1,1)}.$$

We now consider the first integral in (34). Since $\|w^I(x,\cdot)\|_{L^2(-1,1)}$ is still a polynomial of degree p in x, we can obtain an estimate by using the mean value theorem and Lemma 5.1.

$$\int_{-1}^{-1+\epsilon} \frac{\|w^I(x,\cdot)\|^2_{L^2(-1,1)}}{1-x^2}dx = \int_{-1}^{-1+\epsilon} (\frac{d}{dx}\|w^I(\tilde{x},\cdot)\|_{L^2(-1,1)})^2\frac{(1+x)^2}{(1-x^2)}dx$$

$$\leq p^4 \max_x \|w^I(x,\cdot)\|^2_{L^2(-1,1)} \int_{-1}^{-1+\epsilon}(1+x)dx \leq \max_x \|w^I(x,\cdot)\|^2_{L^2(-1,1)} p^4 \frac{\epsilon^2}{2}$$

$$\leq C \max_x \|u(x,\cdot)\|^2_{L^2(-1,1)}.$$

We bound the last integral in (34) in the same way:

$$\int_{1-\epsilon}^{1} \frac{\|w^I(x,\cdot)\|^2_{L^2(-1,1)}}{1-x^2}dx \leq C \max_x \|u(x,\cdot)\|^2_{L^2(-1,1)},$$

and have proved that

$$\int_{F_5} \frac{|w(x,y)|^2}{1-x^2}dxdy \leq C(1+\log p)\max_x \|u(x,\cdot)\|^2_{L^2(-1,1)}.$$

iii) Estimates for the third term in (33) are the same as for the second term after exchanging x and y. We now divide $w = w^I + w^{II}$ into two parts vanishing on the two opposite edges E_2 and E_4 of F_5 (see Figure 7):

$$\begin{aligned} w^I = \; & u - [u(V_1)\varphi_0(x)\varphi_0(y) + u(V_2)\varphi_0^-(x)\varphi_0(y) + u(V_3)\varphi_0(x)\varphi_0^-(y) + u(V_4)\varphi_0^-(x)\varphi_0^-(y))] \\ & - \textstyle\sum_j \alpha_j^{(2)}\Phi_j(x)\varphi_j(y) - \sum_j \alpha_j^{(4)}\Phi_j(x)\varphi_j^-(y) \\ w^{II} = \; & - \textstyle\sum_j \alpha_j^{(1)}\varphi_j(x)\Phi_j(y) - \sum_j \alpha_j^{(3)}\varphi_j^-(x)\Phi_j(y). \end{aligned}$$

As in step ii), we can prove that

$$\int_{F_5} \frac{|w(x,y)|^2}{1-y^2}dxdy \leq C(1+\log p)\max_y \|u(\cdot,y)\|^2_{L^2(-1,1)}.$$

We conclude by combining the estimates from step i), ii), and iii) and by applying Lemma 5.3 and obtain

$$\|u - \tilde{I}^W u\|^2_{H^{1/2}_{00}(F_5)} \leq C(1+\log p)(\max_x \|u(x,\cdot)\|^2_{L^2(-1,1)} + \max_y \|u(\cdot,y)\|^2_{L^2(-1,1)})$$

$$\leq C(1+\log p)^2\|u\|^2_{H^1(\Omega_{ref})}.$$

$\square$

LEMMA 5.8. *Let $\mathcal{F}_k$ be the special face function for the face F_k, defined in Subsection 4.4. Then*

$$\|\mathcal{F}_k\|^2_{H^{1/2}_{00}(F_k)} \leq C(1+\log p).$$

Proof. On F_k, $\mathcal{F}_k = 1 - \tilde{I}^W 1$. Therefore we can repeat the proof of Lemma 5.7 with $u \equiv 1$ on $\partial\Omega_{ref}$. We are now able to avoid the second $(1+\log p)$ factor.

□

A bound for the full interpolation error $u - I^W u$, and of u_{F_k}, is now obtained by combining Lemmas 5.7, 5.8, (29), and Poincaré's inequality:

LEMMA 5.9.

$$\|u_{F_k}\|^2_{H^{1/2}_{00}(F_k)} = \|u - I^W u\|^2_{H^{1/2}_{00}(F_k)} \leq C(1+\log p)^2 |u|^2_{H^1(\Omega_{ref})}.$$

5.2. Proof of main theorem. We are now in a position to prove Theorem 3.1. In order to obtain a bound for the condition number $\kappa(T)$, we provide an upper and lower bound for the Rayleigh quotient (1). To this end, it is enough to prove the local bounds

$$\frac{\sum_i \tilde{a}^{(j)}_i(u_i,u_i)}{C_1(1+\log p)^2} \leq a^{(j)}(u,u) \leq C_2 \sum_i \tilde{a}^{(j)}_i(u_i,u_i) \tag{36}$$

$$\forall u \in \tilde{V}^p, \qquad u = \sum_{i=0}^{N} u_i, \qquad u_i \in V_i,$$

where $\tilde{V}^p$ is the subspace of discrete harmonic functions of V^p and $a^{(j)}(u,v)$ is the contribution to the bilinear form $a(u,v)$ from the substructure Ω_j, etc. We refer to Section 2 for the necessary background.

i) The lower bound follows from

$$(1+\log p)\|u - \overline{u}_W\|^2_{L^2(W)} + \sum_{k=1}^{6} \|u_{F_k}\|^2_{H^{1/2}_{00}(F_k)} \leq C_1(1+\log p)^2 |u|^2_{H^1(\Omega_{ref})}.$$

The required estimates are provided by Lemmas 5.3 and 5.9.

ii) If we shift u by a constant such that $\overline{u}_W = 0$, then the upper bound in (36) is equivalent to

$$|u|^2_{H^1(\Omega_{ref})} \leq C_2((1+\log p)\|u\|^2_{L^2(W)} + \sum_{k=1}^{6} \|u_{F_k}\|^2_{H^{1/2}_{00}(F_k)}).$$

This bound is obtained by applying Corollary 5.5, Lemmas 5.6, 5.8, and (29):

$$|u|^2_{H^1(\Omega_{ref})} = |\sum_{k=1}^{6} u_{F_k} + I^W u|^2_{H^1(\Omega_{ref})}$$

$$\leq 7(\sum_{k=1}^{6} |u_{F_k}|^2_{H^1(\Omega_{ref})} + |I^W u|^2_{H^1(\Omega_{ref})})$$

$$\leq C(\sum_{k=1}^{6} \|u_{F_k}\|^2_{H^{1/2}_{00}(F_k)} + (1+\log p)\|u\|^2_{L^2(W)}).$$

$\square$

Remark. If we do not scale the part of the preconditioner corresponding to the wire basket with a factor proportional to $(1+\log p)$, then, we obtain a bound of a condition number proportional to $(1+\log p)^3$.

6. Matrix form of the preconditioner. In this section, we will first consider the matrix representation of the stiffness matrix and the preconditioner for the special basis that has been used in our analysis and then, in a separate subsection, consider the changes needed in a much more general case.

6.1. The case of the special basis. We order the interior basis functions first, and then those related to the faces, and finally the wire basket basis functions. For the time being, we use the basis functions introduced in Subsection 4.3, i.e. those introduced prior to the corrections with the components constructed from the special functions $\mathcal{F}_k$.

Since all face and wire basket basis functions are discrete harmonic, the stiffness matrix is of the form

$$\begin{pmatrix} D_I & 0 & 0 \\ 0 & S_{FF} & S_{FW} \\ 0 & S_{FW}^T & S_{WW} \end{pmatrix} .$$

The leading block D_I, which corresponds to the interior spaces of all the elements, is diagonal, because the Φ_i are H^1-orthonormal. The diagonal element corresponding to the interior basis function $\Phi_i\Phi_j\Phi_k$ is equal to

$$|\Phi_i\Phi_j\Phi_k|^2_{H^1} = \frac{\lambda_i+\lambda_j+\lambda_k}{\lambda_i\lambda_j\lambda_k}.$$

We note that, with all other basis functions discrete harmonic, the submatrix associated with the interface unknowns already forms a Schur complement S. The contribution to S, from the element Ω_j, can be written as

$$S^{(j)} = \begin{pmatrix} S_{FF}^{(j)} & S_{FW}^{(j)} \\ S_{FW}^{(j)T} & S_{WW}^{(j)} \end{pmatrix} .$$

The preconditioner $\widehat{S}$, for the reduced system of interface variables, is obtained by subassembling local contributions $\widehat{S}^{(j)}$, constructed independently element by element. We first change the basis of the wire basket space introducing the edge and vertex basis functions defined by (25), while keeping the face basis functions the same. The transformation matrix from the old to the new basis takes the form

$$\begin{pmatrix} I & 0 \\ R^{(j)} & I \end{pmatrix}.$$

$S^{(j)}$ is then transformed into

$$\begin{pmatrix} I & 0 \\ R^{(j)} & I \end{pmatrix} \begin{pmatrix} S_{FF}^{(j)} & S_{FW}^{(j)} \\ S_{FW}^{(j)^T} & S_{WW}^{(j)} \end{pmatrix} \begin{pmatrix} I & R^{(j)^T} \\ 0 & I \end{pmatrix} = \begin{pmatrix} S_{FF}^{(j)} & \text{nonzero} \\ \text{nonzero} & \widetilde{S}_{WW}^{(j)} \end{pmatrix}.$$

We construct the local preconditioner $\widehat{S}^{(j)}$ by:

• eliminating the coupling between face and wire basket spaces;

• replacing $S_{FF}^{(j)}$ by its block diagonal part $\widehat{S}_{FF}^{(j)}$ with one block for each face. In our special basis $\widehat{S}_{FF}^{(j)}$ is diagonal because each pair of basis functions, associated with the same face, are $a(\cdot,\cdot)-$orthogonal;

• replacing the wire basket block $\widetilde{S}_{WW}^{(j)}$ by a much simpler matrix $\widehat{S}_{WW}^{(j)}$ that corresponds to the special bilinear form $\tilde{a}_0(\cdot,\cdot)$ chosen for the wire basket space.

In the h-version algorithm of Smith [50], $\widehat{S}_{WW}^{(j)}$ is just a rank-one perturbation of a multiple of the identity matrix. In the case of our special basis, we begin by considering the mass matrix M associated with the L^2-norm over the wire basket; $\widehat{S}_{WW}^{(j)}$ will be a rank-one perturbation of the diagonal part D of this mass matrix. We will now show that the mass matrix is increasingly diagonally dominant, with increasing p, and always spectrally equivalent to its diagonal part D.

Let, for the time being, W denote the wire basket of the reference element. Let further $\underline{z}$ be the vector of wire basket coefficients of the constant function 1, and let $\underline{u}_W$ be that of u. Then, the mass matrix M is defined by

$$\underline{u}_W^T M \underline{u}_W = \|u\|_{L^2(W)}^2 .$$

We also find, by a simple computation, that

$$\inf_c \|u-c\|_{L^2(W)}^2 = \int_W u^2 ds - \frac{(\int_W u ds)^2}{\int_W ds} = \underline{u}_W^T \Big(M - \frac{(M\underline{z})\cdot(M\underline{z})^T}{\underline{z}^T M \underline{z}}\Big)\underline{u}_W .$$

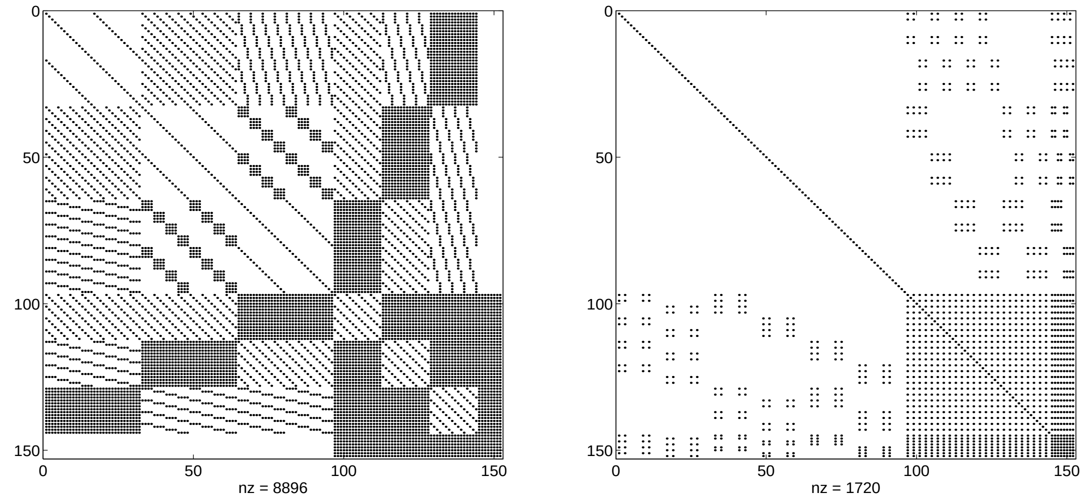
The wire basket block $\widehat{S}_{WW}$ of the preconditioner, is obtained by replacing the matrix M by its diagonal D and by introducing an appropriate scaling factor:

$$\widehat{S}_{WW} = C(1+\log p)\Big(D - \frac{(D\underline{z})\cdot(D\underline{z})^T}{\underline{z}^T D\underline{z}}\Big) . \tag{37}$$

The matrix M is almost diagonal, because all the edge basis functions are $L^2(W)-$orthogonal. The edge and vertex basis functions are also orthogonal to each other, but the vertex basis functions are not mutually orthogonal. However, by Lemma 4.2 and a computation similar to that of the proof of Lemma 5.6, we have

$$3\Big(1-\frac{1}{p+1}\Big)\sum_{k=1}^{8} u^2(V_k)\|\varphi_0\|_{L^2(-1,1)}^2 \le \|\sum_{k=1}^{8} u(V_k)v^{(k)}\|_{L^2(W)}^2$$

FIG. 8. *Sparsity pattern of $S^{(j)}$ (left) and of $\widehat{S}^{(j)}$ (right), for $p = 5$.*



$$\leq 3(1 + \frac{1}{p+1}) \sum_{k=1}^{8} u^2(V_k) \|\varphi_0\|^2_{L^2(-1,1)}.$$

Therefore, for $p \geq 1$,

$$\frac{3}{2} \underline{u}_W^T D \underline{u}_W \leq \underline{u}_W^T M \underline{u}_W \leq \frac{9}{2} \underline{u}_W^T D \underline{u}_W \ .$$

This completes our description of $\tilde{a}_0(\cdot, \cdot)$, the bilinear form for the wire basket space.

We can now return to the old basis:

$$\widehat{S}^{(j)} = \begin{pmatrix} I & 0 \\ -R^{(j)} & I \end{pmatrix} \begin{pmatrix} \widehat{S}^{(j)}_{FF} & 0 \\ 0 & \widehat{S}^{(j)}_{WW} \end{pmatrix} \begin{pmatrix} I & -R^{(j)T} \\ 0 & I \end{pmatrix}. \tag{38}$$

Since the actions of $R^{(i)}$ and $R^{(j)}$ on the common face F_{ij} are the same, the preconditioner can be obtained by subassembly:

$$\widehat{S} = \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} \begin{pmatrix} \widehat{S}_{FF} & 0 \\ 0 & \widehat{S}_{WW} \end{pmatrix} \begin{pmatrix} I & -R^T \\ 0 & I \end{pmatrix}.$$

Therefore

$$\widehat{S}^{-1} S = \sum_i R_{F_i} \widehat{S}^{-1}_{F_i F_i} R^T_{F_i} S + R_0 \widehat{S}^{-1}_{WW} R_0^T S,$$

where $R_0 = (R, I)$; see Dryja, Smith, and Widlund [20]. We have thus obtained an additive preconditioner, with independent parts associated with each face and the wire basket; see Fig. 8 for the sparsity patterns of $S^{(j)}$ and the preconditioner $\widehat{S}^{(j)}$ using our basis functions.

6.2. Other bases. The basis functions used in our proof of Theorem 3.1, and in the previous subsection, are not necessarily ideal in computational practice. There are, e.g., a number of advantages of using a hierarchical basis in particular if the order of the elements are determined only during a run by using a posteriori error bounds. Tensor products of integrated Legendre polynomials have been used for such purposes; see Babuška, Craig, Mandel, and Pitkäranta [1] and Mandel [36]–[41]. Another basis, given in terms of fundamental Lagrange interpolating polynomials and the Gauss-Lobatto-Legendre quadrature points, also have their strong advocates; see Bernardi and Maday [7] and Maday and Patera [35].

We can use the fact that our algorithm is defined primarily by our subspaces and bilinear forms. We will now explore how to use our algorithm given any basis that, restricted to Ω_{ref}, can be decomposed into an interior, six face, and a wire basket subspace. We could of course make a complete change of basis but here we are instead interested in trying to keep the given bases of the subspaces to the extent possible.

Let us assume that the element stiffness matrix $K^{(i)}$ has already been computed. We note that, whatever the basis of the interior subspace, we can factor the submatrix $K_{II}^{(i)}$ which corresponds to the interior variables using the Choleski algorithm. The resulting triangular factor implicitely introduces a new orthogonal basis of the interior subspace.

These Choleski factors can also be used to compute the Schur complement which corresponds to the interface variables. This effectively produces new iterface basis functions which are discrete harmonic. During this process, all the face and wire basket spaces maintain their identity; for each of the original face subspaces, there is a well defined discrete harmonic counterpart. The values of the interior basis functions are not changed during this process nor are the values on $\partial\Omega_{ref}$ of the interface basis functions.

For each basis function of any given edge space, we can compute a corresponding element in our special edge spaces using the following idea. We first solve two discrete, two-dimensional elliptic problems, which are the discrete counter parts of (10). The Dirichlet boundary conditions for each of these problems is given by the basis element on the given edge, continued by zero to the rest of the boundary of the face. The identity of the edge space is maintained since the new basis functions remain zero on the same four faces as before; they are also discrete harmonic, being linear combinations of discrete harmonic functions.

The vertex basis functions (22), which have been introduced in this paper, are defined in terms of a simple variational problem. φ_0 can be expressed in terms of elements of the current edge space by solving this variational problem. The resulting formulas form part of the mapping between the two bases. Similarly, we need to orthogonalize the edge functions in the L^2 sense in order to design an effective bilinear form $\tilde{a}_0(\cdot,\cdot)$.

We also need to write the special functions $\mathcal{F}_j$ in terms of the basis at hand; they can be found using exactly the same recipe as before.

Given the resulting linear transformations, representing changes of bases between the given and the special sets of subspaces, we can again express the contribution to the preconditioner from an individual element, and the entire preconditioner, in terms

TABLE 1
Local condition numbers with the wire basket block $\widehat{S}_{WW}$

p	2	3	4	5	6	7	8	9	10	11
optimal $\kappa(\widehat{S}^{-1}S)$	7.14	11.83	14.41	16.16	19.17	20.76	23.30	24.66	26.82	28.00
λ_{max}	2.211	2.345	2.262	2.230	2.220	2.192	2.176	2.188	2.174	2.183
λ_{min}	0.309	0.198	0.157	0.138	0.116	0.106	0.093	0.089	0.081	0.078
natural $\kappa(\widehat{S}^{-1}S)$	8.36	12.39	14.67	16.35	19.24	20.76	23.31	24.69	26.89	28.10
λ_{max}	1.792	1.977	2.055	2.095	2.137	2.181	2.204	2.236	2.250	2.272
λ_{min}	0.214	0.159	0.140	0.128	0.111	0.105	0.095	0.091	0.084	0.081

TABLE 2
Local condition numbers with the original wire basket block $\tilde{S}_{WW}$

p	2	3	4	5	6	7	8	9	10	11
$\kappa(\widehat{S}^{-1}S)$	5.33	7.21	9.98	12.08	14.20	15.96	17.74	19.23	20.79	22.08
λ_{max}	1.736	1.839	1.954	1.985	2.021	2.042	2.060	2.072	2.083	2.091
λ_{min}	0.326	0.255	0.196	0.164	0.142	0.128	0.116	0.108	0.100	0.095

of triangular matrices and the block diagonal matrix given in the right hand side of formula (38).

7. A numerical study of the condition number. As we have previously pointed out, an upper bound for the condition number of the problem on all of Ω can be obtained by considering a preconditioner for a local problem on the reference element. It is therefore possible to compute this bound solely from the eigenvalues of a matrix pencil defined by the contributions to the stiffness matrix, and the preconditioner, from an individual element. Both of these matrices are singular with the same null space; only the space orthogonal to this one-dimensional space is relevant in our analysis. We have carried out a series of MATLAB 4.0 experiments, which closely parallel similar work by Smith [51] for the case of piecewise linear elements. We note that Smith also reports on full-scale experiments on multi processor systems with his algorithm. In our tables, S denotes the part of the stiffness matrix attributable to the discrete harmonic part of the space and $\widehat{S}$ is its preconditioner. It follows from the general theory for iterative substructuring methods that $\kappa(\widehat{S}^{-1}S)$, the ratio of the extreme non-zero generalized eigenvalues, provides an upper bound for the condition number of the entire preconditioned operator.

In Table 1, we provide the local condition numbers and extreme non-zero eigenvalues of $\widehat{S}^{-1}S$, where the wire basket block $\widehat{S}_{WW}$ is the scaled rank-one perturbation of D given by (37). We consider two choices of a scale factor $\delta(p)$ of the bilinear form $\tilde{a}_0(\cdot,\cdot)$. We refer to them as the optimal and natural scalings. In the first case, we determine, for each degree p, the optimal scaling $\delta(p) = C(1 + \log p)$ by minimizing over C. In the second case, we use $\delta(p) = 1$; this is the natural scaling. We conclude, just as Smith,

TABLE 3
Least square approximation of $\kappa(\widehat{S}^{-1}S)$ in Table 1

optimal scaling	a_0	a_1	a_2	a_3	$\|error\|_{L^\infty}$	$\|error\|_{L^2}$
linear	-2.102	12.184			1.35	2.34
quadratic	3.179	4.575	2.406		0.72	1.22
cubic	-2.444	17.710	-6.730	1.956	0.62	0.98
natural scaling	a_0	a_1	a_2	a_3	$\|error\|_{L^\infty}$	$\|error\|_{L^2}$
linear	-0.811	11.590			1.49	2.80
quadratic	5.918	1.894	3.066		0.69	1.17
cubic	0.457	14.649	-5.806	1.900	0.57	0.94

TABLE 4
Least square approximation of $\kappa(\widehat{S}^{-1}S)$ in Table 2

	a_0	a_1	a_2	a_3	$\|error\|_{L^\infty}$	$\|error\|_{L^2}$
linear	-3.345	10.181			1.62	2.59
quadratic	3.445	0.397	3.095		0.26	0.34
cubic	5.553	-4.528	6.520	-0.734	0.13	0.21

that the natural scaling is only slightly worse than the optimal. We have also considered a scale factor of the form $C_1 + C_2 \log p$ and minimized over C_1 and C_2. The resulting condition numbers are very close to those of the optimal ones in Table 1 and we have found that $C_1 \approx C_2 \approx C$.

In Table 2, we give the local condition numbers and extreme non-zero eigenvalues of $\widehat{S}^{-1}S$, obtained when keeping the original wire basket block $\tilde{S}_{WW}$. This corresponds to using the original bilinear form $a(\cdot,\cdot)$ on the wire basket space. The resulting preconditioner is computationally less interesting than the previous one, because it requires the assembly and solution of a coarse problem with a much less sparse matrix.

It is interesting to note that the condition number estimates obtained with this original wire basket block quite closely approach those obtained by Smith [51] for the traditional seven point finite difference scheme, with the same number of degrees of freedom, as the size of the local problem increases. Thus, for the cases which correspond to $p = 7, 8$ and 9, Smith reports condition numbers of $15.86, 17.59$ and 19.23, respectively.

A least square approximation of the data of Tables 1 and 2 indicates a $\log^2 p$ growth of the condition numbers. The coefficients a_i of the linear ($n = 1$), quadratic ($n = 2$) or cubic ($n = 3$) least square approximation $f_n(p) = \sum_{i=0}^{n} a_i(\log p)^i$, are given in Tables 3 and 4. We note that the introduction of the additional parameter a_3 does not appreciably improve the fit and that in one case $a_3 < 0$.

8. Appendix: Proof of Lemma 4.1. We recall that $\varphi_0(x) \in P^p$, that $\varphi_0(1) = 1$, $\varphi_0(-1) = 0$, and that this polynomial has a minimal L^2-norm. Any polynomial satisfying these boundary conditions can be represented as

$$\frac{1+x}{2}[1-2(1-x)q(x)], \quad q(x) \in P^{p-2}.$$

We expand $2q(x)$ in the Legendre basis

$$2q(x) = \sum_{n=0}^{p-2} \alpha_n L_n(x).$$

Using the classical formula

$$xL_n(x) = \frac{n+1}{2n+1} L_{n+1}(x) + \frac{n}{2n+1} L_{n-1}(x), \tag{39}$$

we find that

$$1-2(1-x)q(x) = \sum_{n=0}^{p-1} t_n L_n.$$

Here $t_0 = 1 - \alpha_0 + \frac{1}{3}\alpha_1$ and

$$t_n = \frac{n}{2n-1}\alpha_{n-1} - \alpha_n + \frac{n+1}{2n+3}\alpha_{n+1}, \qquad n = 1, \cdots, p-1, \tag{40}$$

with the convention $\alpha_{-1} = \alpha_{p-1} = \alpha_p = 0$. Applying (39) once more, we obtain

$$2\varphi_0(x) = (1+x)[1-2(1-x)q(x)] = \sum_{n=0}^{p} r_n L_n(x),$$

where

$$r_n = \frac{n}{2n-1} t_{n-1} + t_n + \frac{n+1}{2n+3} t_{n+1}, \tag{41}$$

with the convention $t_{-1} = t_p = t_{p+1} = 0$. Therefore

$$\frac{1}{2}\|2\varphi_0\|^2_{L^2(-1,1)} = r_0^2 + r_1^2\frac{1}{3} + r_2^2\frac{1}{5} + \cdots + r_p^2\frac{1}{2p+1}.$$

Substituting (40) into (41), the r_n's can be expressed in terms of the α_n's :

$$\begin{aligned} r_n = \frac{n}{2n-1}\Big(\frac{n-1}{2n-3}\alpha_{n-2} & \quad -\alpha_{n-1} & \quad +\frac{n}{2n+1}\alpha_n\Big) \\ +\frac{n}{2n-1}\alpha_{n-1} & \quad -\alpha_n & \quad +\frac{n+1}{2n+3}\alpha_{n+1} \\ +\frac{n+1}{2n+3}\Big(\frac{n+1}{2n+1}\alpha_n & \quad -\alpha_{n+1} & \quad +\frac{n+2}{2n+5}\alpha_{n+2}\Big) = \end{aligned}$$

$$\frac{n(n-1)}{(2n-1)(2n-3)}\alpha_{n-2} + \Big(\frac{n^2}{(2n-1)(2n+1)} - 1 + \frac{(n+1)^2}{(2n+1)(2n+3)}\Big)\alpha_n$$

$$+\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\alpha_{n+2}.$$

This system decouples: the r_n's with even n depend only on the α_n's with even n and the r_n's with odd n depend only on the α_n's with odd n. Since all derivatives with respect to the α_i's vanish at the minimum, a simple calculation and an induction argument lead to the formula

$$r_n = \begin{cases} (2n+1)r_0 & \text{if } n \text{ is even} \\ (2n+1)r_1/3 & \text{if } n \text{ is odd} \end{cases} \tag{42}$$

In order to find the value of r_0, we write the system for the even r_n's as

$$\begin{array}{lcr}
r_0 = & 1-\frac{2}{3}\alpha_0+\frac{2}{15}\alpha_2 & = r_0 \\
r_2 = & \frac{2}{3}\alpha_0-\frac{10}{21}\alpha_2+\frac{4}{21}\alpha_4 & = 5r_0 \\
\vdots & \vdots & \vdots \\
r_n = & \frac{n(n-1)}{(2n-1)(2n-3)}\alpha_{n-2}+[\frac{n^2}{(2n-1)(2n+1)}-1+\frac{(n+1)^2}{(2n+1)(2n+3)}]\alpha_n + & \\
 & +\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\alpha_{n+2} & = (2n+1)r_0 \\
\vdots & \vdots & \vdots \\
r_p = & \frac{p(p-1)}{(2p-1)(2p-3)}\alpha_{p-2} & = (2p+1)r_0
\end{array}$$

By adding all these equations, we obtain for p even

$$1 = r_0(1+5+9+\cdots+(4\frac{p}{2}+1)) = r_0\frac{(p+1)(p+2)}{2}. \tag{43}$$

In the same way, adding the equations of the system for the odd r_n's,

$$1 = \frac{r_1}{3}(3+7+11+\cdots+(4\frac{p-1}{2}+1)) = \frac{r_1}{3}\frac{p(p+1)}{2}. \tag{44}$$

If p is odd, the roles of r_0 and $r_1/3$ in (43) and (44) are reversed.

By (42), we now have found the components r_n of φ_0 in the Legendre basis. Moreover,

$$\frac{1}{2}\|2\varphi_0\|^2_{L^2(-1,1)} = r_0^2(1+5+9+\cdots+(2p+1)) + \frac{r_1^2}{3^2}(3+7+11+\cdots+(2(p-1)+1)) =$$

$$= r_0+\frac{r_1}{3} = \frac{2}{(p+1)(p+2)}+\frac{2}{p(p+1)} = \frac{4}{p(p+2)}$$

and thus $\|\varphi_0\|^2_{L^2(-1,1)} = \frac{2}{p(p+2)}$. Finally, in order to prove that $|\varphi_0(x)| \le 1$, we use (43) and (44) to rewrite r_0 and $r_1/3$ as

$$r_0 = \frac{1}{\sum_{even\ k=0}^{p}(2k+1)}, \qquad \frac{r_1}{3} = \frac{1}{\sum_{odd\ k=1}^{p}(2k+1)}$$

and recall that the Legendre polynomials satisfy $|L_n(x)| \le 1, x \in [-1,1]$. Thus

$$|\varphi_0(x)| \le \frac{1}{2}\sum_n |r_n||L_n(x)| \le \frac{1}{2}\sum_n r_n = \frac{1}{2}(\sum_{even\ n}(2n+1)r_0 + \sum_{odd\ n}(2n+1)\frac{r_1}{3}) = 1.$$

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