Domain Decomposition Algorithms for the p-version Finite Element Method for Elliptic Problems

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Abstract

Domain decomposition algorithms based on the Schwarz framework were originally proposed for the h-version finite element method for elliptic problems. In this thesis, we study some Schwarz algorithms for the p-version finite element method, in which increased accuracy is achieved by increasing the degree p of the elements while the mesh is fixed. These iterative algorithms, often of conjugate gradient type, are both parallel and scalable, and therefore very well suited for massively parallel computing.

We consider linear, scalar, self adjoint, second order elliptic problems and quadrilateral elements in the finite element discretization. For a class of overlapping methods, we prove a constant bound, independent of the degree p, the mesh size H and the number of elements N, for the condition number of the iteration operator. This optimal result holds in two and three dimensions for additive and multiplicative schemes, as well as variants on the interface.

We consider then local refinement for the same class of overlapping methods in two dimensions. Optimal bounds are obtained under certain hypothesis on the choice of refinement points, while in general almost optimal bounds with logarithmic growth in p are obtained. In the analysis of these local refinement methods, we prove some results of independent interest, such as a polynomial discrete Sobolev inequality and a bounded decomposition of discrete harmonic polynomials.

Iterative subtructuring methods in two dimensions are also considered. We use the additive Schwarz framework to prove almost optimal bounds as in the h-version finite element method.

Results of numerical experiments, confirming the theoretical results, are conducted in two dimensions for model problems.

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I dedicate this thesis to my parents.

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Chapter 1

Introduction

1.1 Overview

Domain decomposition is a major focus of contemporary research in numerical analysis of partial differential equations, see the proceedings of the international conferences held every year since 1987; [42], [29], [30], [43], [31], [71]. Among the reasons for considering domain decomposition are parallel computing, modeling of different physical phenomena in different subregions and complicated geometries.

In this thesis, we restrict our attention to boundary value problems for self-adjoint elliptic PDEs. When discretized by the finite element method, elliptic problems lead to large and sparse linear systems. In industrial practice, these systems are often solved by direct methods. However, the use of iterative methods like the conjugate gradient can offer real advantages. Domain decomposition methods are iterative methods, often of conjugate gradient type, based on the idea of decomposing the domain into smaller subdomains. In each iteration, the original problem is restricted to the subdomains and the subproblems obtained are solved in parallel, approximately or exactly. An additional global coarse subproblem must be introduced in order to obtain global transportation of information and fast convergence in case of many subdomains. These algorithms can therefore take advantage of the new generations of parallel computers, see Gropp [47].

In recent years, a general variational framework has been developed to construct and analyze domain decomposition methods in terms of subspaces and projections, see Dryja and Widlund [36], [39] and Widlund [87]. This allows a unified analysis of both Schwarz methods (where there is overlap of the subregions) and substructuring methods

(where there is no overlap). In particular, the so called additive Schwarz method allows us to analyze interesting iterative substructuring methods without using finite element extension theorems.

In this thesis, we apply the Schwarz framework to construct and analyze domain decomposition methods for the p-version finite element discretization of elliptic problems. In the standard finite element methods, increased accuracy is obtained by refining the mesh and using fixed low order polynomials on each element. This is the h-version finite element method. In the p-version, we instead keep the mesh fixed and increase the accuracy by increasing the degree p of each element. The hp-version combines the two approaches. When the exact solution of the continuous problem is smooth, the p and hp-version have convergence rates that are more than algebraic and exponential, respectively. If the solution is not smooth, the convergence rate is at least the same as that of the h-version and it is faster if the singularities are on interelement boundaries, which is often the case of interest in applications, see Babuška [1], Babuška and Szabo [7], Babuška and Elman [2].

In this introductory chapter, we review some basic definitions and results about Sobolev spaces, the p-version finite element method and iterative methods for linear systems of equations.

In Chapter 2, we describe first the classical alternating Schwarz method and then the abstract Schwarz theory in Hilbert spaces.

In Chapter 3, we introduce our first algorithm for the p-version finite element method. It is an additive Schwarz method (ASM) which uses overlapping subregions associated with each interior mesh point. We prove that the condition number of the iteration operator of this method is bounded by a constant independent of the degree p, the mesh size H and the number of subdomains N. This optimal bound guarantees that the method is parallelizable and scalable. The same result holds for three and n dimensional problems using a proof by induction. We then consider a variant of the algorithm on the interface. First the interior variables are eliminated and the reduced Schur complement is then solved by ASM. This is also an optimal algorithm. The general Schwarz theory also allows us to prove optimal results for multiplicative algorithms, with or without acceleration. The chapter ends with some numerical experiments in two dimensions.

In Chapter 4, we consider local refinement for the ASM of Chapter 3. In this case

some interior points are selected and associated local refinement subregions are introduced. In two dimensions, we prove optimal and almost optimal bounds depending on the boundary of the refinement region. Some technical tools of independent interest are obtained, in particular a polynomial discrete Sobolev inequality and a result on the decomposition of discrete harmonic polynomials. In three dimensions, we have been able to complete only the first part of the analysis regarding refinement regions with regular boundaries. The general case is closely connected to the classical analysis of iterative substructuring methods, involving trace and extension theorems. For the p-version in three dimensions, these tools have been the subject of intense research and only very recently have some positive results been announced; see Belgacem [10] and the references in Chapter 4. Numerical experiments in the plane conclude the chapter.

In Chapter 5, we study some iterative substructuring methods in the additive Schwarz framework. Only the two dimensional case is considered. Almost optimal bounds are obtained for the condition number of the iteration operator. Three dimensional iterative substructuring methods are currently the subject of much research, for both h and p-version finite elements. In the p-version case, the Schwarz analysis is more difficult, mainly because polynomial basis functions no longer have local support and major technical tools such as extension theorems are just becoming available. Many problems remain in this area and there are many possibilities for future research. Numerical experiments with model problems are reported at the end of the chapter.

In the concluding chapter, we briefly comment on future directions of research and on connections with other fields.

1.2 Functional analysis tools

1.2.1 Sobolev spaces

Let Ω be an open, bounded and Lipschitz set in R^n. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u | D^\alpha u \in L^p(\Omega), \text{ for all } |\alpha| \leq m\} ,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index and D^α is the distributional derivative

$$D^\alpha = \frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1}, \cdots, \partial x_n^{\alpha_n}}.$$

With the norm

$$\|u\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \ ,$$

$W^{m,p}$ becomes a Banach space for $1 \leq p \leq \infty$. It is often useful to work with the seminorm

$$|u|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega)}.$$

When $p = 2$, $W^{m,2} = H^m$ becomes a Hilbert space with the inner product

$$(u, v)_{H^m} = \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v)_{L^2}.$$

Since we will consider second order elliptic problems in this thesis , we will mainly be working with $H^1(\Omega)$ and related spaces. An important example of such spaces are the Sobolev spaces of fractional order. If $0 < s < 1\,(s \in R)$ and $1 \leq p < \infty$, we can define a family of intermediate spaces between $L^p(\Omega)$ and $W^{1,p}(\Omega)$:

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{s+\frac{n}{p}}} \in L^p(\Omega \times \Omega)\}.$$

These are Banach spaces with the norm

$$\|u\|^p_{W^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)} \ ,$$

where the last term is the seminorm

$$|u|^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{ps+n}} dx\, dy.$$

As before, we use the notation $H^s(\Omega) = W^{s,2}(\Omega)$. Sobolev spaces of fractional order can also be defined by interpolation between L^p and $W^{1,p}$ or by Fourier transform if $p = 2$ and $\Omega = R^n$, see Lions and Magenes [52]. Finally, the Sobolev space $W^{s,p}(\Omega)$ with $s \in R$ not integer, $s > 1$ is defined as follows. We write $s = m + \sigma$, with $m =$ integer part of s and define

$$W^{s,p}(\Omega) = \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega),\ \forall \alpha \text{ with } |\alpha| = m\}.$$

We will need also Sobolev spaces defined on the boundary $\partial\Omega$. These spaces can in general be defined on manifolds which are part of boundaries of regular open set of R^n,

see Grisvard [45]. We are interested here only in the definition of $H^s(\partial\Omega)$ for $0 < s < 1$. First, we define

$$L^2(\partial\Omega) = \{u : \int_{\partial\Omega} |u|^2 \, ds < \infty\},$$

with the natural norm

$$\|u\|_{L^2(\partial\Omega)} = (\int_{\partial\Omega} |u|^2 \, ds)^{1/2}.$$

Then, we can define

$$H^s(\partial\Omega) = \{u \in L^2(\partial\Omega) : |u|_{H^s(\partial\Omega)} < \infty\},$$

where

$$|u|^2_{H^s(\partial\Omega)} = \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} ds_x ds_y .$$

The norm for this space is given by

$$\|u\|^2_{H^s(\partial\Omega)} = \|u\|^2_{L^2(\partial\Omega)} + |u|^2_{H^s(\partial\Omega)} .$$

We remark that the standard definitions of Sobolev norms we have given are appropriate for domains Ω with diameter $O(1)$. We will often work in this thesis with domains of diameter $O(H)$, in which case the Sobolev norms contain certain scale factors obtained from the standard definition by a change of variables. For example, for $p = 2$

$$\|u\|^2_{H^1(\Omega)} = H^{n-2}(|u|^2_{H^1(\Omega)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega)}),$$

$$\|u\|^2_{H^1(\partial\Omega)} = H^{n-2}(|u|^2_{H^1(\partial\Omega)} + \frac{1}{H}\|u\|^2_{L^2(\partial\Omega)}).$$

1.2.2 Trace and extension theorems

One of the most important consequences of Sobolev embedding theorem is the continuity (up to the boundary) of functions of $W^{m,p}(\Omega)$ when $mp > n$. In this case, we have an obvious definition of boundary value, or trace on $\partial\Omega$, of these smooth functions. This can be generalized to functions in $H^s(\Omega)$ and, more generally, to functions in $W^{s,p}(\Omega)$ under specific hypotheses on the boundary of Ω, see Nečas [67] or Lions and Magenes [52]. We consider for simplicity $1/2 < s < 3/2$, and we assume only that Ω is a Lipschitz region. For smooth domains, the result also holds for $s \geq 3/2$.

Theorem 1.1 (Trace) *Let $1/2 < s < 3/2$. The trace map $\gamma : u \to u|_{\partial\Omega}$ defined from $C_0^\infty(\Omega)$ to $L^2(\partial\Omega)$ can be extended by density to a linear, continuous operator from $H^s(\Omega)$ to $L^2(\partial\Omega)$. Moreover, the range of γ is $H^{s-1/2}(\partial\Omega)$:*

$$\gamma : H^s(\Omega) \stackrel{onto}{\to} H^{s-1/2}(\partial\Omega),$$

and

$$\|\gamma u\|_{H^{s-1/2}(\partial\Omega)} \leq C(\Omega, s)\|u\|_{H^s(\Omega)}.$$

When $s = 1$, the kernel of γ is the important space $H_0^1(\Omega)$:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \gamma u = 0 \text{ on } \partial\Omega\}.$$

This space can also be defined as the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ with respect to the H^1-norm. This is a proper subspace of $H^1(\Omega)$ and it is the maximal subspace for which the extension by zero to the complement of Ω defines a bounded operator into $H^1(R^n)$. Let now Γ be a nonempty proper subset of $\partial\Omega$. The closure of $C_0^\infty(\Gamma)$ in $H^{1/2}(\Gamma)$ with respect to the $H^{1/2}(\Gamma)$-norm, which by analogy we could denote by $H_0^{1/2}(\Gamma)$, can be proved to be equal to $H^{1/2}(\Gamma)$. On the other hand, we obtain a new subspace $H_{00}^{1/2}(\Gamma) \subset H^{1/2}(\partial\Omega)$ when considering the maximal subspace of $H^{1/2}(\Gamma)$ for which the extension by zero to the complement of Γ defines a bounded operator into $H^{1/2}(\partial\Omega)$. A norm on $H_{00}^{1/2}(\Gamma)$ is given by

$$\|u\|_{H_{00}^{1/2}(\Gamma)} = \|\tilde{u}\|_{H^{1/2}(\partial\Omega)},$$

where $\tilde{u}$ is the extension of u by zero on $\partial\Omega \setminus \Gamma$. For $n = 2$, if we parametrize the curve Γ as $\{x(s) : 0 \leq s \leq l\}$, we can define an equivalent norm by

$$\|u\|^2_{H_{00}^{1/2}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + \int_0^l \frac{|u(x(s))|^2}{s} ds + \int_0^l \frac{|u(x(s))|^2}{l-s} ds.$$

Similar formulas also hold for $n > 2$; see Grisvard [45]. $H_{00}^{1/2}(\Gamma)$ is a proper subspace of $H^{1/2}(\Gamma)$: the last two terms in the definition of the norm cannot be bounded in terms of $\|u\|^2_{H^{1/2}(\Gamma)}$. By using the K-method of interpolation, see Lions and Magenes [52], we can also use the definitions

$$H^{1/2}(\Gamma) = [L^2(\Gamma), H^1(\Gamma)]_{1/2}$$

and

$$H_{00}^{1/2}(\Gamma) = [L^2(\Gamma), H_0^1(\Gamma)]_{1/2} .$$

Since the trace map γ maps $H^s(\Omega)$ continuously onto $H^{s-1/2}(\partial\Omega)$, we can apply the open mapping theorem to obtain an extension theorem:

Theorem 1.2 (Extension) *With the same hypotheses as in the Trace Theorem 1.1, the trace map γ has a continuous right inverse E*

$$E : H^{s-1/2}(\partial\Omega) \to H^s(\Omega).$$

Therefore, E satisfies

$$\gamma E g = g, \qquad \forall g \in H^{s-1/2}(\partial\Omega)$$

and

$$\|Eg\|_{H^s(\Omega)} \leq C(\Omega, s)\|g\|_{H^{s-1/2}(\partial\Omega)} .$$

For a proof and details, see Nečas [67].

Trace and extension theorems are very important tools in the analysis of domain decomposition methods. In particular, when considering finite element spaces, it is important to obtain extension theorems with bounds independent of the discretization parameters. For the h-version finite element method, Widlund [83] proved an extension theorem in R^n analogous to Theorem 1.2 with the constant C independent of h. For the p-version, in one and two dimensions, polynomial extension theorems have been obtained by Maday [54], Bernardi and Maday [11], Babuška, Craig, Mandel and Pitkäranta [8]. A three dimensional extension theorem for the p-version finite element method has been announced very recently by Belgacem [10]; see Chapter 4 for more details.

1.2.3 Some inequalities

We state the following important inequalities for a region Ω of diameter H. For proofs and details, see Nečas [67] and Lions and Magenes [52]. It is often very important to establish the equivalence of certain norms. Friedrichs' inequality proves that the H^1-seminorm is equivalent to the H^1-norm on $H_0^1(\Omega)$.

Lemma 1.1 (Friedrichs) *There exists a positive constant C such that*

$$\|u\|_{L^2(\Omega)} \leq C H^2 |u|_{H^1(\Omega)}\,, \qquad \forall u \in H^1_0(\Omega)\,.$$

By using the following inequality, we can prove that the H^1-norm and seminorm are equivalent on the quotient space $H^1(\Omega)/Q_0(\Omega)$, where $Q_p(\Omega)$ is the space of polynomials of degree $\leq p$ on Ω.

Lemma 1.2 (Poincaré) *There exists a positive constant $C(\Omega)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C(\Omega) H^2 (|u|^2_{H^1(\Omega)} + \frac{1}{H^{2+n}} (\int_\Omega u dx)^2), \qquad \forall u \in H^1(\Omega)\,.$$

We will study elliptic problems with Dirichlet conditions on part of the boundary $\Gamma_D \subset \partial\Omega$. The natural Sobolev space in this case is $H^1_D(\Omega) = \{v \in H^1(\Omega) : \gamma v = 0 \,\text{on}\, \Gamma_D\}$. The equivalence of H^1-norm and seminorm on this space is given by the following inequality.

Lemma 1.3 (Poincaré-Friedrichs) *Let Γ be a subset of $\partial\Omega$ with positive measure. Then there exists a positive constant $C(\Omega, \Gamma)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C(\Omega, \Gamma) H^2 (|u|^2_{H^1(\Omega)} + \frac{1}{H} \int_\Gamma |u|^2 dx), \qquad \forall u \in H^1(\Omega)\,.$$

1.3 The p-version finite element method

In this thesis, we consider linear, self adjoint, elliptic problems on a bounded Lipschitz region $\Omega \subset R^n$, $n = 2$ or 3. A discretization of the problem is obtained applying the p-version finite element method. This is one of three standard versions of the method. In the standard one, known as h-version, low order polynomial elements are used (often the degree p of the elements is 1 or 2) and the mesh is refined in order to increase accuracy. Instead, in the p-version the degree of the piecewise polynomial elements is increased in order to achieve the desired accuracy, while the mesh is fixed. The hp-version combines these two approaches. While the standard h-version has been known and studied extensively in the last four decades, the p and hp-versions are more recent developments, stimulated by singular problems in structural mechanics and mechanics of solids. Experimental programs and the first convergence proofs appeared in the late seventies, see Szabo and Mehta [82] and Szabo [80], while theoretical analysis of the optimal convergence rate and approximation results appeared in the eighties, see Babuška,

Szabo and Katz [6], Babuška and Suri [4], and Dorr [32], [33]. For an overview and a good introduction on the p-version, see Babuška and Suri [5], Babuška [1], Babuška and Szabo [7].

1.3.1 The model problem

For simplicity, we consider the following problem in R^2 on a bounded Lipschitz domain Ω:

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & u_0 & \text{on } \partial\Omega_D \\ \frac{\partial u}{\partial n} & = & g & \text{on } \partial\Omega_N. \end{array} \right. \tag{1.1}$$

We suppose that the boundary $\partial\Omega = \partial\Omega_D \bigcup \partial\Omega_N$ is a piecewise smooth curve $\Gamma = \bigcup_{i=1}^{M} \Gamma_i$. The Γ_i are closed arcs (edges) with endpoints A_i, A_{i+1} . We can consider Ω to be a curvilinear polygon with vertices A_i . This problem is a classic example of an elliptic problem on a nonsmooth domain. The structure of its solutions, with possible vertex singularities, is well understood in two dimensions, while only partial results are available in the three dimensional case, in the presence of vertex and edge singularities; see Grisvard [45] and Kondrat'ev and Oleinik [50] for details and proofs. Here we follow the exposition of Babuška and Suri [4].

Introducing local polar coordinates (r_i, θ_i) with the origin at the vertex A_i, we can write the solution of (1.1) in the form

$$u = u_1 + u_2 + \sum_{i=1}^{M} u_3^i \, , \tag{1.2}$$

where u_1 is related to the solution of (1.1) with homogeneous Dirichlet conditions on $\partial\Omega_D$, u_2 is related to the nonhomogeneous Dirichlet conditions and the u_3^i's describe the singular behavior of the solution due to the corners of Ω or to the change of boundary conditions. More precisely, let

$$H_D^1(\Omega) = \{ u \in H^1(\Omega) | \, \gamma u = 0 \text{ on } \partial\Omega_D \}$$

and

$$H_D^k(\Omega) = H^k(\Omega) \bigcap H_D^1(\Omega),$$

We can then specify the structure of the terms in (1.2):

$$u_1 \in H_D^q(\Omega), \;\; q > 1,$$

$$u_2 \in H^k(\Omega),\ u = u_0 \text{ on } \partial\Omega_D,\ k > 3/2,$$

$$u_3^i = \sum_{l=1}^{n_i} C_l^i \,|\log r_i|^{\gamma_l^i}\, r_i^{\alpha_l^i}\, \Phi_l^i(\theta_i)\, \chi^i(r_i) \;\in H_D^1(\Omega)$$

where $\alpha_l^i > 0$, $\alpha_{l+1}^i > \alpha_l^i$, $\gamma_l^i > 0$, $\Phi_l^i(\theta_i)$ and $\chi^i(r_i)$ are C^∞ functions, $\chi^i(r_i) = 1$ for $0 < r_i < \rho^i < 1/4$, and $\chi^i(r_i) = 0$ for $r_i > 2\rho^i$.

We will assume, without loss of generality, that $u_0 = 0$. If not, we can always subtract from u an harmonic function w that equals u_0 on Γ_D. The standard variational formulation of this problem is:
Find $u^* \in V = H_D^1(\Omega)$ such that

$$a(u^*, v) = F(v)\,, \qquad \forall\ v \in\ V\,, \tag{1.3}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx \qquad \text{and} \qquad F(v) = \int_\Omega f v dx + \int_{\Gamma_N} g v ds\,.$$

Our analysis works equally well for any general self adjoint, continuous, coercive, bilinear form

$$b(u,v) = \int_\Omega \sum_{ij=1,2} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx_1 dx_2\,,$$

since the H_0^1-norm and the one induced by $b(\cdot,\cdot)$ are equivalent:

$$c|u|_{H_0^1}^2 \le b(u,u) \le C|u|_{H_0^1}^2\,.$$

A triangulation of the region Ω is introduced by dividing it into non-overlapping brick-like elements $\Omega_i\,, i = 1, \cdots, N_e$. For simplicity, we will analyze square and brick-shaped elements; however, using affine mappings onto a reference square or cube, our analysis also works for general quadrilateral elements. We suppose that the original region is a union of such elements and we denote the mesh size by H.

Let Q_p to be the set of polynomials of degree less then or equal to p in each variable, i.e. in two dimensions

$$Q_p = span\{x^i y^j\,:\, 0 \le i,j \le p\}\,,$$

and discretize the problem with continuous, piecewise, degree p polynomial finite elements:

$$V^p = \{\phi \in C^0(\Omega) : \phi|_{\Omega_i} \in Q_p,\ i = 1, \cdots, N_e\}\,.$$

Then the discrete problem takes the form:
Find $u_p^* \in V_D^p = \{v \in V^p \ : v = 0 \text{ on } \Gamma_D\}$ such that

$$a(u_p^*, v_p) = F(v_p)\,, \qquad \forall\ v_p \in\ V_D^p\,. \tag{1.4}$$

1.3.2 Approximation results

The basic approximation properties of the space V^p have been established mainly by Babuška et al. in a series of papers; see [6] and [4] for proofs and details.

Let $Q = [-1,1]^2$ be the reference square and γ_i, $i = 1,2,3,4$, be the sides of Q and γ_5 be the diagonal $x_1 = x_2$ of Q.

Theorem 1.3 *Let $u \in H^k(Q)$. Then there exists a sequence $z_p \in V^p(Q), p = 0,1,2,\cdots$ such that*

$$\|u - z_p\|_{H^q(Q)} \le Cp^{-(k-q)}\|u\|_{H^k(Q)} \tag{1.5}$$

for $k \ge 0$, $q = 0,1,\cdots,k$;

$$\|u - z_p\|_{H^0(\gamma_i)} \le Cp^{-(k-1/2)}\|u\|_{H^k(Q)},\ \ i = 1,\cdots,5, \tag{1.6}$$

for $k > 1/2$;

$$\|u - z_p\|_{H^1(\gamma_i)} \le Cp^{-(k-3/2)}\|u\|_{H^k(Q)},\ \ i = 1,\cdots,5, \tag{1.7}$$

for $k > 3/2$;

$$|(u - z_p)(x)| \le Cp^{-(k-1)}\|u\|_{H^k(Q)}, \tag{1.8}$$

for $k > 1$ and any $x \in Q$.
In general the constants C depend on k, not on u and p.

The proof is based on elaborated estimates of truncated Fourier series expansions of u; see Babuška and Suri [4].

1.3.3 Rate of convergence

In a first paper, which appeared in 1981 (see [6]), Babuška et al. proved that the rate of convergence of the p-version finite element method was optimal up to an arbitrary small $\epsilon > 0$, namely

$$\|u - u_p\|_{H^1} \le C(\epsilon)p^{-(k-1)+\epsilon}\|u\|_{H^k}$$

when $u \in H^k$ and

$$\|u - u_p\|_{H^1} \leq C(\epsilon) p^{-2\alpha+\epsilon}$$

when the solution u has a singularity of type $u \approx r^\alpha$, $\alpha > 0$ and the vertex of the elements is at the origin. Following computational experiments indicating that the term ϵ could be removed, they were able to establish, in a paper which appeared in 1987, the following optimal bounds, see [4].

Theorem 1.4 *Let $u \in H^k(\Omega)$, $k > 3/2$, be the solution of (1.1). Then there exists $u_p \in V^{p+1}$, $u_p = u_0$ on Γ_D , such that*

$$\|u - u_p\|_{H^1(\Omega)} \leq C p^{-(k-1)} \|u\|_{H^k(\Omega)}$$

where C depends on the triangulation of Ω and on k, but is independent of u and p.

Theorem 1.5 *Let $u \in H^k(\Omega)$, $k > 1$, be the solution of (1.1) such that*

$$u = u_1 + u_2, \qquad u_1 \in H_D^{k_1}(\Omega), \qquad u_2 \in H^{k_2}(\Omega),\ k_2 > 3/2,\ k_1 \leq 3/2\,.$$

Then there exists $u_p \in V^{p+1}$, $u_p = u_0$ on Γ_D , such that

$$\|u - u_p\|_{H^1(\Omega)} \leq C p^{-(k-1)} \|u\|_{H^k(\Omega)}$$

where $k = \min(k_1, k_2)$.

There remains to consider the case $u = u_3^i$ in the decomposition (1.2), i.e.

$$u = r^\alpha |\log r|^\gamma \chi(r) \Phi(\theta)$$

for $\alpha > 0$, $\gamma > 0$ and χ and Φ smooth functions satisfying certain technical hypotheses, see [4]. Then

Theorem 1.6 (Singular solutions) *Under the previous hypotheses, there exists $u_p \in V^p$, such that*

$$\|u - u_p\|_{H^1(\Omega)} \leq C |\log p|^\gamma p^{-2\alpha}\ ,$$

where C is independent of p.

These results can be summarized in the following way. Assume that the solution u of the model problem (1.1) can be written in the form (1.2), with $k > 3/2$, and assume that u_p is the p-version finite element solution of the discrete problem. Then,

$$\|u - u_p\|_{H^1(\Omega)} \leq C p^{-\mu} |\log p|^{\nu} R.$$

Here, letting $\alpha = \min_i \alpha_1^i$,

$$\mu = \min_i (q - 1, k - 1, 2\alpha_1^i) = \min(q - 1, k - 1, 2\alpha),$$

$$\nu = \begin{cases} \max\{\gamma_1^j : \alpha_1^j = \alpha\} & \text{if } \mu = 2\alpha \\ 0 & \text{otherwise,} \end{cases}$$

and

$$R = \|u_1\|_{H^q(\Omega)} + \|u_2\|_{H^k(\Omega)} + \sum_{i,l} |C_l^i|.$$

1.3.4 Advantages of the p-version finite element methods

The choice of the finite element method version, h, p or hp, to be used in a specific case, depends mainly on the properties of the exact solution u. Szabo [81] and Babuška and Szabo [7] classify the exact solution u into three categories.

Category A: u is analytic on each finite element, including the boundaries of each finite element.

Category B: u is analytic on each finite element, including the boundaries of each finite element, with the exception of some of the vertices (singular points).

Category C: the mesh cannot be constructed so that singular points are at vertices or the locations where abrupt changes occur in the derivatives of u, such as material interfaces, are at interelement boundaries. This is usually because the locations of singular points are solution-dependent.

According to these authors, for problems in category A, the most effective method is the p-version, since in this case the strain energy of the approximation error decreases exponentially with p. For problems in category B, the best method is the hp-version. The mesh is graded so that the sizes of the elements decrease in geometric progression

toward the singular point and the polynomial degree of the elements is increased. The strain energy of the error decreases exponentially, provided that there is a sufficient number of elements in the vicinity of the singular points. For problems in category C, the h-version with adaptive mesh refinement is considered the best approach.

The large majority of solid mechanics problems solved in engineering practice are problems in linear elastostatics and elastodynamics. These and many nonlinear problems belong to category A or B. Therefore, the use of p and hp-version finite element methods is of great importance in engineering design and analysis. The p and hp-version are closely related to spectral methods, in particular to the spectral element method; see Patera []. Traditionally, spectral methods have been used in fluid mechanics problems in simple domains and in the context of smooth solutions. With the introduction of new methods like the spectral element method, it has become difficult to clearly distinguish the two approaches; see the concluding Chapter.

1.3.5 The basis functions for the p-version finite element method

In contrast to the h-version finite element method, there are only a few commercial implementations of the p-version. The first experimental code called COMET-X was developed at Washington University in St. Louis in the mid-seventies. In Bergamo, Italy, the Instituto Sperimentale Modelli e Strutture (ISMES) developed a three dimensional finite element code called FIESTA, with some p-version capabilities in the early 1980s. The commercial system PROBE by Noetic Tech., St. Louis, was first released in 1985, implementing p and hp-version for two dimensional problems and successively extended to three dimensional problems. The Aeronautical Research Institute of Sweden has developed a new three dimensional program called STRIPE.

In this thesis, we adopt the standard choice of a hierarchical basis for V^p, well-known in the literature and used for example in PROBE, consisting of nodal, side and interior functions for $n = 2$. For $n = 3$, it includes also face functions. For a more complete description of this and other possible bases for V^p, see Babuška and Elman [2] and Babuška, Griebel, and Pitkäranta [9]. Let us introduce this basis on the reference square $Q = [-1, 1]^2$. For $j \geq 2$, let

$$\phi_j(x) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{x} L_{j-1}(s)ds \,,$$

where L_j is the Legendre polynomial of degree j. The different basis functions are the following.

a) Nodal functions:

$$\phi^{N,1}(x,y) = \tfrac{1}{4}(1+x)(1-y) \quad \phi^{N,2}(x,y) = \tfrac{1}{4}(1+x)(1+y)$$
$$\phi^{N,3}(x,y) = \tfrac{1}{4}(1-x)(1+y) \quad \phi^{N,4}(x,y) = \tfrac{1}{4}(1-x)(1-y).$$

b) Side functions, $2 \leq j \leq p$:

$$\phi_j^{S,1}(x,y) = \tfrac{1}{2}(1+x)\phi_j(y) \quad \phi_j^{S,2}(x,y) = \tfrac{1}{2}\phi_j(x)(1+y)$$
$$\phi_j^{S,3}(x,y) = \tfrac{1}{2}(1-x)\phi_j(y) \quad \phi_j^{S,4}(x,y) = \tfrac{1}{2}\phi_j(x)(1-y).$$

c) Interior functions, $2 \leq j,k \leq p$:

$$\phi_{j,k}^{I}(x,y) = \phi_j(x)\phi_k(y).$$

Each nodal function is associated to a vertex of the reference square Q. Each side function is associated to one side of Q and is zero on the three other sides. There are $4(p-1)$ of them in V^p. The interior functions are zero on all four sides of Q (they are bubble functions) and there are $(p-1)^2$ of them, completing the set of $(p+1)^2$ basis functions needed for V^p. This basis is hierarchical, meaning that the set of basis functions for V^p is contained in the set of basis functions for V^{p+1}.

There are many other bases for the p-version finite element method, but in this thesis we will only use the one just introduced. The stiffness matrix associated to this basis is quite ill conditioned and the linear system representing the finite element problem needs good preconditioners (see the next section for a definition of a preconditioner). We illustrate here the conditioning of the stiffness matrix in the simplest case of one square element. In the following table, we give the condition numbers of the stiffness matrix and the submatrices associated to the interior and side functions and to their couplings (I=interior, S=side, N=nodal). Since the first eigenvalue vanish, we define the condition number as the ratio of the maximum and the minimum positive eigenvalues. The stiffness submatrix associated with the nodal functions is of course singular, therefore it is not given in the table. It is clear from the table that the condition number of the global stiffness matrix grows like p^3 and that this is due mainly to the coupling between interior and side functions. This is the main motivation for eliminating first the interior functions and using the Schur complement approach in devising domain decomposition algorithms for the p-version finite element method; see Section 4 in Chapter 3.

p	I+S+N	I	S	S+N	I+N	I+S
1	1.5			1.5		
2	22.25	1	1.3	5.32	1.5	22.25
3	36.63	4.2	2.33	5.32	5.25	25.55
4	103.73	10.35	2.68	5.53	12.77	103.73
5	159.29	20.31	2.83	5.53	25.06	118.62
6	321.16	35.56	2.90	5.54	43.87	321.16
7	462.53	57.80	2.94	5.54	71.32	353.68
8	783.61	89.05	2.96	5.54	109.86	783.60

Table 1.1: Condition number of stiffness matrix and submatrices

1.4 Iterative methods for linear systems of equations

The discretization of our model problem with the p-version finite element method produces a linear system of equations

$$Ax = b \,, \tag{1.9}$$

where the $M \times M$ matrix A is symmetric, positive definite and relatively sparse. The system (1.9) can be solved directly or iteratively. In this thesis, we will consider iterative solvers of conjugate gradient type preconditioned with various Schwarz methods.

Most of the engineering and commercial applications use direct solvers. Iterative solvers, which have been mainly academic projects until recently, are beginning to be used in large scale applications. In [59], Mandel concisely compares the two categories in the table 1.2:

Even if iterative methods do not involve fill-in, their performance depends on the numerical values of the problem and in particular on the spectrum of the iteration operator and on its condition number. Mandel then summarizes the requirements for a practical iterative method, that should be:

- Faster and require less storage than existing direct methods.
- Robust: handle distorted geometries, strong anisotropies, etc. It might slow down for hard problems, but must not fail.
- Predictable: give an apriori estimate of the solution cost.

	Direct	**Iterative**
Solution	exact	approximate
Performance	nonzero structure	numerical values
Solution cost	predictable	unknown in advance
New right-hand side	cheap	repeat all
Irregular geometries	no problems	slows down
Solution time	degrades $\sim NDOF^2$ or worse	for smart methods $\sim NDOF$
Storage required	fill-in requires $\sim NDOF^{1.5}$	only original data + small data structures
Current usage	commercial use	academic projects

Table 1.2: Comparison of direct and iterative methods

- Fully automated: select solution strategy using available data, transparently to the user.

We believe that iterative solvers for the p-method based on Schwarz techniques can eventually achieve these goals (at least the first three, the fourth being more a matter of engineering and implementation).

1.4.1 Conjugate gradient method and preconditioning

Since the stiffness matrix A is symmetric and positive definite, the standard choice in most domain decomposition methods is the preconditioned conjugate gradient method. For a detailed presentation of this and other iterative methods, see Golub and Van Loan [44]. An essential feature of the conjugate gradient method that allows its application to domain decomposition algorithms, is that an explicit representation of A is not needed: it is enough to be able to apply it to a given vector. In its basic form, with a tolerance $\epsilon > 0$, the conjugate gradient method is:

Set $k = 0$ and choose x_0
$r_0 = b - Ax_0$
While $|r_k| \geq \epsilon |r_0|$
    $k = k + 1$
    if k =1

$p_1 = r_0$

else

$\beta_k = (r_{k-1}, r_{k-1})/(r_{k-2}, r_{k-2})$

$p_k = r_{k-1} + \beta_k p_{k-1}$

endif

$\alpha_k = (r_k, r_k)/(p_k, Ap_k)$

$x_k = x_k + \alpha_k p_k$

$r_k = r_k - \alpha_k A p_k$

end

Here (x, y) is the A-inner product $x^T Ay$. It is a well known result that in the absence of round-off errors, the conjugate gradient method gives the exact solution after at most M steps, since the search directions p_k are A-conjugate:

$$p_i A p_j^T = 0, \ \ i \neq j.$$

The subspace

$$W_k(A, r_0) = \text{span}\{r_0, Ar_0, \cdots, A^{k-1} r_0\} = \text{span}\{p_0, \cdots, p_{k-1}\}$$

is called the Krylov subspace related to the matrix A and r_0. An important result is that $\|x - x_k\|_A$ is the norm of the difference between the initial error $x - x_0$ and its projection on W_k. Using results from approximation theory, it is possible to prove that after n steps, the reduction in the A-norm of the error satisfies the bound:

$$\|x - x_n\|_A \leq 2(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1})^n \|x - x_0\|_A \ .$$

Here $\kappa(A)$ is the condition number of A:

$$\kappa(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

When A comes from the discretization of an elliptic problem, $\kappa(A)$ is usually large: for example, in the h-version finite element method, $\kappa(A) = O(h^{-2m})$ for problems of order $2m$. A preconditioner B is then introduced, with the properties that

- $\kappa((B^{-1/2})^T A B^{-1/2}) << \kappa(A)$,
- the system $Bz = d$ is computationally easier to solve than $Az = d$.

The conjugate gradient method is therefore applied to the preconditioned linear system

$$B^{-1}Ax = B^{-1}b.$$

Standard preconditioners used in practice are diagonal scaling and incomplete factorizations (e.g. Cholesky). In this thesis, we will study domain decomposition preconditioners based on the Schwarz framework.

The condition number of A can be approximated during the conjugate gradient iteration by using a variant of an eigenvalues algorithm due to Lanczos [51]. Let R_k be a matrix with columns chosen as the normalized residual vectors. It is possible to prove that

$$R_k^T A R_k = J_k = \begin{pmatrix} 1/\alpha_0 & -\sqrt{\beta_0/\alpha_0} & & \\ -\sqrt{\beta_0/\alpha_0} & 1/\alpha_1 + \beta_0/\alpha_0 & -\sqrt{\beta_1/\alpha_1} & \\ & -\sqrt{\beta_1/\alpha_1} & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

is tridiagonal. J_k is the matrix representation of the restriction of the operator A to the space spanned by the residual vectors $r_0, \cdots, r_{k-1}$. Since J_k is symmetric and tridiagonal, approximations of the eigenvalues of A can be obtained from the eigenvalues of J_k at a small extra cost. Moreover, the eigenvalues of J_k interlace those of J_{k+1}, therefore we obtain improved estimates of the extreme eigenvalues at each step. See Parlett [68] for a full discussion of the Lanczos method.

1.4.2 GMRES

In the application of some multiplicative Schwarz schemes, we need to solve linear systems that are positive definite, but no longer symmetric. Schwarz methods can in fact be applied to nonsymmetric and indefinite problems, see Cai and Widlund [24], [25]. In this case, the acceleration of the algorithm with an iterative method can no longer be done by the conjugate gradient method. Alternative iterative methods have been developed in the last decade. In this thesis, we will consider the generalized minimum residual method (GMRES), that has performanced well for many nonsymmetric and indefinite

problems. For a more complete description, see Eisenstat, Elman, and Schultz [41] and Saad and Schultz [73].

In the n-th iteration of GMRES, a correction vector x_n is computed in the Krylov space W_n minimizing the norm of the residual

$$\min_{z \in W_n} \|b - A(x_0 + z)\|.$$

It can be proved that in exact arithmetic, the exact solution is reached in at most M iterations, where M is the size of A. A basic form of the GMRES algorithm is:

Choose x_0
$r_0 = b - Ax_0$; $p_0 = r_0$
For $i = 0,1,\cdots$, until convergence do
$a_i = (r_i, Ap_i)/(Ap_i, Ap_i)$
$x_{i+1} = x_i + a_i p_i$
$r_{i+1} = r_i - a_i A p_i$
$p_{i+1} = r_{i+1} + \sum_{j=0}^{i} b_j^{(i)} p_j$
where for $j \leq i$, $b_j^{(i)} = -(Ar_{i+1}, Ap_j)/(Ap_j, Ap_j)$.
end for;

The rate of convergence of GMRES can be characterized by the two quantities

$$c_A = \inf_{x \neq 0} \frac{(x, Ax)}{(x, x)}$$

and

$$C_A = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

In [41], Eisenstat, Elman, and Schultz have established that if $c_A > 0$, then GMRES converges and at the $n - th$ iteration the residual is bounded by

$$\|r_n\| \leq (1 - \frac{c_A^2}{C_A^2})^{n/2} \|r_0\|.$$

Chapter 2

Domain decomposition methods

2.1 The Classical Alternating Schwarz method

This method is believed to be the oldest domain decomposition method. It was introduced by H. A. Schwarz in 1870 in the case of two overlapping subregions, see [74]. In order to find the solution u to the problem

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega , \end{cases}$$

on the plane region

$$\Omega = \Omega_1 \bigcup \Omega_2 ,$$

a sequence of functions $\{u_n\}$ converging to u is computed. The sequence is constructed in the following way.

Choose an initial guess u_0 . Each step consists of two fractional steps:

i) solve

$$\begin{cases} -\Delta u_{n+1/2} &= f & \text{in } \Omega_1, \\ u_{n+1/2} &= u_n & \text{on } \partial\Omega_1 . \end{cases}$$

to find $u_{n+1/2}$;

ii) solve

$$\begin{cases} -\Delta u_{n+1} &= f & \text{in } \Omega_2, \\ u_{n+1} &= u_{n+1/2} & \text{on } \partial\Omega_2 . \end{cases}$$

to find u_{n+1} .

The convergence of the sequence u_n was first proved by Schwarz using the maximum

principle. We can write the method in a variational form, using the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \ :$$

i) find $\delta u_{n+1/2} \in V_1 = H_0^1(\Omega_1)$ such that

$$a(\delta u_{n+1/2}, v) = a(u - u_n, v)\,, \qquad \forall\ v \in V_1$$

and update $u_{n+1/2} = u_n + \delta u_{n+1/2}$;
ii) find $\delta u_{n+1} \in V_2 = H_0^1(\Omega_2)$ such that

$$a(\delta u_{n+1}, v) = a(u - u_{n+1/2}, v)\,, \qquad \forall\ v \in V_2$$

and update $u_{n+1} = u_{n+1/2} + \delta u_{n+1}$.
Define the projections $P_i : V \to V_i$ by

$$a(P_i v, \phi) = a(v, \phi)\,, \qquad \forall \phi \in V_i \tag{2.1}$$

and the error by $e_n = u - u_n$. In the two fractional steps, we have:

$$e_{n+1/2} = (I - P_1)e_n$$

$$e_{n+1} = (I - P_2)e_{n+1/2}$$

and therefore the error propagation equation is

$$e_{n+1} = (I - P_2)(I - P_1)e_n.$$

Thus the algorithm proceeds by serially projecting the error onto the orthogonal complements of the subspaces.

2.2 Abstract Schwarz framework

Schwarz methods can be introduced in a general variational setting for nonsymmetric and indefinite problems, see Cai and Widlund [24], [25], mixed problems, see Mathew [61], [63], [62], systems of equations, see Smith [76], problems of order higher than two, see Zhang [89]. Here we limit our discussion to symmetric, positive definite, scalar elliptic problems.

Let V be a finite dimensional Hilbert space with inner product $a(\cdot,\cdot)$ and f a continuous linear functional on V. We want to solve the discrete problem:
Find $u \in V$ such that

$$a(u,v) = f(v)\,, \qquad \forall\ v \in\ V\ . \tag{2.2}$$

If we introduce a basis in V, this is just a symmetric, positive definite linear system. A Schwarz method is defined by a decomposition of V into $N+1$ subspaces:

$$V = V_0 + V_1 + \cdots + V_N\ .$$

In most applications to elliptic problems, these subspaces are related to a decomposition of the domain Ω into subdomains Ω_i . For example, in the classical Dryja-Widlund algorithm introduced in [35], $V_i = H_0^1(\Omega_i') \bigcap V$, where $V = V^h$ is the h-version finite element space and Ω_i' are extensions of the subdomains Ω_i. The subspace V_0 has a special role: it corresponds to a coarse mesh and provides global transportation of information in each iteration.

For each subspace, we introduce the symmetric, positive definite bilinear form

$$b_i(\cdot,\cdot) : V_i \times V_i \rightarrow R$$

and the operator

$$T_i : V \rightarrow V_i\ ,$$

that can be regarded as an approximate projection onto V_i. $T_i v \in V_i$ is the solution of the following problem on V_i:

$$b_i(T_i v, w) = a(v,w), \qquad \forall w \in V_i.$$

This formalism allows the consideration of inexact solvers for the subproblems. If we use exact solvers, then $b_i(\cdot,\cdot) = a(\cdot,\cdot)$ and $T_i = P_i$, the usual projection onto V_i . Note that the approximate projection $T_i u$ of the exact solution u can be computed without knowing u by solving

$$b_i(T_i u, v) = a(u,v) = f(v), \qquad \forall v \in V_i,$$

since f is given.

2.2.1 Multiplicative methods

Multiplicative Schwarz methods generalize to N subregions the classical alternating Schwarz method.

Classical multiplicative Schwarz algorithm:

i) compute $g_i = T_i u$, for $i = 0, 1, \cdots, N$;

ii) given u_n , compute u_{n+1} in $N+1$ fractional steps:

$$u_{n+\frac{i+1}{N+1}} = u_{n+\frac{i}{N+1}} + (g_i - T_i u_{n+\frac{i}{N+1}}), \ \ i = 0, 1, \cdots, N.$$

In general these $N+1$ steps are sequential. However, if some subspaces V_i are mutually a-orthogonal, then we can solve some of these subproblems concurrently (in parallel). In the standard case when the subspaces V_i are associated to subdomains Ω_i forming a decomposition of Ω, we can define an undirected graph in which the nodes represent subdomains and the edges intersections of subdomains. This graph can be colored such that no connected nodes have the same color. Subspaces of the same color are then considered together as one subspace and the total number of sequential steps in each iteration is decreased. Numerical experiments show that minimizing the number of colors improves the convergence, see Cai, Gropp and Keyes [23].

It is easy to see that the error $e_i = u - u_i$ satisfies

$$e_{i+1} = E_N e_i \ ,$$

where E_N is the multiplicative Schwarz operator

$$E_N = (I - T_N)(I - T_{N-1}) \cdots (I - T_1)(I - T_0) \ .$$

The convergence rate of the multiplicative algorithm is therefore determined by the reduction in the error in each iteration:

$$|E_N v|_a \leq \gamma |v|_a \ .$$

Important theorems giving explicit bounds for γ have been proved by Bramble, Pasciak, Wang and Xu [21] and Xu [88].

Theorem 2.1 *Let there exist*

i) a constant C_0 such that

$$a(v,v) \leq C_0^2 \sum_{i=0}^{N} a(T_i v, v), \qquad \forall v \in V;$$

ii) a constant $1 \leq \omega < 2$ such that for $i = 0, 1, \cdots, N$,

$$a(v,v) \leq \omega b_i(v,v), \qquad \forall v \in V_i\,;$$

iii) constants ϵ_{ij} , for $i, j = 1, \cdots, N$ such that

$$a(v_i, v_j) \leq \epsilon_{ij} a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2}, \qquad u_i \in V_i,\ u_j \in V_j\,.$$

Then

$$|E_N v|_a \leq \gamma |v|_a\,,$$

with

$$\gamma = \sqrt{1 - \frac{2-\omega}{C_0^2(1 + 2\omega^2 \rho(\mathcal{E})^2)}}\,.$$

Here $\rho(\mathcal{E})$ is the spectral radius of the matrix $\mathcal{E} = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

Assumption *ii)* and the definition of T_i imply that $\|T_i\|_a \leq \omega$, see the proof given later in Theorem 2.2. The inequalities in assumption *iii)* are known as strengthened Cauchy-Schwarz inequalities. We remark that *iii)* does not make any assumption on the coarse space, while in *i)* and *ii)* the coarse space is included.

The classical multiplicative Schwarz algorithm can be accelerated by different iterative methods. Since the polynomial $I - E_N$ does not contain any constant term, we can compute

$$g = (I - E_N)u$$

without knowing the exact solution u. We remark that for a given vector v, the matrix-vector product $v_N = (I - E_N)v$ can be computed recursively by

$$\left\{ \begin{array}{rcl} v_0 & = & T_0 v \\ v_1 & = & v_0 + T_1(v - v_0) \\ & \vdots & \\ v_N & = & v_{N-1} + T_N(v - v_{N-1})\,. \end{array} \right.$$

Then, we can consider the following algorithm:

Accelerated multiplicative Schwarz algorithm:

i) compute $g = (I - E_N)u$;

ii) solve the nonsymmetric operator equation

$$(I - E_N)v = g$$

by a conjugate gradient-type iterative method, such as GMRES.

Another possibility, since we are considering the case when the T_i are symmetric and positive definite, is given by the symmetrized multiplicative Schwarz: the operator E_N is symmetrized by doubling the number of fractional steps and reversing the order of the subspaces. Since in this case the last forward and the first backward steps are the same, we can avoid performing one of the two. We can then use the standard CG to accelerate the convergence.

Symmetrized multiplicative Schwarz algorithm:

i) compute $g = (I - E_N^T E_N)u$;

ii) solve the symmetric operator equation

$$(I - E_N^T E_N)v = g$$

by the conjugate gradient method.

2.2.2 Additive methods

In the multiplicative Schwarz algorithm, each iteration involves $N+1$ or $2N+1$ fractional steps, depending on whether the classical or symmetrized version is considered. Even if with a good coloring strategy some fractional steps can be performed concurrently, in general there can be a relatively large number of sequential fractional steps. In order to remove this limitation, Dryja and Widlund introduced the additive Schwarz method, see [35], [36], [34]. Independent work on additive Schwarz methods can also be found in Matsokin and Nepomnyaschikh [64], [66]. In recent years, the method has been generalized in several directions, see Dryja and Widlund [39], [40].

The basic idea of the method is to work with the simplest possible polynomial T in the operators T_i's, namely

$$T = T_0 + T_1 + \cdots + T_N \, .$$

T is known as the additive Schwarz operator. Since T does not contain any constant term, we can compute $g = Tu = \sum_i T_i u$ without knowing the exact solution u of (2.2). The algorithm can then be written:

Additive Schwarz algorithm:

i) compute $g = Tu$;

ii) solve the operator equation

$$Tv = g$$

by the conjugate gradient method.

The convergence properties of the algorithm depend on the spectrum of T. It is therefore of fundamental importance to be able to estimate the condition number of T. In order to do this, we use two lemmas from Zhang [89].

Lemma 2.1 *The additive Schwarz operator $T = \sum_i T_i$ is invertible and*

$$a(T^{-1}v, v) = \min_{\sum v_i = v} \sum_i b_i(v_i, v_i),$$

with $v_i \in V_i$. The minimum is achieved at $v_i = T_i T^{-1} v$.

Proof. We use the properties of T_i and the Cauchy-Schwarz inequality:

$$a(T^{-1}v, v) = \sum_i a(T^{-1}v, v_i) = \sum_i b_i(T_i T^{-1}v, v_i) \leq$$

$$\leq (\sum_i b_i(T_i T^{-1}v, T_i T^{-1}v))^{1/2} (\sum_i b_i(v_i, v_i))^{1/2} =$$

$$= (\sum_i a(T^{-1}v, T_i T^{-1}v))^{1/2} (\sum_i b_i(v_i, v_i))^{1/2} = (a(T^{-1}v, v))^{1/2} (\sum_i b_i(v_i, v_i))^{1/2} .$$

Therefore

$$(a(T^{-1}v, v)) \leq \sum_i b_i(v_i, v_i).$$

Equality holds if and only if $v_i = T_i T^{-1} v$.

□

As direct consequence of Lemma 2.1, we obtain:

Lemma 2.2 *Let $T = \sum_i T_i$. Then*

$$\lambda_{min}^{-1}(T) = \lambda_{max}(T^{-1}) = \max_v \frac{a(T^{-1}v, v)}{a(v, v)} = \max_v \min_{\sum v_i = v} \frac{\sum_i b_i(v_i, v_i)}{a(v, v)}$$

and

$$\lambda_{max}^{-1}(T) = \lambda_{min}(T^{-1}) = \min_v \frac{a(T^{-1}v, v)}{a(v, v)} = \min_v \min_{\sum v_i = v} \frac{\sum_i b_i(v_i, v_i)}{a(v, v)}.$$

We can now prove the main result of this section.

Theorem 2.2 *Let there exist*

i) a constant C_0 such that $\forall v \in V$ there exists a decomposition $v = \sum_{i=0}^N v_i$, $v_i \in V_i$, such that

$$\sum_{i=0}^N b_i(v_i, v_i) \leq C_0^2 a(v, v);$$

ii) a constant ω such that for $i = 0, 1, \cdots, N$,

$$a(v, v) \leq \omega b_i(v, v), \qquad \forall v \in V_i\,;$$

iii) constants ϵ_{ij} , for $i, j = 1, \cdots, N$ such that

$$a(v_i, v_j) \leq \epsilon_{ij} a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2}, \qquad v_i \in V_i,\ v_j \in V_j\,.$$

Then

$$C_0^{-2} a(v, v) \leq a(Tv, v) \leq (\rho(\mathcal{E}) + 1)\omega a(v, v), \qquad v \in V.$$

Again, $\rho(\mathcal{E})$ is the spectral radius of the matrix $\mathcal{E} = \{\epsilon_{ij}\}_{i,j=1}^N$.

Proof. The left inequality is a direct consequence of Lemma 2.2: by assumption *i*), we have $\lambda_{min}(T) \geq C_0^{-2}$. In order to prove the right inequality, we first remark that assumption *ii*) implies that $\|T_i\|_a \leq \omega$:

$$a(T_i v, T_i v) \leq \omega b_i(T_i v, T_i v) = \omega a(v, T_i v)$$

$$\leq \omega a(v, v)^{1/2} a(T_i v, T_i v)^{1/2}\,.$$

Therefore

$$a(T_i v, T_i v) \leq \omega^2 a(v, v)\,.$$

Let us now prove the right inequality by using assumption $ii)$ and $iii)$:

$$a(\sum_{i=1}^{N} T_i v, \sum_{i=1}^{N} T_i v) = \sum_{i,j=1}^{N} a(T_i v, T_j v)$$

$$\leq \sum_{i,j=1}^{N} \epsilon_{ij} a(T_i v, T_i v)^{1/2} a(T_j v, T_j v)^{1/2} \leq \rho(\mathcal{E}) \sum_{i=1}^{N} a(T_i v, T_i v)$$

$$\leq \rho(\mathcal{E})\omega \sum_{i=1}^{N} a(v, T_i v) \leq \rho(\mathcal{E})\omega a(v, v)^{1/2} a(\sum_{i=1}^{N} T_i v, \sum_{i=1}^{N} T_i v)^{1/2}.$$

Therefore

$$a(\sum_{i=1}^{N} T_i v, \sum_{i=1}^{N} T_i v) \leq \rho(\mathcal{E})^2 \omega^2 a(v, v),$$

from which follows

$$a(\sum_{i=1}^{N} T_i v, v) \leq \rho(\mathcal{E})\omega a(v, v).$$

This last inequality, added to

$$a(T_0 v, v) \leq \omega a(v, v)$$

completes the proof.

□

We have seen that assumption $i)$ and Lemma 2.2 imply that $\lambda_{min}(T) \geq C_0^{-2}$. This result is often known as Lions' Lemma and it is a very important tool in estimating the minimum eigenvalue of T. See Lions [53], for the multiplicative case when $N = 2$, and see Nepomnyaschikh [66] and Widlund [86] for the general case. Theorem 2.2 allows us to estimate the condition number of T by

$$\kappa(T) \leq C_0^2 \omega(\rho(\mathcal{E}) + 1).$$

Comparing Theorem 2.2 and 2.1, we can see that the same parameters C_0, ω and $\rho(\mathcal{E})$ determine bounds for the convergence rate of the additive and multiplicative variant of the Schwarz method. However, there is no general theory comparing the two convergence

rates and little can be said in the case of many subspaces. In the case of two subspaces the relation between the two versions is well understood, see Bjørstad [13] and Bjørstad and Mandel [14]. For numerical comparison of the two versions in the case of many subspaces using massively parallel computers, see Bjørstad and Skogen [16], Skogen [75], Gropp and Smith [46]. In these experiments, it appears that multiplicative versions have a better convergence rate than additive ones, but they are more expensive. The architectures of the various parallel computers available and the nature of the applications will probably determine which version to prefer.

Chapter 3

Overlapping ASM for the p-version finite element method

3.1 Introduction

In this chapter, we study some domain decomposition methods using p-version finite elements in the Schwarz framework introduced in the previous chapter. We consider linear, self-adjoint, second order elliptic problems and brick-shaped elements in the finite element discretization. We show that the condition number of the iteration operator of these algorithms is bounded by a constant independent of p, H and N. The basic result, illustrated in the next section for an additive Schwarz method (ASM), appeared first in Pavarino [70]. It was inspired by a similar method for the h-version described in Bramble et al. [18] and by results of Dryja and Widlund [36]. Next, we consider a variant of the method on the interface and some multiplicative variants.

We recall that our model problem (1.1) is given on a bounded Lipschitz region Ω. Dirichlet boundary conditions are given on Γ_D, a closed subset of $\partial\Omega$ of positive measure, and Neumann conditions are given on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. A triangulation of the region Ω is introduced by dividing it into non-overlapping brick-like elements Ω_i , $i = 1, \cdots, N_e$. Using affine mappings onto a reference square or cube, our analysis also works for general quadrilateral elements. We assume that the original region is a union of such elements and we denote the mesh size by H. Where Ω is locally L-shaped, we call the corresponding non-convex vertex on the boundary an L-point. For every element Ω_i in three dimensions, we require that at most three faces belong to the boundary $\partial\Omega$. Similarly, in

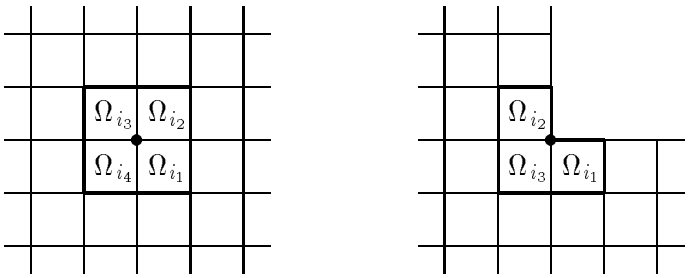


Figure 3.1: Regular and L-shaped substructures Ω_i'

two dimensions, we require that at most two sides of an element belong to the boundary. We also assume that the other endpoint of each interior edge connected with an L-point, is an interior point. This is always possible by choosing an appropriate mesh size H and is not a restriction on the shape of Ω. Our analysis of L-points can be generalized, in the Dirichlet case, to cracks, i.e. to edges on the boundary of Ω that do not belong to the boundary of the interior of the closure of Ω. Thus, Ω can be a non-convex polyhedron with cracks.

Let Q_p be the set of polynomials of degree less than or equal to p in each variable, i.e. in three dimensions

$$Q_p = span\{x^i y^j z^k \,:\, 0 \le i,j,k \le p\}\,.$$

We discretize the problem with continuous, piecewise, degree p polynomial finite elements:

$$V^p = \{\phi \in C^0(\Omega) : \phi|_{\Omega_i} \in Q_p,\ i = 1, \cdots, N_e\}.$$

Then the discrete problem takes the form:
Find $u_p^* \in V_D^p = \{v \in V^p \,: v = 0 \text{ on } \Gamma_D\}$ such that

$$a(u_p^*, v_p) = F(v_p)\,, \qquad \forall\ v_p \in\ V_D^p\,. \tag{3.1}$$

3.2 Overlapping ASM using square elements in two dimensions

We now consider square elements Ω_i in two dimension. We start by considering the simplest case, with homogeneous Dirichlet boundary conditions everywhere. At the end of this section, we will illustrate how to modify the method and the analysis to include Neumann and mixed boundary conditions.

Let N be the number of interior nodes. Our finite element space is represented as the sum of $N+1$ subspaces

$$V_D^p = V_0^p + V_1^p + \cdots + V_N^p \, .$$

The first space V_0^p serves the same purpose as the coarse space in the h-version. Here:

- $V_0^p = V_D^1$, i.e. the space of continuous, piecewise Q_1 functions on the mesh defined by the elements Ω_i;
- $V_i^p = V^p \bigcap H_0^1(\Omega_i')$ where Ω_i' is the $2H \times 2H$ open square centered at the i-th vertex. In other words Ω_i' is the interior of $\overline{\Omega}_{i_1} \bigcup \overline{\Omega}_{i_2} \bigcup \overline{\Omega}_{i_3} \bigcup \overline{\Omega}_{i_4}$, see figure 3.1.

As in the h-version, the algorithm consists in solving, by an iterative method, the equation

$$P u_p = (P_0 + P_1 + ... + P_N) u_p = g_p \, , \tag{3.2}$$

where the projections $P_i : V_D^p \rightarrow V_i^p$ are defined by

$$a(P_i v_p, \phi_p) = a(v_p, \phi_p) \, , \qquad \forall \phi_p \in V_i^p . \tag{3.3}$$

$g_p = \sum_{i=0}^{N} P_i u_p^*$ can be computed without knowing the solution u_p^* of (1.4) by

$$a(P_i u_p^*, \phi_p) = f(\phi_p) \, , \qquad \forall \phi_p \in V_i^p . \tag{3.4}$$

The following is the main result of this chapter. It is first given here for two dimensions and then extended to three in the next section.

Theorem 3.1 *The operator P of the additive algorithm defined by the spaces V_i^p satisfies the estimate $\kappa(P) \leq$ const. independent of $p, H,$ and N.*

Proof. The idea of the proof is similar to a result given in Dryja and Widlund [36] for the h-version, but the technical details are quite different. A constant upper bound for the spectrum of P is obtained directly by noting that for $i \geq 1$

$$a(P_i u_p, u_p) = a(P_i u_p, P_i u_p) = a_{\Omega_i'}(P_i u_p, P_i u_p) \leq a_{\Omega_i'}(u_p, u_p).$$

Each point is covered by no more than four subregions Ω_i' and the norm of P_0 is equal to one; therefore $\lambda_{max} \leq 5$.

A lower bound is obtained by using Theorem 2.2. We have to define a partition of the finite element function $u_p = \sum_{i=0}^{N} u_{p,i},\ u_{p,i} \in V_i^p$, and obtain a good bound of the constant C_0^2. To define the first term $u_{p,0}$, we can use either the L^2- projection onto V_0^p or a standard construction by smoothing and interpolation, see Bramble and Xu [22] or Strang [79]. In both cases, there exists a linear map $\hat{I}_1 : V^p \rightarrow V_0^p$, which satisfies

$$\|u_p - \hat{I}_1 u_p\|^2_{L^2(\Omega)} \leq C_1 H^2 |u_p|^2_{H^1(\Omega)} \tag{3.5}$$

and

$$|\hat{I}_1 u_p|^2_{H^1(\Omega)} \leq C_2 |u_p|^2_{H^1(\Omega)}. \tag{3.6}$$

Let

$$u_{p,0} = \hat{I}_1 u_p, \qquad w_p = u_p - u_{p,0}\,.$$

The construction of $u_{p,i}$ requires two technical tools: a special partition of unity $\{\theta_i\}$ and an interpolation operator I_p .

Step 1: Construction of the partition of unity $\{\theta_i\}$. In order to define $\{\theta_i\}$, we consider linear combinations of the standard basis functions for Q_1 :

$$\theta_i \in V_1\,, \qquad supp(\theta_i) = \Omega_i'\,, \qquad 0 \leq \theta_i \leq 1\,, \qquad \sum_{i=1}^{N} \theta_i(x,y) = 1\,.$$

We define θ_i by specifying its values at the nine nodes of $\overline{\Omega}_i'$ and making it zero at all other nodes. At the interior node x_i , $\theta_i(x_i) = 1$. At each node x_j on the boundary $\partial\Omega_i'$, two boundary edges Γ_{j_1} , $\Gamma_{j_2} \in \partial\Omega_i'$ meet. Let

$$\theta_i(x_j) = \begin{cases} 1 & \text{if both } \Gamma_{j_1} \text{ and } \Gamma_{j_2} \text{ are on the boundary of } \Omega \\ 0 & \text{otherwise,} \end{cases}$$

Some cases are illustrated in figure 3.2. It is easy to see that this construction does

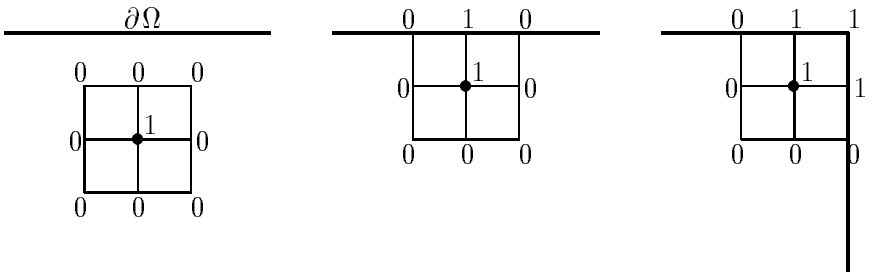


Figure 3.2: Partition of unity functions θ_i for 2-dim regular subregions

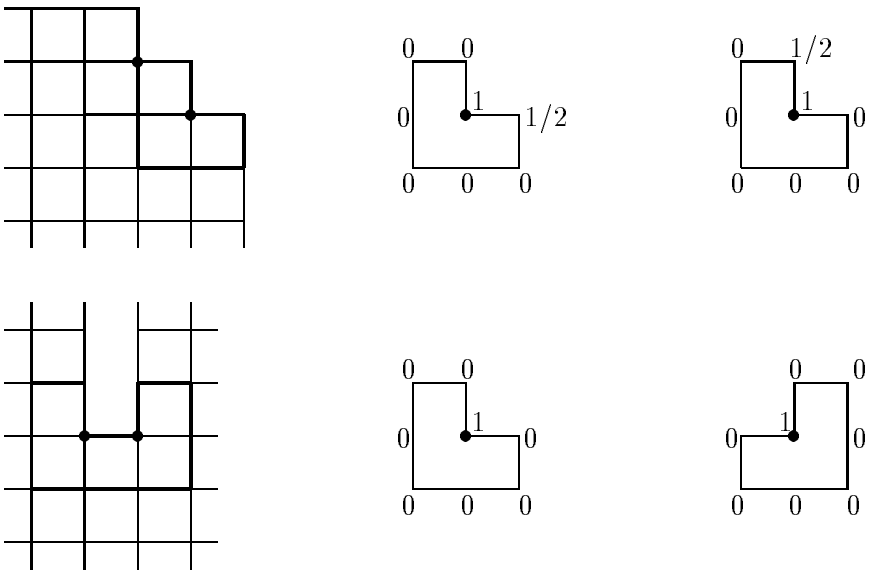


Figure 3.3: Partition of unity functions θ_i for 2-dim L-shaped subregions

not work for L-points. Therefore, we introduce some additional subspaces. In the case of Dirichlet boundary conditions, these additional spaces are introduced only in the proof to obtain a good partition of unity; they are not actually needed in the algorithm, because their functions can be decomposed into a sum of functions belonging to regular subspaces. This will be shown in Lemma 3.2 at the end of this section. The situation is different in the case of Neumann boundary conditions: if we want an optimal algorithm, these special spaces have to be included in the algorithm, see the remark regarding other boundary conditions below. For every L-point x_i, let

$$V_{L_i}^p = V^p \bigcap H_0^1(\Omega'_{L_i}),$$

where Ω'_{L_i} is the L-shaped subregion consisting of the three elements sharing the L-point x_i, see figure 3.1. Assuming that these L-shaped subregions do not overlap, we construct θ_i in the same way as in the regular case. If they overlap, then θ_i can be constructed in each special case, see figure 3.3 for the cases with two L-points. Let N_L be the number of points on $\partial\Omega$ with additional subspaces introduced for L-points. In our proof, we will work with $N + N_L$ subspaces V_i^p, corresponding to regular and L-shaped subregions, respectively.

Step 2: Construction of the interpolation operator I_p. It is easy to see that $\theta_i w_p$ is an element of V^{p+1} vanishing outside Ω'_i. Since in our partition we need an element of V_i^p, we interpolate $\theta_i w_p$ back into V_i^p. We define this interpolation operator I_p on one of the elements Ω_{i_j} of Ω'_i; on the others the construction is completely analogous. We transform this element into the reference square $[-1,1]\times[-1,1]$. Our partition of unity function is a linear combination of the four standard V^1 basis functions

$$\theta_i = \frac{1}{4}(x \pm 1)(y \pm 1)\,,$$

Let $u_{p,i} = I_p(\theta_i w_p)$, i.e. the polynomial in Q_p which interpolate $\theta_i w_p$ at the $(p+1)^2$ points (x_n, x_m), where the $x'_n s$ are the zeros of the polynomial

$$\mathcal{L}_{p+1}(x) = \int_{-1}^{x} L_p(s)\,ds\,. \tag{3.7}$$

Here $L_p(s)$ is the Legendre polynomial of degree p. This definition makes sense for $p \geq 1$ because $\mathcal{L}_{p+1}$ has $p+1$ distinct real zeros in $[-1,1]$. In fact $\mathcal{L}_{p+1}(\pm 1) = 0$ and $p-1$ roots interleave those of L_p, which, as is well known, has p distinct real zeros in $[-1,1]$. We

define $\mathcal{L}_0 = 1$. We remark that while this definition of the interpolation operator is local, we obtain an element in V_i^p. In fact, $u_{p,i}$ is continuous across element boundaries, because on each edge there are $p+1$ interpolation points uniquely determining a polynomial of degree p. Since I_p is a linear operator, we have

$$\sum_{i=1}^{N+N_L} u_{p,i} = u_p - u_{p,0} \, .$$

We note that

$$I_p|_{Q_p} = identity \, .$$

Since $|\cdot|_{H^1}$ is a seminorm on Q_p, it is natural to introduce the quotient space $\hat{Q}_p = Q_p/Q_0$, on which $|\cdot|_{H^1}$ is a norm. Clearly $dim Q_p = (p+1)^2$, while $dim\hat{Q}_p = (p+1)^2 - 1$. We now need to establish that the interpolation operator is uniformly bounded in the H^1-seminorm.

Lemma 3.1 *The interpolation operator $I_p : \hat{Q}_{p+1}([-1,1]^2) \to \hat{Q}_p([-1,1]^2)$ is uniformly bounded in the H^1-norm, i.e.*

$$|I_p(f)|_{H^1} \leq const.|f|_{H^1} \, , \qquad \forall f \in \hat{Q}_{p+1}([-1,1]^2) \, .$$

Proof. If f is a function of x only, then $I_p(f)$ is a function of x only and

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} = \frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}}.$$

Similarly, if f is a function of y only

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} = \frac{\|\frac{\partial}{\partial y} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial y}\|^2_{L^2}}.$$

In general, if both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not vanish identically, it is easy to see that

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} = \frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2} + \|\frac{\partial}{\partial y} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2} + \|\frac{\partial f}{\partial y}\|^2_{L^2}} \leq$$

$$\leq \frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial y} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial y}\|^2_{L^2}} \, . \tag{3.8}$$

We consider the two terms separately. By symmetry, we need only to study the first term. We form a basis for $\hat{Q}_{p+1}$ using the polynomials

$$\phi_{0,j}(x,y) = \frac{1}{\sqrt{2}} \frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|_{L^2}} , \qquad 1 \le j \le p+1,$$

$$\phi_{i,0}(x,y) = \frac{1}{\sqrt{2}} \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|_{L^2}} , \qquad 1 \le i \le p+1,$$

and

$$\phi_{i,j}(x,y) = \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|_{L^2}} \frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|_{L^2}} , \qquad 1 \le i,j \le p+1. \tag{3.9}$$

We can disregard the space spanned by the $\phi_{0,j}$'s because every function f_0 in that space will not contribute to the x-term we are considering. In fact, if $f = f_0 + \tilde{f}$, then

$$\|\frac{\partial}{\partial x} I_p f\|_{L^2} = \|\frac{\partial}{\partial x} I_p \tilde{f}\|_{L^2} \qquad \text{and} \qquad \|\frac{\partial}{\partial x} f\|_{L^2} = \|\frac{\partial}{\partial x} \tilde{f}\|_{L^2} .$$

In the resulting space $\tilde{Q}_{p+1}$ of dimension $p' = (p+2)(p+1)$, we choose the order

$$\phi_{10}, \phi_{11}, \cdots, \phi_{1p+1}, \phi_{20}, \phi_{21}, \cdots, \phi_{2p+1}, \cdots, \phi_{p+1,0}, \phi_{p+1,1}, \cdots, \phi_{p+1,p+1}.$$

If, for simplicity, we relabel the basis as $\{\phi_k(x,y)\,,\ k = 1, 2, \cdots, p'\}$, we obtain

$$f(x,y) = \sum_{k=1}^{p'} \alpha_k \phi_k(x,y) ,$$

$$I_p(f) = \sum_{k=1}^{p'} \alpha_k I_p(\phi_k) , \qquad \text{with} \qquad I_p(\phi_k) = \begin{cases} 0 & \text{if } \phi_k \in \tilde{Q}_{p+1} - \tilde{Q}_p \\ \phi_k & \text{if } \phi_k \in \tilde{Q}_p . \end{cases}$$

The last relations follow from the fact that the coordinates of the interpolation nodes (x_n, x_m) are the zeros of $\mathcal{L}_{p+1}$ and $I_p|_{\tilde{Q}_p}$ is the identity. Hence

$$\|\frac{\partial f}{\partial x}\|^2_{L^2} = (\sum_{k=1}^{p'} \alpha_k \frac{\partial \phi_k}{\partial x}, \sum_{l=1}^{p'} \alpha_l \frac{\partial \phi_l}{\partial x})_{L^2} = \sum_{k,l} \alpha_k \alpha_l (\frac{\partial \phi_k}{\partial x}, \frac{\partial \phi_l}{\partial x})_{L^2} =$$

$$= \alpha^T S_x \alpha ,$$

where $(S_x)_{k,l} = (\frac{\partial \phi_k}{\partial x}, \frac{\partial \phi_l}{\partial x})_{L^2}$ is a symmetric, positive definite matrix of order p'. Similarly,

$$\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2} = \alpha^T B S_x B \alpha ,$$

where

$$B = \begin{pmatrix} I & & & & & & & \\ & 0 & & & & & & \\ & & I & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & I & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

is a projection matrix onto $\tilde{Q}_p$. Here the identity matrices are all of order $p+1$, the last zero block is of order $p+2$ and the other zeros are of order 1. We find

$$\frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}} = \frac{\alpha^T B S_x B \alpha}{\alpha^T S_x \alpha} .$$

The proof of Lemma 3.1 follows from a bound for the eigenvalues λ of the generalized eigenvalue problem

$$B S_x B \alpha = \lambda S_x \alpha . \tag{3.10}$$

The structure of S_x can be made explicit by using the orthogonality of the Legendre polynomials and the formula

$$\int_{-1}^{x} L_n(s)\, ds \; = \frac{L_{n+1}(x) - L_{n-1}(x)}{2n+1} , \qquad n \geq 1 . \tag{3.11}$$

In fact, if $\phi_k(x,y) = \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|}\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|}$ and $\phi_l(x,y) = \frac{\mathcal{L}_n(x)}{\|L_{n-1}\|}\frac{\mathcal{L}_m(y)}{\|\mathcal{L}_m\|}$, we find

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_k(x,y), \frac{\partial}{\partial x}\phi_l(x,y))_{L^2_{xy}} = (\frac{L_{i-1}(x)}{\|L_{i-1}\|}\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|}, \frac{L_{n-1}(x)}{\|L_{n-1}\|}\frac{\mathcal{L}_m(y)}{\|\mathcal{L}_m\|})_{L^2_{xy}} =$$

$$= (\frac{L_{i-1}(x)}{\|L_{i-1}\|}, \frac{L_{n-1}(x)}{\|L_{n-1}\|})_{L^2_x} (\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_m(y)}{\|\mathcal{L}_m\|})_{L^2_y} .$$

This expression differs from zero iff

$$n = i \qquad \text{and} \qquad m = \begin{cases} j-2 \\ j \\ j+2 \end{cases} ,$$

and therefore each row of S_x has at most three nonzero elements. They are

$$(S_x)_{kl} = \begin{cases} (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_{j-2}}{\|\mathcal{L}_{j-2}\|})_{L^2_y} \\ (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}) = 1 \\ (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_{j+2}}{\|\mathcal{L}_{j+2}\|})_{L^2_y} \end{cases} .$$

The only exceptions to this rule occur when one of the indexes is $0, 1$ or 2. In this case we cannot use formula (3.11), but we can use $\mathcal{L}_0 = 1 = L_0$ and $\mathcal{L}_1(x) = \int_{-1}^{x} ds = x + 1 = L_1 + L_0$. The exceptional elements are then

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,0}, \frac{\partial}{\partial x}\phi_{i,1}) = 1 \cdot (\frac{1}{\sqrt{2}}, \frac{\mathcal{L}_1(y)}{\|\mathcal{L}_1\|})_{L^2_y} = \sqrt{\frac{3}{2}} = c_0 \ .$$

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,0}, \frac{\partial}{\partial x}\phi_{i,2}) = 1 \cdot (\frac{1}{\sqrt{2}}, \frac{\mathcal{L}_2(y)}{\|\mathcal{L}_2\|})_{L^2_y} = -\sqrt{\frac{5}{6}} = b_0 \ .$$

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,1}, \frac{\partial}{\partial x}\phi_{i,2}) = 1 \cdot (\frac{\mathcal{L}_1(y)}{\|\mathcal{L}_1\|}, \frac{\mathcal{L}_2(y)}{\|\mathcal{L}_2\|})_{L^2_y} = -\sqrt{\frac{5}{8}} = c_1 \ .$$

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,1}, \frac{\partial}{\partial x}\phi_{i,3}) = 1 \cdot (\frac{\mathcal{L}_1(y)}{\|\mathcal{L}_1\|}, \frac{\mathcal{L}_3(y)}{\|\mathcal{L}_3\|})_{L^2_y} = -\sqrt{\frac{7}{40}} = b_1 \ .$$

This shows that S_x has the structure

$$S_x = \begin{pmatrix} A_{p+2} & & & \\ & A_{p+2} & & \\ & & \ddots & \\ & & & A_{p+2} \end{pmatrix} .$$

Each block A_{p+2} is symmetric, pentadiagonal and of order $p+2$

$$A_{p+2} = \begin{pmatrix} 1 & c_0 & b_0 & & & & & & & \\ c_0 & 1 & c_1 & b_1 & & & & & & \\ b_0 & c_1 & 1 & 0 & b_2 & & & & & \\ & b_1 & 0 & 1 & 0 & ... & & & & \\ & & b_2 & 0 & 1 & ... & ... & & & \\ & & & ... & ... & ... & & b_{p-2} & & \\ & & & & ... & & 1 & 0 & b_{p-1} & \\ & & & & & b_{p-2} & 0 & 1 & 0 & \\ & & & & & & b_{p-1} & 0 & 1 & \end{pmatrix} .$$

The elements c_0, c_1 and b_0, b_1 have been defined above and

$$b_j = (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_{j+2}}{\|\mathcal{L}_{j+2}\|})_{L^2_y} \qquad j \geq 2 \ .$$

By using (3.11) and $\|L_n\|^2_{L^2[-1,1]} = \frac{2}{2n+1}$, we can compute the b_j's explicitly .

$$\|\mathcal{L}_j\|^2 = \frac{4}{(2j-3)(2j-1)(2j+1)} \ , \qquad j \geq 2 \ , \qquad \|\mathcal{L}_1\| = \sqrt{\frac{8}{3}} \ ,$$

$$(\mathcal{L}_j, \mathcal{L}_{j+2}) = -\frac{2}{(2j-1)(2j+1)(2j+3)}\,, \qquad j \geq 1\,, \qquad (\mathcal{L}_1, \mathcal{L}_2) = -\frac{2}{3}$$

and therefore

$$b_j = -\frac{1}{2}\sqrt{\frac{(2j-3)(2j+5)}{(2j-1)(2j+3)}}\,, \qquad j \geq 2\,. \tag{3.12}$$

Our generalized eigenvalue problem (3.10) can now be written as

$$\begin{pmatrix} B_{p+2} & & & & \\ & B_{p+2} & & & \\ & & \ddots & & \\ & & & B_{p+2} & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{p+1} \end{pmatrix} =$$

$$= \lambda \begin{pmatrix} A_{p+2} & & & & \\ & A_{p+2} & & & \\ & & \ddots & & \\ & & & A_{p+2} & \\ & & & & A_{p+2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{p+1} \end{pmatrix}\,,$$

where

$$B_{p+2} = \begin{pmatrix} A_{p+1} & \\ & 0 \end{pmatrix}\,.$$

The last zero is a scalar and the α_i's are vectors of length $p+2$. This is equivalent to $\alpha_{p+1} = 0$ and

$$\begin{pmatrix} A_{p+1} & \\ & 0 \end{pmatrix} \alpha_i = \lambda A_{p+2}\alpha_i\,, \qquad 1 \leq i \leq p\,. \tag{3.13}$$

But

$$A_{p+2} = \begin{pmatrix} A_{p+1} & b \\ b^T & 1 \end{pmatrix}\,,$$

with $b^T = (0, \cdots, 0, b_{p-1}, 0) = b_{p-1}e^T$, where e is the $p-th$ column of the identity matrix of order $p+1$. Therefore, (3.13) is equivalent to

$$\begin{pmatrix} A_{p+1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} A_{p+1} & b \\ b^T & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}\,,$$

i.e.

$$A_{p+1}v = \lambda(A_{p+1} - b_{p-1}^2 ee^T)v\,. \tag{3.14}$$

Since ee^T has rank 1, we see immediately that we have p eigenvalues equal to 1, corresponding to the eigenvectors v with $v_p = 0$. In order to find the only non-trivial

eigenvalue, we apply the Shermann-Morrison formula (see Golub and Van Loan, [44], p. 51)

$$(A + uv^T)^{-1} = A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} uv^T A^{-1} \tag{3.15}$$

to $A_{p+1} - b_{p-1}^2 ee^T$. Clearly, A_{p+1} is positive definite, because $\alpha^T A_{p+1} \alpha$ defines a L^2-norm of a function with components $\{\alpha_i\}$, and therefore it is invertible. We obtain

$$(A_{p+1} - b_{p-1}^2 ee^T)^{-1} = A_{p+1}^{-1} + \frac{b_{p-1}^2}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} A_{p+1}^{-1} ee^T A_{p+1}^{-1} .$$

With this formula, we have reduced (3.14) to the standard eigenvalue problem

$$Mv = (A_{p+1} - b_{p-1}^2 ee^T)^{-1} A_{p+1} v = \lambda v , \tag{3.16}$$

with

$$M = I + \frac{b_{p-1}^2}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} A_{p+1}^{-1} ee^T .$$

By substituting $v \in \text{nullspace}(A_{p+1}^{-1} ee^T)$ into (3.16), we see that the eigenspace corresponding to $\lambda = 1$ has dimension p. We obtain the non-trivial eigenvalue by substituting $v = A_{p+1}^{-1} e$ into (3.16):

$$Mv = A_{p+1}^{-1} e + \frac{b_{p-1}^2 e^T A_{p+1}^{-1} e}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} A_{p+1}^{-1} e = \lambda A_{p+1}^{-1} e = \lambda v ,$$

with

$$\lambda = 1 + \frac{b_{p-1}^2 e^T A_{p+1}^{-1} e}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} = \frac{1}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} . \tag{3.17}$$

In order to obtain an upper bound for λ, we need some properties of A_{p+1}^{-1} and the sequence of b_j's . Since from (3.12), $b_{p-1}^2 < \frac{1}{4}$, we find that

$$\lambda < \frac{1}{1 - \frac{1}{4} e^T A_{p+1}^{-1} e} .$$

What remains is to find a bound on $e^T A_{p+1}^{-1} e =$ the $p-$th diagonal element of A_{p+1}^{-1} . Let $a_p = det(A_p)$. By Cramer's rule,

$$e^T A_{p+1}^{-1} e = \frac{\hat{a}_p}{a_{p+1}} , \tag{3.18}$$

where the cofactor $\hat{a}_p$ is the determinant of the matrix obtained from A_{p+1} by deleting the p-th row and column. Applying Laplace's Theorem for the expansion of determinants to a_{p+1} and $\hat{a}_p$, it is easy to prove the following recurrence relations:

$$a_{p+1} = a_p - b_{p-2}^2(a_{p-2} - b_{p-3}^2 a_{p-3}) , \qquad p \geq 4, \tag{3.19}$$

$$\hat{a}_p = a_{p-1} - b_{p-2}^2 a_{p-2} , \qquad p \geq 4 . \tag{3.20}$$

(3.19) can be written as

$$\frac{a_p - a_{p+1}}{b_{p-2}^2} = a_{p-2} - b_{p-3}^2 a_{p-3} , \tag{3.21}$$

which shows that

$$a_p > a_{p+1} \qquad \text{iff} \qquad a_{p-2} > b_{p-3}^2 a_{p-3} . \tag{3.22}$$

Since A_{p+1}^{-1} is positive definite. substituting (3.20) into (3.18) and using (3.21), we get

$$0 < eA_{p+1}^{-1}e = \frac{a_{p-1} - b_{p-2}^2 a_{p-2}}{a_{p+1}} = \frac{a_{p+1} - a_{p+2}}{b_{p-1}^2 a_{p+1}} = \frac{1}{b_{p-1}^2}(1 - \frac{a_{p+2}}{a_{p+1}}) . \tag{3.23}$$

This implies that $1 - \frac{a_{p+2}}{a_{p+1}} > 0$, i.e. $a_{p+1} > a_{p+2}$, $\forall p$. By (3.22) we then obtain $a_{p+2} > b_{p+1}^2 a_{p+1}$, i.e. $\frac{a_{p+2}}{a_{p+1}} > b_{p+1}^2$, $\forall p$. Hence

$$e^T A_{p+1}^{-1} e < \frac{1}{b_{p-1}^2}(1 - b_{p+1}^2) .$$

Since $\lim_{p\to\infty} b_p^2 = \frac{1}{4}$ and $b_p^2 < \frac{1}{4}$, for every $\epsilon > 0$ we have $b_p^2 > \frac{1}{4}(1-\epsilon)$ for p large enough. This gives us the desired bound

$$e^T A_{p+1}^{-1} e < \frac{4}{1-\epsilon}(1 - \frac{1}{4}(1-\epsilon)) = \frac{3+\epsilon}{1-\epsilon} = 3 + \epsilon'$$

for p large enough, and finally

$$\lambda < \frac{1}{1 - \frac{1}{4}(3+\epsilon')} = \frac{4}{1-\epsilon'} .$$

In other words, $\lambda < const.$ uniformly in p . Numerical experiments in MATLAB show that a stronger result is actually true : $\lim_{p\to\infty} \lambda = 2$.

In conclusion, we have found a bound for the x-term in (3.8):

$$sup \frac{\| \frac{\partial}{\partial x} I_p(f) \|_{L^2}^2}{\| \frac{\partial f}{\partial x} \|_{L^2}^2} \leq const. .$$

Reasoning in the same way for the y-term, we find from formula (3.8)

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} \leq const.,$$

which completes the proof of Lemma 3.1.

□

We can now conclude the proof of Theorem 3.1 by applying Lemma 3.1 to bound the H^1-norm of each component $u_{p,i} = I_p(\theta_i w_p)$ over a single element Ω_k .

$$|u_{p,i}|^2_{H^1(\Omega_k)} \leq C|\theta_i w_p|^2_{H^1(\Omega_k)} =$$

$$= C(\|\frac{\partial \theta_i}{\partial x} w_p + \theta_i \frac{\partial w_p}{\partial x}\|^2_{L^2(\Omega_k)} + \|\frac{\partial \theta_i}{\partial y} w_p + \theta_i \frac{\partial w_p}{\partial y}\|^2_{L^2(\Omega_k)}) \leq$$

$$\leq 2C(\|\frac{\partial \theta_i}{\partial x} w_p\|^2_{L^2(\Omega_k)} + \|\theta_i \frac{\partial w_p}{\partial x}\|^2_{L^2(\Omega_k)} + \|\frac{\partial \theta_i}{\partial y} w_p\|^2_{L^2(\Omega_k)} + \|\theta_i \frac{\partial w_p}{\partial y}\|^2_{L^2(\Omega_k)}) .$$

On a square element Ω_k of side H , $|\frac{\partial \theta_i}{\partial x}|$ and $|\frac{\partial \theta_i}{\partial y}|$ are bounded by $1/H$ and, by construction, $\|\theta_i\|_{L^\infty} \leq 1$. Therefore

$$|u_{p,i}|^2_{H^1(\Omega_k)} \leq 2C(\frac{2}{H^2}\|w_p\|^2_{L^2(\Omega_k)} + |w_p|^2_{H^1(\Omega_k)}) .$$

Since at most 4 components $u_{p,i}$ are nonzero for any element Ω_k , we obtain, when summing over i,

$$\sum_{i=1}^{N+N_L} |u_{p,i}|^2_{H^1(\Omega_k)} \leq 8C(\frac{2}{H^2}\|w_p\|^2_{L^2(\Omega_k)} + |w_p|^2_{H^1(\Omega_k)}) ,$$

and summing over all the elements Ω_k

$$\sum_{i=1}^{N+N_L} |u_{p,i}|^2_{H^1(\Omega)} \leq 8C(\frac{2}{H^2}\|w_p\|^2_{L^2(\Omega)} + |w_p|^2_{H^1(\Omega)}) . \tag{3.24}$$

Using equations (3.5) and (3.6), we conclude

$$\sum_{i=1}^{N+N_L} |u_{p,i}|^2_{H^1(\Omega)} \leq 8C(2C_1 + C_2)|u_p|^2_{H^1(\Omega)} = const\, |u_p|^2_{H^1(\Omega)} .$$

By Lemma 3.2, proved below, we decompose the functions belonging to the N_L additional subspaces and distribute them among neighboring regular subspaces. If we still denote by $u_{p,i}$ the terms of the decomposition, we obtain

$$\sum_{i=1}^{N} |u_{p,i}|^2_{H^1(\Omega)} \leq const\, |u_p|^2_{H^1(\Omega)} .$$

By Theorem 2.2, Theorem 3.1 is then proved.

□

As mentioned before, we can remove any additional subspace V_i^p, corresponding to an L-point, from the algorithm in the Dirichlet case. In fact, the following Lemma proves that each function in such spaces can be decomposed, in an acceptable way, into a sum of two functions belonging to regular subspaces.

Lemma 3.2 *Let $v \in V_{L_i}^p$ and Ω_i' be represented as in figure 3.4, with a and b internal points. Then there exists two functions $v_a \in V_a^p$ and $v_b \in V_b^p$ such that $v = v_a + v_b$ and*

$$a(v_a, v_a) + a(v_b, v_b) \leq C\, a(v, v).$$

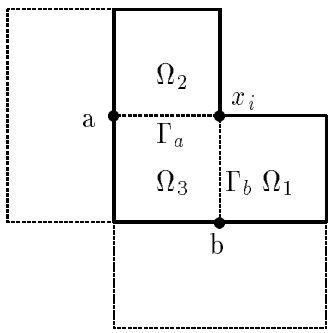


Figure 3.4: Decomposition of $V_{L_i}^p$ in 2-dim

Proof. Define

$$v_a = \begin{cases} v \text{ in } \Omega_2 \bigcup \Gamma_a \\ \text{reflection across } \Gamma_a \text{ from } \Omega_2 \text{ onto } \Omega_3 . \end{cases}$$

Since the support of v_a is contained in $\Omega_2 \bigcup \Omega_3 \bigcup \Gamma_a$, we have $v_a \in V_a^p$, and

$$|v_a|_{H^1(\Omega_a)} = 2|v_a|_{H^1(\Omega_2)} = 2|v|_{H^1(\Omega_2)} \leq 2|v|_{H^1(\Omega_i')} .$$

$v - v_a$ vanishes on Γ_a and by the triangle inequality $|v - v_a|_{H^1(\Omega_i')} \leq 3|v|_{H^1(\Omega_i')}$. We can therefore define $v_b = v - v_a$.

□

Remark: Neumann and mixed boundary conditions Away from the boundary, we use the previous regular spaces $V_i^p = V^p \bigcap H_0^1(\Omega_i')$. Consider now a subregion Ω_i' : it is the union of four squares if it is a regular subregion and it is a union of three squares if x_i is an L-point. We have to describe the case when $\partial\Omega_i'$ intersects $\partial\Omega$.
a) If $\partial\Omega_i' \bigcap \Gamma_N = \emptyset$, we are in the Dirichlet case considered before.
b) If $\partial\Omega_i' \bigcap \Gamma_N \neq \emptyset$, we define a *Neumann* subspace associated with Ω_i' as

$$V_i^p = V^p \bigcap H_N^1(\Omega_i'),$$

where $H_N^1(\Omega_i') = \{v \in H^1(\Omega_i') : v = 0 \, \mathrm{on} \, \partial\Omega_i' \backslash \Gamma_N\}$. With this modification, we can repeat the analysis given before for the Dirichlet case. The partition of unity is constructed in the same way. If we want to prove a constant bound, we cannot eliminate the subspaces corresponding to points x_i with $\partial\Omega \bigcap \Gamma_N \neq \emptyset$, i.e. with part of the boundary subject to a Neumann condition. However, it is proved in the next chapter, that Lemma 3.2 can be generalized to such a case with a bound growing logarithmically with p. Therefore, the Neumann subspaces, regular or not, have to be included in the algorithm if we want an optimal bound.

3.3 Overlapping ASM using cubic elements in three dimensions

Theorem 3.1 can be extended to dimension three and brick-shaped elements and, by induction, to an arbitrary dimension. We assume for simplicity that the region $\Omega \subset R^3$ is the union of non-overlapping cubic elements Ω_i of side H. We consider the simplest case, with homogeneous Dirichlet boundary conditions on $\partial\Omega$; the extension to Neumann and mixed boundary conditions can be carried out as in the two dimensional case. If N is the number of interior nodes, we represent V_D^p as

$$V_D^p = V_0^p + V_1^p + \cdots + V_N^p \, ,$$

where

- $V_0^p = V_D^1$

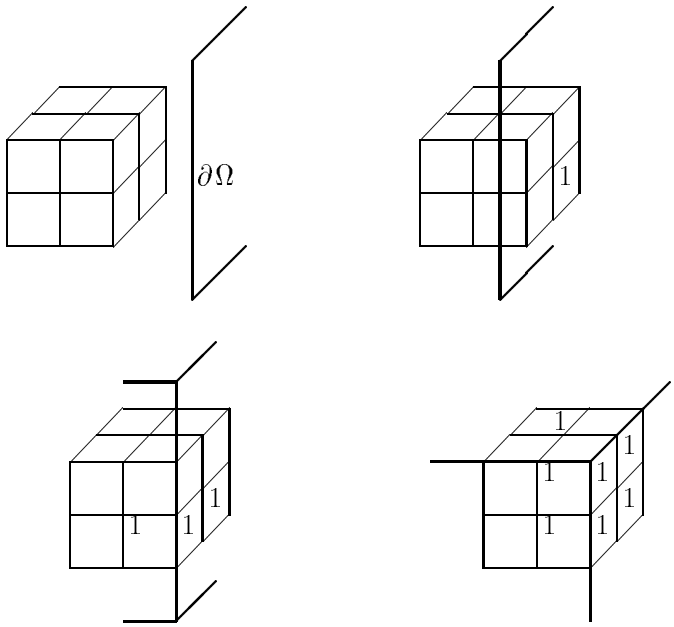


Figure 3.5: Partition of unity functions θ_i for 3-dim regular subregions

- $V_i^p = V^p \cap H_0^1(\Omega_i')$.

Now Ω_i' is the open cube of side $2H$ centered at the i-th interior node.

We now prove the main theorem in dimension three. The upper bound is obtained as before. Using the same notation as in the two dimensional case, we define a partition of the finite element function u_p , required by Theorem 2.2. $u_{p,0}$ is defined in the same way by L^2-projection or by smoothing and interpolation. The piecewise trilinear partition of unity $\{\theta_i\}$, needed to define $u_{p,i}$, is constructed in a similar way by specifying the values of θ_i at the mesh points of Ω_i'. In the closure of each subregion, there are 27 mesh points: the center x_i and the 26 nearest neighboors x_j . We define: $\theta_i(x_i) = 1$ and

$$\theta_i(x_j) = \begin{cases} 1 & \text{if all the faces meeting at } x_j \text{ are on } \partial\Omega \\ 0 & \text{otherwise,} \end{cases}$$

Some cases are illustrated in figure 3.5. Again, this construction does not work for L-points. In two dimensions, there is essentially one kind of L-shaped region, obtained by removing one square from the union of four. In three dimensions, the situation is

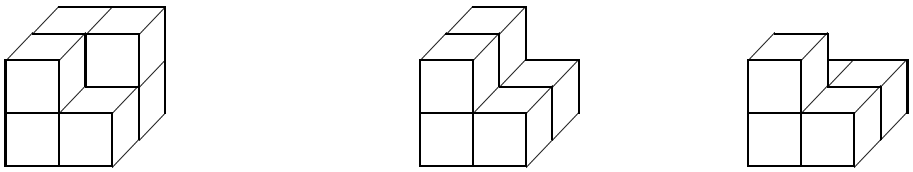


Figure 3.6: Some 3-dim L-shaped subregions

more complicated, because there are three kinds of L-shaped subregions, obtained by removing one, two or three cubes from the union of eight (we consider only Lipschitz subregions), see figure 3.6. It is not difficult to see that we can generalize the construction of partition of unity functions for two dimensional L-shaped subregions to these three dimensional cases . In our analysis, we introduce additional subspaces $V_{L_i}^p$ associated to the L-points. Because of our hypothesis on the interior edges meeting at an L-point, we can then decompose theses additional subspaces into regular ones, in the Dirichlet case. This is proved in Lemma 3.4 at the end of this section. Therefore, they are only auxiliary tools in the proof and are not needed in the algorithm.

We then proceed to the definition of the components $u_{p,i}$ using the interpolation operator I_p in three dimensions. On the reference cube $[-1,1]^3$, θ_i is a linear combination of the eight standard V^1 basis functions

$$\theta_i = \frac{1}{8}(x \pm 1)(y \pm 1)(z \pm 1)\,.$$

Recalling that $w_p = u_p - u_{p,0}$, we define $u_{p,i} = I_p(\theta_i w_p)$ as the polynomial in Q_p interpolating $\theta_i w_p$ at the $(p+1)^3$ points (x_l, x_m, x_n) , where the x_n's are the zeros of the integrals of the Legendre polynomials $\mathcal{L}_{p+1}$ defined in (3.7). Working again with the quotient spaces $\hat{Q}_p = Q_p/Q_0$, we establish

Lemma 3.3 *The interpolation operator $I_p : \hat{Q}_{p+1}([-1,1]^3) \to \hat{Q}_p([-1,1]^3)$ is uniformly bounded in the H^1-norm, i.e.*

$$|I_p(f)|_{H^1} \leq const.|f|_{H^1}\,, \qquad \forall f \in \hat{Q}_{p+1}([-1,1]^3)\,.$$

Proof. Since

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} \le \frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial y} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial y}\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial z} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial z}\|^2_{L^2}} \,, \tag{3.25}$$

we need only consider one of the three terms, for example the x-term. As before, if in one term the denominator is zero, we can prove that (3.25) is still valid after dropping that term. We form a basis using the polynomials

$$\phi_{i,j,l}(x,y,z) = \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|_{L^2}} \frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|_{L^2}} \frac{\mathcal{L}_l(z)}{\|\mathcal{L}_l\|_{L^2}} \,, \qquad 1 \le i,j,l \le p+1 \,.$$

and the same modification with the constant term $\frac{1}{\sqrt{2}}$ when one index is zero; cf.(3.9). We do not consider polynomials that do not depend on x, because they do not contribute to the x-term we are considering. We order the remaining basis functions in such a way that we can use the results obtained in the two dimensional case. For every fixed l, we have a two dimensional subspace and we order the basis of this subspace as in the two dimensional case. We do this for $l = 0, 1, \cdots, p+1$. Relabeling the $\phi_{i,j,l}$'s as a one dimensional array, we have

$$\|\frac{\partial f}{\partial x}\|^2_{L^2} = \alpha^T C_x \alpha \,, \qquad \|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2} = \alpha^T D C_x D \alpha \,.$$

We are interested in an upper bound for the eigenvalues of the generalized eigenvalue problem

$$DC_xD\alpha = \lambda C_x \alpha \,. \tag{3.26}$$

Here the stiffness matrix C_x, of order $p' = (p+2)^2(p+1)$, has the structure

$$C_x = \begin{pmatrix} S_x & c_0 S_x & b_0 & & & & \\ c_0 S_x & S_x & c_1 S_x & b_1 S_x & & & \\ b_0 S_x & c_1 S_x & S_x & 0 & b_2 S_x & & \\ & b_1 S_x & 0 & S_x & 0 & \ddots & \\ & & b_2 S_x & 0 & S_x & \ddots & b_{p-1} S_x \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & b_{p-1} S_x & 0 & S_x \end{pmatrix}$$

and the interpolation matrix D, of the same size, has the structure

$$D = \begin{pmatrix} B & & & & & \\ & B & & & & \\ & & B & & & \\ & & & \ddots & & \\ & & & & B & \\ & & & & & 0 \end{pmatrix} .$$

The matrices S_x and B have already been defined in Section 3. Clearly, DC_xD is the matrix

$$\begin{pmatrix} \tilde{S_x} & c_0\tilde{S_x} & b_0\tilde{S_x} & & & & \\ c_0\tilde{S_x} & \tilde{S_x} & c_1\tilde{S_x} & b_1\tilde{S_x} & & & \\ b_0\tilde{S_x} & c_1\tilde{S_x} & \tilde{S_x} & 0 & b_2\tilde{S_x} & & \\ & b_1\tilde{S_x} & 0 & \tilde{S_x} & 0 & \ddots & \\ & & b_2\tilde{S_x} & 0 & \tilde{S_x} & \ddots & b_{p-1}\tilde{S_x} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & b_{p-1}\tilde{S_x} & 0 & \tilde{S_x} \end{pmatrix} ,$$

where $\tilde{S_x} = BS_xB$. Therefore equation (3.26) gives us the block equations

$$\begin{aligned} \tilde{S_x}(\alpha_1 + c_0\alpha_2 + b_0\alpha_3) &= \lambda S_x(\alpha_1 + c_0\alpha_2 + b_0\alpha_3) , \\ \vdots \qquad & \qquad \vdots \\ \tilde{S_x}(b_{i-2}\alpha_{i-2} + \alpha_i + b_i\alpha_{i+2}) &= \lambda S_x(b_{i-2}\alpha_{i-2} + \alpha_i + b_i\alpha_{i+2}) , \\ \vdots \qquad & \qquad \vdots \\ \tilde{S_x}(b_{p-1}\alpha_{p-1} + \alpha_{p+1}) &= \lambda S_x(b_{p-1}\alpha_{p-1} + \alpha_{p+1}) . \end{aligned}$$

These are all generalized eigenvalue problems of the form

$$\tilde{S_x}v = \lambda S_x v , \tag{3.27}$$

where v is a linear combination of some α_i . But this is the same generalized eigenvalue problem considered in the two dimensional case; see eq. (3.10). We then apply the two dimensional result and conclude that the eigenvalues λ are bounded by a constant independent of p. Reasoning in the same way for the terms in y and z, we complete the proof of Lemma 3.3.

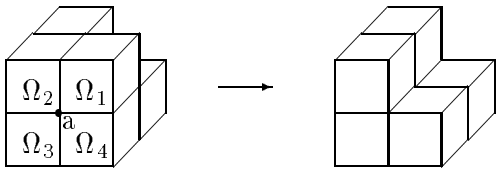


Figure 3.7: First step in the decomposition of $V_{L_i}^p$ in 3-dim

□

In order to complete the proof of Theorem 3.1 in three dimensions, we just repeat the arguments given in the two dimensional case. We first apply Lemma 3.3 to bound the H^1-norm of each component $u_{p,i} = I_p(\theta_i w_p)$ over a single element Ω_k. We then sum over i, recalling that at most 8 components $u_{p,i}$ are nonzero for any element Ω_k and finally we sum over all elements. We conclude the proof using Lemma 3.4, proved below, to remove the additional subspaces, and equations (3.5) and (3.6).

□

The additional subspaces $V_{L_i}^p$ associated to L-points, can be decomposed into a sum of regular subspaces in case of Dirichlet boundary conditions. The following Lemma proves this result using the idea of extension by reflection as in the two dimensional Lemma 3.2. We consider the more general case of Ω_i' formed by seven elements as in figure 3.7. Other cases follow from this one.

Lemma 3.4 *Let $v \in V_{L_i}^p$ and Ω_i' be represented as in figure 3.7, with a, b, and c internal points. Then there exists three functions $v_a \in V_a^p$ and $v_b \in V_b^p$ and $v_c \in V_c^p$ such that $v = v_a + v_b + v_c$ and*

$$a(v_a, v_a) + a(v_b, v_b) + a(v_c, v_c) \leq C a(v, v).$$

Proof. Define

$$v_a = \begin{cases} v \text{ in } \Omega_1 \bigcup F_{12} \bigcup F_{14} \\ \text{reflection across } F_{12} \text{ from } \overline{\Omega}_1 \text{ onto } \overline{\Omega}_2 \\ \text{reflection across } F_{14} \text{ from } \overline{\Omega}_1 \text{ onto } \overline{\Omega}_4 \\ \text{reflection across } F_{23} \text{ from } \overline{\Omega}_2 \text{ onto } \overline{\Omega}_3 \end{cases}$$

Clearly $v_a \in V_a^p$ and

$$|v_a|_{H^1(\Omega_a)} = 4|v_a|_{H^1(\Omega_1)} = 4|v|_{H^1(\Omega_1)} \leq 4|v|_{H^1(\Omega_i')} .$$

$v - v_a$ vanishes on F_{12} and F_{14} and it has support in $\Omega_i' \setminus \Omega_1$. This subregion is now an L-shaped plane region times $[0,1]$ and we can proceed as in the two dimensional case.

□

Remark: The n dimensional case. The result can be extended to any dimension by induction. The only nontrivial part is the proof of the lemma about the interpolation operator. The induction step from dimension n to $n+1$ is essentially analogous to the arguments in the proof of Lemma 3.3. We consider one term at a time and order the basis in the following way: first fix the $(n+1)$-th index to be equal to 1 and order the resulting subspace in the same way as in the case of n variables; then fix the $(n+1)$-th index to be equal to 2 and repeat the process, until the $(n+1)$-th index is equal to $p+1$. With this choice, the stiffness and projection matrices have a block structure that allow us to reduce the $(n+1)$ dimensional generalized eigenvalue problem to one of dimension n.

3.4 Overlapping ASM on the interface

The basis functions of V^p can be hierarchically ordered into nodal, side, face, and interior functions. On a reference, brick-like element, these polynomials vanish on 3, 4, 5 and all faces, respectively; see Mandel [59]. Let us order first the unknowns associated with the interior basis functions and then the ones associated with the interface $\Gamma = \bigcup_i \partial\Omega_i$. The vector of unknowns can then be written $x = (x_I, x_B)$ and the linear system corresponding to our discrete problem 1.4 becomes:

$$\begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix} \begin{pmatrix} x_I \\ x_B \end{pmatrix} = \begin{pmatrix} b_I \\ b_B \end{pmatrix} .$$

Eliminating the interior variables x_I from the system, we are left with the Schur complement equation

$$Sx_B = \tilde{b}_B,$$

where

$$S = K_{BB} - K_{IB}^T K_{II}^{-1} K_{IB} \qquad \text{and} \qquad \tilde{b}_B = b_B - K_{IB}^T K_{II}^{-1} b_I.$$

The interior unknowns x_I are then the solution of the system

$$K_{II} x_I = b_I - K_{IB} x_B,$$

which naturally decouples element by element and can be solved in parallel. The reduced Schur complement is solved with the overlapping ASM introduced in the previous Section. Now only the interface unknowns are involved and the Schur complement is better conditioned than the original system.

It is easy to see that the variational formulation of the discrete problem is now: Find $u_p^* \in \tilde{V}_D^p$ such that

$$a(u_p^*, v_p) = f(v_p)\,, \qquad \forall\ v_p \in\ \tilde{V}_D^p\,, \tag{3.28}$$

where $\tilde{V}_D^p$ is the subspace of the discrete harmonic functions of V_D^p. A function $v \in V^p$ is discrete harmonic if

$$a(v, \phi) = 0,$$

for every $\phi \in V^p$ that vanishes on the interface Γ. In other terms, $v = (v_I, v_B)$ is discrete harmonic if

$$K_{II} v_I + K_{IB} v_B = 0.$$

The algorithm is defined by the following decomposition of $\tilde{V}_D^p$:

$$\tilde{V}_D^p = \tilde{V}_0^p + \tilde{V}_1^p + \cdots + \tilde{V}_N^p\,,$$

where

- $\tilde{V}_0^p = V_0^p$
- $\tilde{V}_i^p = \tilde{V}^p \bigcap H_0^1(\Omega_i')$ and the Ω_i''s are defined as before;

In terms of projections $\tilde{P}_i : \tilde{V}_D^p \rightarrow \tilde{V}_i^p$, the method solves the linear operator equation

$$\tilde{P}u_p = (\tilde{P}_0 + \tilde{P}_1 + \cdots + \tilde{P}_N)u_p = g_p \,, \tag{3.29}$$

by an iterative method such as the conjugate gradient. We can then prove the analogue of Theorem 3.1:

Theorem 3.2 *The operator $\tilde{P}$ of the additive algorithm defined by the spaces $\tilde{V}_i^p$ satisfies the estimate $\kappa(P) \leq$ const. independent of p, H and N.*

Proof. The proof of the upper bound is the same as in Theorem 3.1. To obtain a lower bound, we use the decomposition of a function of $\tilde{V}_D^p \subset V_D^p$ obtained in the proof of Theorem 3.1:

$$\tilde{u}_p = \sum_{i=0}^{N} u_{p,i} \qquad \text{and} \qquad \sum_{i=0}^{N} a(u_{p,i}, u_{p,i}) \leq C_0^2 a(\tilde{u}_p, \tilde{u}_p).$$

We then restrict each component $u_{p,i}$ to Γ and extend it as a discrete harmonic polynomial $\tilde{u}_{p,i} \in \tilde{V}_i^p$. Since the discrete harmonic extension is the one that minimizes the energy, we have obtained the desired decomposition.

□

3.5 Overlapping Multiplicative Schwarz methods

We consider now some multiplicative schemes (MSM) based on the space decomposition introduced before:

$$V_D^p = V_0^p + V_1^p + \cdots + V_N^p \,,$$

with:

- $V_0^p = V_D^1$,
- $V_i^p = V^p \bigcap H_0^1(\Omega_i') \,.$

In terms of projections $P_i : V_D^p \rightarrow V_i^p$ the algorithm can be written:

MSM algorithm:

i) compute $g_i = P_i u$, for $i = 0, 1, \cdots, N$;

ii) given u_n , compute u_{n+1} in $N+1$ fractional steps:

$$u_{n+\frac{i+1}{N+1}} = u_{n+\frac{i}{N+1}} + (g_i - P_i u_{n+\frac{i}{N+1}}), \ i = 0, 1, \cdots, N.$$

Using the estimates for the constant $\rho(\mathcal{E})$ and C_0 obtained in the previous sections for the additive scheme, we can easily apply Theorem 2.1 and obtain:

Theorem 3.3 *The MSM defined by the spaces V_i^p is convergent and the convergence rate is independent of p, H and N. This holds in any dimension and also for the algorithms accelerated with GMRES or CG.*

For a more detailed description of the accelerated versions of MSM, see Chapter 2. As for the additive case, we can consider these multiplicative schemes (accelerated or not) on the interface, namely, we can apply them to solve the Schur complement system obtained after eliminating the interior unknowns. The results proved in the additive case can then be translated into the multiplicative case by considering the discrete harmonic subspaces $\tilde{V}_i^p$ of V_i^p :

Theorem 3.4 *The MSM on the interface defined by the spaces $\tilde{V}_i^p$ is convergent and the convergence rate is in in two dimensionsdependent of p, H and N. This holds in any dimension and also for the algorithms accelerated with GMRES or CG.*

3.6 Numerical experiments in two dimensions

In this section, we describe some numerical experiments with the additive Schwarz p-methods introduced previously. In these experiments, we are not concerned with the issue of parallel or efficient implementation of the method; the main goal here is to verify our theoretical results and to get a sense of the sharpness of the bounds.

For general considerations about parallel implementation of domain decomposition algorithms, see Keyes and Gropp [49], [48], Gropp [47]. For numerical experiments with h-version Schwarz methods and different parallel architectures, see Bjørstad, Moe, and Skogen [15], Bjørstad and Skogen [16], Moe [65], Skogen [75], Gropp and Smith [46], Smith [78]. A detailed analysis of the parallel implementation of the hp-version

finite element method can be found in Babuška, Elman, and Markley [3]. For numerical experiments with other important methods using the p-version finite element method and iterative substructuring ideas, see Babuška, Craig, Mandel, and Pitkäranta [8] and Mandel [56], [57], [58], [55] for three dimensional experiments.

The programs, written in MATLAB, have been run on Sun Sparc workstations. We have considered the Poisson equation in two dimensions with homogeneous Dirichlet boundary conditions. The equation $Pu_p = g_p$ is solved iteratively using the conjugate gradient method. The iteration process is stopped when the relative l^2-norm of the residual is less than 10^{-9}. The eigenvalues of P are approximated by using the Lanczos method. In the following tables, $\kappa(P) = \frac{\lambda_{max}}{\lambda_{min}}$ and $\kappa(K)$ are the condition number of the operator P and of the stiffness matrix K respectively. $l^2 - err$ is the normalized l^2-norm of the error $u - u_p$, for cases when the exact solution u is available.

3.6.1 Overlapping ASM

We consider first a problem with a trigonometric exact solution.
Problem 1:

$$\begin{cases} -\Delta u &= -\frac{2\pi^2}{c^2} sin(\frac{\pi}{c}x) sin(\frac{\pi}{c}y). & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

We consider three decompositions of a square region Ω into 9, 16 and 25 elements ($4, 9, 16$ subregions, respectively). The constant c is chosen such that the right hand side vanishes on $\partial\Omega$. The graph of the solution u is illustrated in figure 3.8 and the numerical results are reported in table 3.1. In this case, λ_{max} is constant, while λ_{min} oscillates and seems to converge as p increases. The condition number is clearly bounded independently of p and the number N of subregions. The number of iterations increases with the number of subregions, but we remark that the theoretical bound for the number of iterations is $\frac{1}{2}\sqrt{\kappa(P)}\log(2 \cdot 10^9) \simeq 21.4$. The algorithm is therefore performing better than expected and this is probably due to the distribution of the other eigenvalues of the spectrum of P. Numerical quadrature is used to compute the right hand side g with tolerance $\simeq 10^{-8}$. This affects the precision reached by the algorithm for $p = 7$ and 8, see Table 3.1.

Next, we consider a problem on an L-shaped region, which can have singular solutions due to the presence of corners in the non convex domain Ω. We decompose the region

into 27 elements and 16 subregions.
Problem 2:

$$\begin{cases} -\Delta u &= 1 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

The graph of the solution u is shown in figure 3.9, and table 3.2 reports the numerical results. Again, λ_{max} is constant. λ_{min} decreases monotonically and $\kappa(P)$ grows very slowly and seems to converge when p is increased.

We also run the algorithm without the coarse space V_0^p , see tables 3.5 and 3.6 for results for problem 1 and 2, respectively. The theory says that in this case the condition number grows proportionally to the number of subdomains across the domain, see Widlund [84]. This is confirmed by our results. Note that in the absence of the coarse space λ_{max} does not change, while λ_{min} decreases considerably. In this case the number of iterations is much better than the theoretical prediction using the condition number.

In figures 3.10, 3.11, 3.12, 3.13, we report also some graphs plotting the convergence history of the basic overlapping ASM for problems 1 for $N = 9, 16$ and $p = 8$, with and without coarse space. It is interesting to note that without a coarse space the convergence of the algorithm is worse only in the first few iterations. After this bad start, the convergence is essentially the same as in the algorithm with coarse space.

3.6.2 Overlapping ASM on the interface

We consider now the additive method on the interface for problem 1, see table 3.3. Here S is the unpreconditioned Schur complement. We remark on the better condition number compared with the previous ASM without interior variables elimination, since the Schur complement is better conditioned. λ_{min} is almost the same, while λ_{max} in now smaller. This is due to the fact that now there is less overlap between subspaces. However, the cost of eliminating the interior variables and the cost of the local interior solvers have to be considered in addition to the conjugate gradient cost. In table 3.4, we consider problem 2 on the same L-shaped region. We similarly obtain an improved condition number and number of iterations; λ_{min} is still the same, while λ_{max} is smaller and no longer constant.

3.6.3 Overlapping MSM

We consider here three multiplicative schemes: the classical Schwarz (MSM), the accelerated version using GMRES (MSGMRES) and the accelerated symmetrized version using conjugate gradient (MSCG). The algorithms are run on the model problems described before and the iteration process is stopped when the normalized l^2-norm of the residual ρ is less than 10^{-9}. Results for problem 1 on a square region and for problem 2 on a L-shaped region, are given in Table 3.7 and Table 3.8 respectively. The convergence history in the two cases for $p = 8$ and $N = 16$ is illustrated in the two graphs of figure 3.14 and 3.15. The theoretical results are clearly confirmed: the number or iterations and the condition number of the MSCG operator are bounded independently of p and N. In comparing these methods, we must take into consideration the different amount of work required in each iteration. In particular, the MSCG operator requires twice the number of fractional steps required by the other two operators. For the easier problem 1, the effect of the acceleration is almost negligible. This shows that for this kind of problems, classical MSM is already a good preconditioner. The advantage of the acceleration becomes visible for the harder problem 2, where MSGMRES and MSCG are still running in a constant small number of iterations, but MSM start needing more iterations for larger p. Considering the amount of work per iteration, MSGMRES seems to be the best method in these experiments. In figure 3.14 and 3.15, we show the convergence history for problem 1 and 2 with $N = 16$ and $p = 8$.

Figure 3.8: Solution u of problem 1

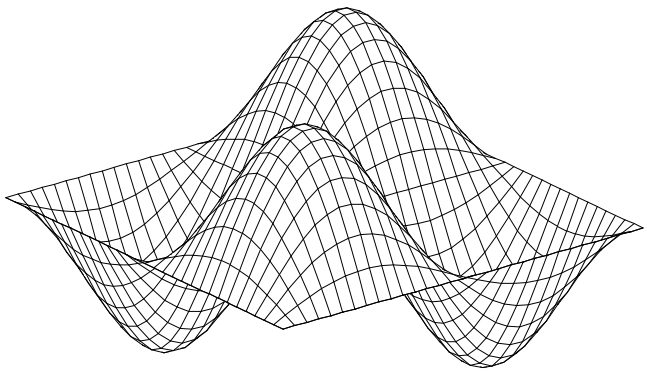
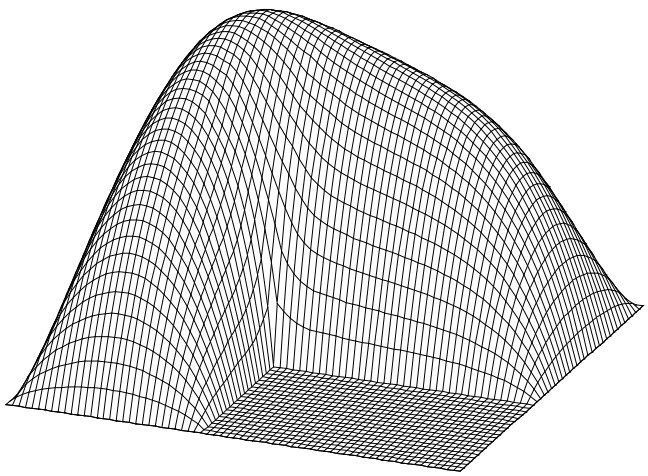


Figure 3.9: Solution u of problem 2



N	p	unkn.	$\kappa(K)$	$\kappa(P)$	λ_{max}	λ_{min}	iter.	$l^2 - err$
	3	64	56.2	4.2119	4	0.9497	10	0.0083
	4	121	154.16	4.1138	4	0.9723	11	8.4248e-4
4	5	196	275.68	4.1414	4	0.9659	12	7.1026e-5
	6	289	539.04	4.1112	4	0.9730	13	5.1892e-6
	7	400	844.81	4.1032	4	0.9748	12	3.4483e-7
	8	529	1399.2	4.0919	4	0.9775	13	3.0349e-8
	3	121	65.75	4.2105	4	0.9501	14	0.0027
	4	225	188.83	4.2467	4	0.9419	15	2.0497e-4
9	5	361	319.48	4.1640	4	0.9606	15	1.2917e-5
	6	529	639.97	4.1830	4	0.9563	15	7.0414e-7
	7	729	972.19	4.1614	4	0.9612	15	3.6437e-8
	8	961	1634.3	4.1705	4	0.9591	16	3.6643e-8
	3	196	71.60	4.1825	4	0.9564	16	0.0011
	4	361	210.16	4.2717	4	0.9364	16	6.7965e-5
16	5	576	311.28	4.2090	4	0.9504	16	3.4208e-6
	6	841	701.34	4.2304	4	0.9455	16	1.4984e-7
	7	1156	959.90	4.2108	4	0.9499	16	5.3302e-8
	8	1521	1776.8	4.2202	4	0.9478	17	6.4125e-8

Table 3.1: ASM for problem 1

p	unkn.	$\kappa(K)$	$\kappa(P)$	λ_{max}	λ_{min}	iter.
3	208	69.99	5.3761	4	0.7440	18
4	385	202.51	5.6790	4	0.7043	19
5	616	339.52	5.8843	4	0.6798	19
6	901	682.67	6.0414	4	0.6621	20
7	1240	884.55	6.1633	4	0.6490	20
8	1633	2255.1	6.2626	4	0.6387	20

Table 3.2: ASM for problem 2 on an L-shaped region

N	p	unkn.	$\kappa(S)$	$\kappa(P)$	λ_{max}	λ_{min}	iter.	$l^2 - err$
	3	28	13.0218	2.4872	2.3621	0.9497	4	0.0083
	4	40	23.4768	2.3548	2.3683	1.0057	5	8.4248e-4
4	5	52	31.3066	2.4511	2.3673	0.9658	7	7.1026e-5
	6	64	45.5085	2.3646	2.3675	1.0012	7	5.1892e-6
	7	76	56.7395	2.4289	2.3674	0.9747	8	3.4483e-7
	8	88	74.5727	2.3648	2.3674	1.0011	7	3.0349e-8
	3	57	15.2793	2.2454	2.2454	1.0000	4	0.0027
	4	81	28.3993	2.2449	2.2449	1.0000	5	2.0497e-4
9	5	105	36.2878	2.2449	2.2449	1.0000	6	1.2917e-5
	6	129	53.6689	2.2449	2.2449	1.0000	6	7.0414e-7
	7	153	65.3747	2.2449	2.2449	1.0000	6	3.6437e-8
	8	177	86.8770	2.2449	2.2449	1.0000	6	3.6643e-8
	3	96	16.6324	3.0655	2.8946	0.9443	11	0.0011
	4	136	31.3978	2.9832	2.9088	0.9751	13	6.7965e-5
16	5	176	39.2160	2.9536	2.9152	0.9870	13	3.4208e-6
	6	216	58.4926	2.9696	2.9187	0.9828	13	1.4984e-7
	7	256	70.4135	2.9602	2.9211	0.9868	13	5.3302e-8
	8	296	94.0759	2.9659	2.9226	0.9854	13	6.4125e-8

Table 3.3: ASM on the interface for problem 1

p	unkn.	$\kappa(S)$	$\kappa(P)$	λ_{max}	λ_{min}	iter.
3	100	16.3266	4.0267	2.9960	0.7440	18
4	142	30.4453	4.2864	3.0191	0.7043	18
5	184	38.6854	4.4587	3.0309	0.6798	18
6	226	57.3283	4.5891	3.0384	0.6621	19
7	268	68.7481	4.6895	3.0435	0.6490	19
8	310	92.6457	4.7708	3.0472	0.6387	19

Table 3.4: ASM on the interface for problem 2 on an L-shaped region

N	degree p	$\kappa(P)$	λ_{max}	λ_{min}	# of iter.
4	3	6.4899	4	0.6163	10
	4	6.5005	4	0.6153	11
	5	6.4868	4	0.6166	12
	6	6.4891	4	0.6164	13
	7	6.4856	4	0.6167	12
	8	6.4869	4	0.6166	14
9	3	9.9679	4	0.4013	14
	4	9.9771	4	0.4009	15
	5	9.9345	4	0.4026	15
	6	9.9371	4	0.4025	16
	7	9.9245	4	0.4030	16
	8	9.9245	4	0.4030	16
16	3	14.5548	4	0.2748	17
	4	14.5560	4	0.2748	18
	5	14.4842	4	0.2762	18
	6	14.4855	4	0.2761	19
	7	14.4638	4	0.2766	19
	8	14.4644	4	0.2765	19

Table 3.5: ASM without coarse space for problem 1

p	$\kappa(P)$	λ_{max}	λ_{min}	iter.
3	11.4241	4	0.3501	21
4	11.4573	4	0.3491	22
5	11.4354	4	0.3498	23
6	11.4459	4	0.3495	23
7	11.4412	4	0.3496	23
8	11.4458	4	0.3495	24

Table 3.6: ASM without coarse space for problem 2

Figure 3.10: Convergence history: ASM for pb.1, $N = 9$, $p = 8$.

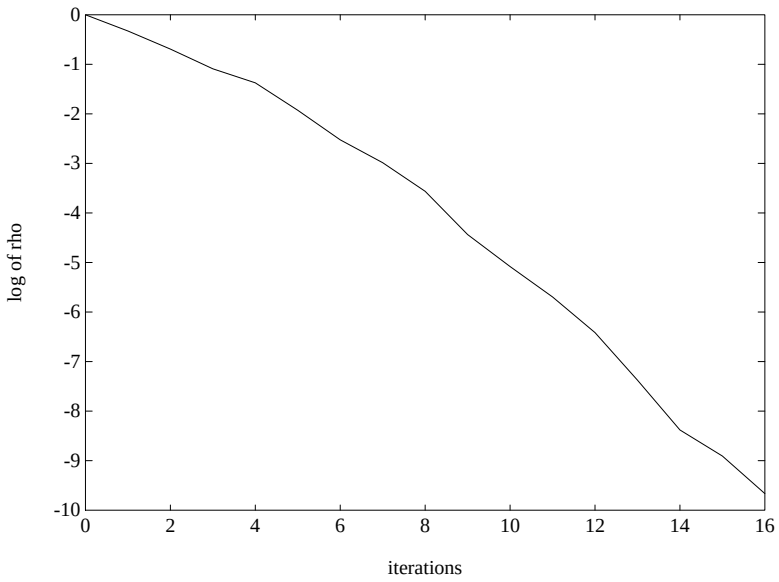


Figure 3.11: Convergence history: ASM for pb.1, $N = 16$, $p = 8$.

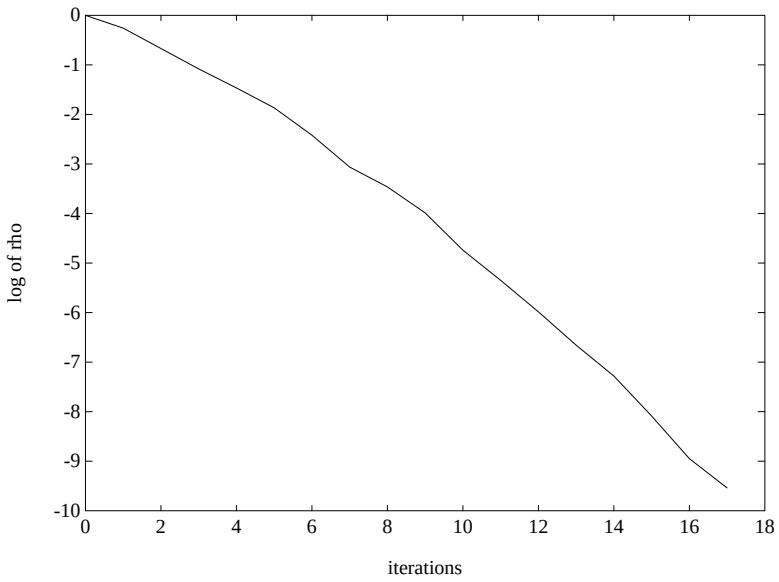


Figure 3.12: Convergence history: ASM for pb.1 without coarse space, $N = 9$, $p = 8$.

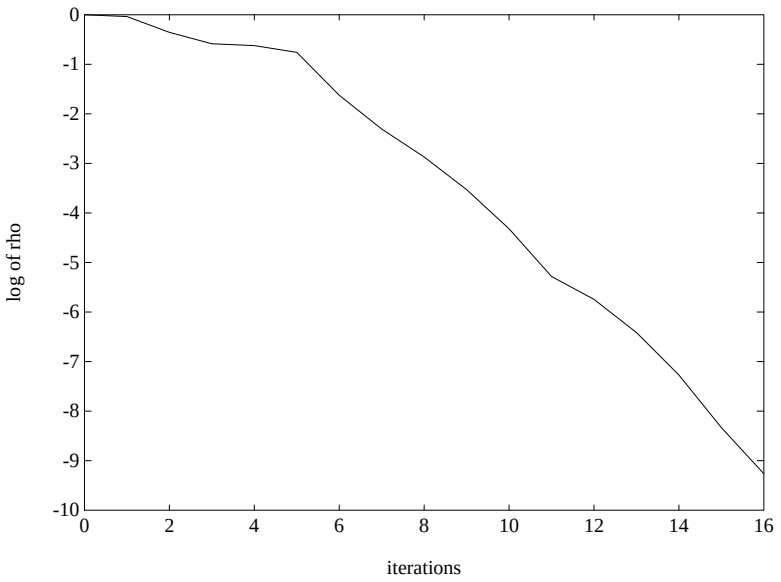
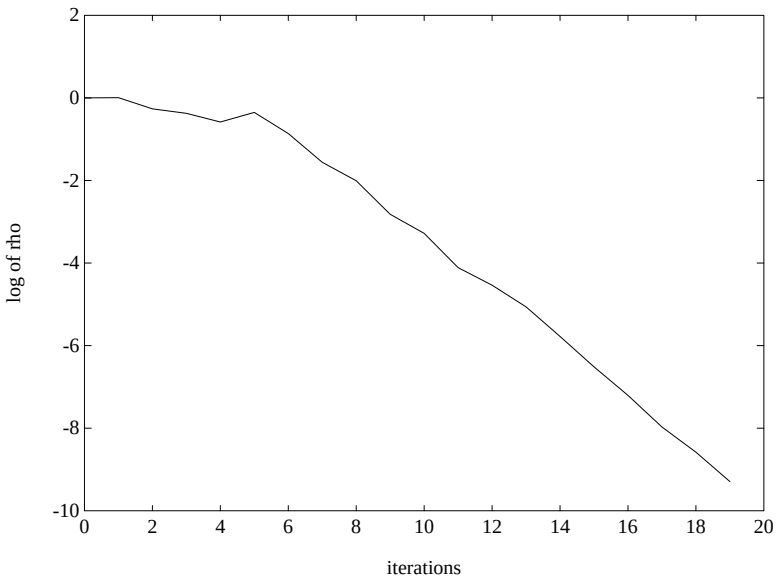


Figure 3.13: Convergence history: ASM for pb.1 without coarse space, $N = 16$, $p = 8$.



N	p	MSGMRES iter.	MSCG iter.	MSCG $\kappa(P) = \lambda_{max}/\lambda_{min}$	MSM iter.	$l^2 - err$
4	3	5	4	1.0097 = 1 / 0.9904	6	0.0083
	4	5	3	1.0065 = 1 / 0.9935	6	8.4248e-4
	5	5	4	1.0076 = 1 / 0.9925	6	7.1026e-5
	6	5	4	1.0086 = 1 / 0.9915	6	5.1892e-6
	7	5	4	1.0086 = 1 / 0.9914	6	3.4483e-7
	8	5	4	1.0088 = 1 / 0.9913	6	3.0349e-8
9	3	6	4	1.0115 = 1 / 0.9886	6	0.0027
	4	6	4	1.0098 = 1 / 0.9903	6	2.0497e-4
	5	6	4	1.0098 = 1 / 0.9903	6	1.2917e-5
	6	6	4	1.0097 = 1 / 0.9904	6	7.0414e-7
	7	6	4	1.0096 = 1 / 0.9904	6	3.6437e-8
	8	6	4	1.0097 = 1 / 0.9904	6	3.6643e-8
16	3	6	4	1.0114 = 1 / 0.9887	6	0.0011
	4	6	4	1.0152 = 1 / 0.9850	6	6.7965e-5
	5	6	4	1.0141 = 1 / 0.9861	6	3.4208e-6
	6	6	4	1.0150 = 1 / 0.9852	6	1.4984e-7
	7	6	4	1.0144 = 1 / 0.9858	6	5.3302e-8
	8	6	4	1.0147 = 1 / 0.9855	6	6.4125e-8

Table 3.7: Multiplicative methods for problem 1

p	MSGMRES iter.	MSCG iter.	MSCG $\kappa(P) = \lambda_{max}/\lambda_{min}$	MSM iter.
3	7	5	1.0772 = 1 / 0.9284	9
4	7	5	1.0997 = 1 / 0.9093	10
5	7	5	1.1167 = 1 / 0.8955	10
6	7	5	1.1304 = 1 / 0.8847	10
7	7	5	1.1414 = 1 / 0.8761	11
8	7	5	1.1505 = 1 / 0.8691	11

Table 3.8: Multiplicative methods for problem 2 on an L-shaped region

Figure 3.14: Convergence history: MSM for pb.1, $N = 16$, $p = 8$.

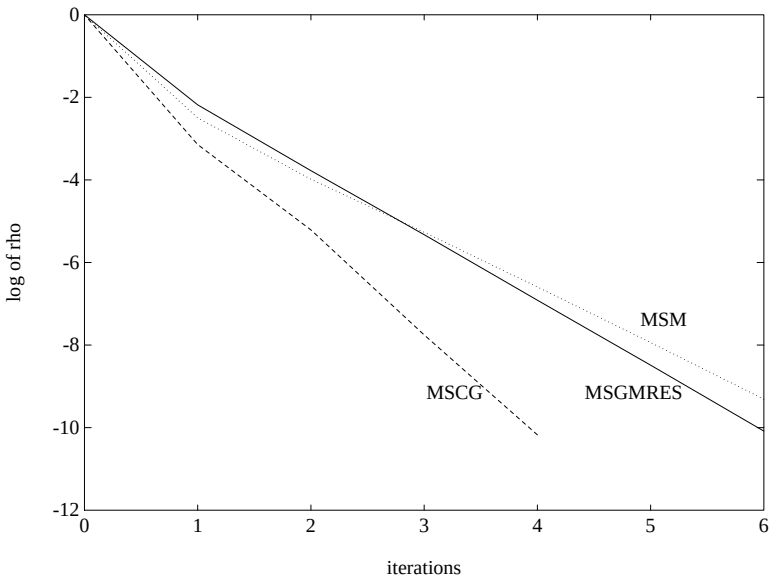
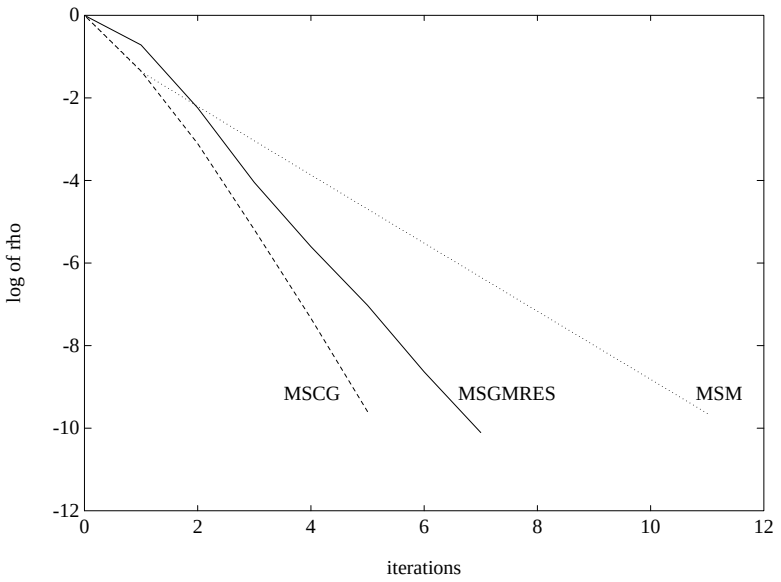


Figure 3.15: Convergence history: MSM for pb.2, $N = 16$, $p = 8$.



Chapter 4

Local refinement for overlapping Schwarz methods

4.1 Introduction

In the previous chapter, we studied a domain decomposition method using p-version finite elements in the Schwarz framework. We showed that the condition number of the ASM iteration operator is bounded by a constant independent of p, H and the number of subdomains. Here we consider a variant of the method based on local refinement. It is inspired by work Bramble, Ewing, Parashkevov, and Pasciak for the h-version finite element method, see [17], [18], and can be of interest in many applications where the accuracy of the numerical solution needs to be increased only in certain parts of the domain. For standard h-version finite elements, local refinement can be introduced by selecting and refining some elements of a coarse triangulation. This process can be applied recursively and multilevel methods considered. For the p-version finite element method considered here, local refinement consists in increasing the order p of the polynomial basis functions only in selected elements of the fixed triangulation.

4.2 An overlapping ASM with local refinement in two dimensions

We consider the same model problem 1.3 for linear, self adjoint, second order elliptic problems, on a bounded Lipschitz region $\Omega \subset R^2$ with homogeneous Dirichlet boundary conditions. The discrete problem is given by the p-version finite element method. As

before, the original region Ω is a union of non-overlapping square elements Ω_i , $i = 1, \cdots, N_e$, with mesh size H. Let N be the number of interior mesh points. With refinement everywhere, we have the finite element space

$$V_D^p = V^p \bigcap H_0^1(\Omega).$$

In the case of local refinement, we select $N^r < N$ interior nodes. Let I^r be the set of refinement indexes. With each selected interior node x_i, we associate a subdomain Ω_i', defined as the $2H \times 2H$ open square centered at x_i. The region of refinement is then

$$\Omega_r = \bigcup \Omega_i' , \qquad i \in I^r$$

and the finite element space

$$V_r^p = V_0^p + V_1^p + \cdots + V_{N_r}^p .$$

- $V_0^p = V_D^1$ is the analogue of the h-version coarse space;
- $V_i^p = V^p \bigcap H_0^1(\Omega_i')$ are the local spaces.

The discrete problem is then:
Find $u_p^* \in V_r^p$ such that

$$a(u_p^*, v_p) = f(v_p) , \qquad \forall \ v_p \in \ V_r^p . \tag{4.1}$$

As before, the algorithm consists in solving, by an iterative method, the equation

$$Pu_p = (P_0 + P_1 + ... + P_{N_r})u_p = g_p , \tag{4.2}$$

with the projections $P_i : V_r^p \to V_i^p$.
The main result of this chapter is:

Theorem 4.1 *Let $\Omega \subset R^2$. The operator P of the additive algorithm defined by the spaces V_i^p satisfies the estimate*

$$\kappa(P) \leq const.$$

if there are no isolated points on $\partial\Omega_r$, and

$$\kappa(P) \leq C(1 + \log p)^2$$

otherwise.

A point on $\partial\Omega_r$ is isolated if it is not a limit (accumulation) point of $\partial\Omega_r$; see the section on numerical experiments for examples. Before proving this theorem, we need to develop some technical results concerning the decomposition of discrete harmonic polynomials as well as review some classical tools in approximation theory such as Markov's inequality. We recall that a polynomial $w \in Q_p$ on an element Ω_i is discrete harmonic if

$$a(w, \phi) = 0, \qquad \forall\, \phi \in Q_p \quad \text{that vanish on } \partial\Omega_i.$$

A piecewise polynomial function $w \in V^p$ defined on Ω is discrete harmonic if it is so on each element, i.e. if

$$a(w, \phi) = 0, \qquad \forall\, \phi \in V^p \quad \text{that vanish on the interface } \Gamma = \bigcup \partial\Omega_i.$$

4.2.1 Preliminary results

We will use the classical Markov's theorem:

Theorem 4.2 *If v is a polynomial of degree p on $I = [-1, 1]$, then*

$$\max_I |v'(x)| \leq p^2 \max_I |v(x)|,$$

with equality only for $v = \pm T_p$, the Chebyshev polynomial of degree p. This inequality is not scale invariant: on the interval I=$[-H, H]$

$$\max_I |v'(x)| \leq \frac{p^2}{H} \max_I |v(x)|,$$

A proof and many useful generalizations can be found in Rivlin [72]. The following result is stated and proved as Lemma 2.2 in Bramble and Xu [22]:

Lemma 4.1 *If D is a bounded domain in R^2 and ∂D is Lipschitz continuous, then*

$$\|w\|_{L^\infty(D)} \leq C(|\log \epsilon|^{1/2} \|w\|_{H^1(D)} + \epsilon \|w\|_{W^{1,\infty}(D)})$$

for every function $w \in W^{1,\infty}(D)$ and any $\epsilon \in (0, 1)$.

With this lemma, we can prove a Sobolev-like inequality for polynomial finite element:

Lemma 4.2 *If $u_p \in Q_p(\Omega_i)$, where Ω_i is an element of diameter H, then*

$$\|u_p\|^2_{L^\infty(\Omega_i)} \leq C(1+\log p)\|u_p\|^2_{H^1(\Omega_i)}.$$

Proof. Apply Lemma 4.1 to $w = u_p$ on one element $D = \Omega_i$:

$$\|u_p\|_{L^\infty} \leq C(|\log\epsilon|^{1/2}\|u_p\|_{H^1} + \epsilon\|u_p\|_{W^{1,\infty}}).$$

The $W^{1,\infty}-$norm is not scale invariant: on an element Ω_i of diameter H

$$\|u_p\|_{W^{1,\infty}(\Omega_i)} = \|u_p\|_{L^\infty(\Omega_i)} + H\|\nabla u_p\|_{L^\infty(\Omega_i)} .$$

By Markov's inequality:

$$\|u_p\|_{W^{1,\infty}(\Omega_i)} \leq (1 + H\frac{p^2}{H})\|u_p\|_{L^\infty(\Omega_i)} = (1+p^2)\|u_p\|_{L^\infty(\Omega_i)}.$$

Choosing $\epsilon = \frac{1}{4Cp^2}$, we obtain:

$$\|u_p\|_{L^\infty(\Omega_i)} \leq C|\log 4C + 2\log p|^{1/2}\|u_p\|_{H^1(\Omega_i)} + \frac{1+p^2}{4p^2}\|u_p\|_{L^\infty(\Omega_i)}$$

and finally

$$\|u_p\|^2_{L^\infty(\Omega_i)} \leq C(1+\log p)\|u_p\|^2_{H^1(\Omega_i)}.$$

□

Corollary 4.1 *If u_p of Lemma 4.2 vanishes at some point in Ω_i, then*

$$\|u_p\|^2_{L^\infty(\Omega_i)} \leq C(1+\log p)\|u_p + \alpha\|^2_{H^1(\Omega_i)},$$

for every constant α.

Proof. Use the inequality

$$\|u_p\|_\infty \leq \|u_p + \alpha\|_\infty + \|\alpha\|_\infty \leq 2\|u_p + \alpha\|_\infty$$

and apply Lemma 4.2.

□

Corollary 4.2 *If β is any value of u_p in Ω_i, then*

$$\|u_p - \beta\|^2_{L^\infty(\Omega_i)} \leq C(1+\log p)|u_p|^2_{H^1(\Omega_i)}.$$

Proof. Consider the inf over α in the estimate of Corollary 4.1. By a quotient space argument we obtain

$$\|u_p\|^2_{L^\infty} \leq C(1+\log p)|u_p|^2_{H^1}.$$

Then, just apply this estimate to $u_p - \beta$.

□

One of the main tools we are going to use is the following p-version analog of Lemma 3.2 in Widlund [84]:

Lemma 4.3 *Let $w \in Q_p$ be a discrete harmonic polynomial on a square element Ω_i, that vanishes at the vertices of Ω_i. Then there exist discrete harmonic polynomials $w_j \in Q_p$, with nonzero boundary values only on one side Γ_j of Ω_i, such that $w = \sum_j w_j$ and*

$$\sum_j |w_j|^2_{H^{1/2}(\partial\Omega_i)} \leq C(1+\log p)^2 |w|^2_{H^1(\Omega_i)}.$$

Proof. For each side Γ_j, we define

$$w_j = \begin{cases} w \text{ on } \Gamma_j \\ 0 \text{ on } \Gamma_i \neq \Gamma_j \\ \text{discrete harmonic extension in } \Omega_i. \end{cases}$$

Clearly $w = \sum_j w_j$. By definition

$$|w_j|^2_{H^{1/2}(\partial\Omega_i)} = \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|w(x(s)) - w(x(s'))|^2}{|x(s)-x(s')|^2}\, dsds'.$$

Since $\partial\Omega_i = \bigcup_j \Gamma_j$ we have:

$$|w_j|^2_{H^{1/2}(\partial\Omega_i)} = \sum_j |w_j|^2_{H^{1/2}(\Gamma_j)} + \sum_{i\neq j}\int_{\Gamma_i}\int_{\Gamma_j} \frac{|w(x(s)) - w(x(s'))|^2}{|x(s)-x(s')|^2}\, dsds'.$$

The first terms can be bounded using the trace theorem:

$$|w_j|^2_{H^{1/2}(\Gamma_j)} = |w|^2_{H^{1/2}(\Gamma_j)} \leq |w|^2_{H^{1/2}(\partial\Omega_i)} \leq C|w|^2_{H^1(\Omega_i)}.$$

Now consider one of the other terms. We parametrize the double integral as

$$\int_0^H \int_{-H}^0 \frac{|w(x(s)) - w(x(s'))|^2}{|x(s) - x(s')|^2}\, ds'ds$$

and bound it by

$$2\int_0^H \int_{-H}^0 \frac{|w(x(s)) - w(x(s'))|^2}{|s - s'|^2}\, ds'ds\ .$$

This integral can be estimated by

$$4\int_0^H \int_{-H}^0 \frac{|w(x(s))|^2}{|s - s'|^2}\, ds'ds + 4\int_0^H \int_{-H}^0 \frac{|w(x(s'))|^2}{|s - s'|^2}\, ds'ds \leq$$

$$\leq 4\int_0^H \frac{|w(x(s))|^2}{s}\, ds - 4\int_{-H}^0 \frac{|w(x(s'))|^2}{s'}\, ds',$$

because

$$\int_{-H}^0 \frac{ds'}{(s - s')^2} = \frac{1}{s - s'}|_{-H}^0 = \frac{1}{s} - \frac{1}{s + H} \leq \frac{1}{s}$$

and similarly

$$\int_0^H \frac{ds}{(s - s')^2} \leq -\frac{1}{s'}.$$

We divide each resulting integral into two:

$$\int_0^H \frac{|w(x(s))|^2}{s}\, ds = \int_0^{H/p^2} \frac{|w(x(s))|^2}{s}\, ds + \int_{H/p^2}^H \frac{|w(x(s))|^2}{s}\, ds.$$

We estimate the first term by using the mean value theorem and Markov's inequality.

$$\int_0^{H/p^2} \frac{|w(x(s))|^2}{s}\, ds = \int_0^{H/p^2} |\frac{dw}{ds}(\tilde{x})|^2 \frac{(x(s) - x(0))^2}{s}\, ds \leq$$

$$\leq (\frac{p^2}{H})^2 \|w\|^2_{L^\infty(\Omega_i)} \int_0^{H/p^2} \frac{(x(s) - x(0))^2}{s}\, ds = (\frac{p^2}{H})^2 \|w\|^2_{L^\infty(\Omega_i)} \int_0^{H/p^2} s\, ds =$$

$$= (\frac{p^2}{H})^2 \|w\|^2_{L^\infty(\Omega_i)} \frac{1}{2} (\frac{H}{p^2})^2 = \frac{1}{2}\|w\|^2_{L^\infty(\Omega_i)},$$

since $x(s) - x(0) = s$. We estimate the second term by using the discrete Sobolev inequality of Corollary 4.2.

$$\int_{H/p^2}^H \frac{|w(x(s))|^2}{s}\, ds \leq \|w\|^2_{L^\infty(\Omega_i)} \log \frac{H}{H/p^2} = 2\log p \|w\|^2_{L^\infty(\Omega_i)} \leq$$

$$\leq C(1 + \log p)^2 |w|^2_{H^1(\Omega_i)}$$

Combining these estimates, we obtain the result.

□

We can prove, in essentially the same way, the result:

Lemma 4.4 *Under the same hypotheses as in Lemma 4.3,*

$$\sum_j \|w_j\|^2_{H^{1/2}_{00}(\Gamma_j)} \leq C(1+\log p)^2 |w|^2_{H^1(\Omega_i)}$$

We will need the following two dimensional extension theorem for polynomial finite element functions. The proof can be found in Babuška et al. [8] or Bernardi and Maday [11].

Theorem 4.3 *Let Ω_i be a square element and $f \in H^{1/2}(\partial\Omega_i)$ be a polynomial of degree p on each side of Ω_i. Then there exists a polynomial $u \in Q_p(\Omega_i)$ such that $u = f$ on $\partial\Omega_i$ and*

$$\|u\|_{H^1(\Omega_i)} \leq C\|f\|_{H^{1/2}(\partial\Omega_i)}.$$

Using this extension theorem, it is possible to prove the following p-version analog of a Lemma of Bramble et al. [18]. It is a generalization of Lemma 3.2 of Chapter 3.

Lemma 4.5 *Consider an interior node x_i with its associated subdomain Ω_i' and let $\Gamma_j, j = 1,2,3,4$, denote the four edges connecting x_i and $\partial\Omega_i'$ (see the figure 4.1). Let $w_i \in V_i^p = V^p \bigcap H_0^1(\Omega_i')$ be a discrete harmonic polynomial that vanishes at the interior node x_i. Let w_{ij} be the discrete harmonic polynomial that equals w_i on Γ_j and vanishes on the remaining edges.*
Then $w_i = \sum_j w_{ij}$ and

$$\sum_j a(w_{ij}, w_{ij}) \leq C a(w_i, w_i)$$

if w_i vanishes on at least one of the Γ_j, otherwise

$$\sum_j a(w_{ij}, w_{ij}) \leq C(1+\log p)^2 a(w_i, w_i).$$

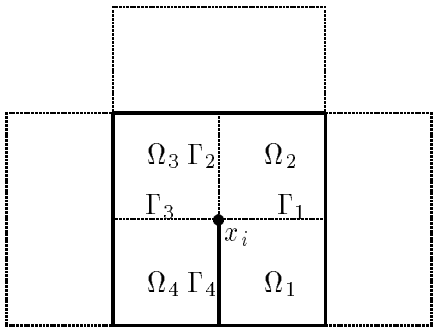


Figure 4.1: Decomposition of w_i in 2-dim

Proof. a) Let suppose that $w_i = 0$ on the edge Γ_4. We start by constructing w_{i1} with support in $\Omega_1 \bigcup \Omega_2 \bigcup \Gamma_1$:

$$w_{i1} = \begin{cases} w_i \text{ on } \Omega_1 \cup \Gamma_1 \\ \text{reflection across } \Gamma_1 \text{ from } \Omega_1 \text{ onto } \Omega_2. \end{cases}$$

Clearly w_{i1} is discrete harmonic, since w_i is, and

$$|w_{i1}|_{H^1(\Omega_i')} = 2|w_{i1}|_{H^1(\Omega_1)} = 2|w_i|_{H^1(\Omega_1)} \leq 2|w_i|_{H^1(\Omega_i')}.$$

Now $w_i - w_{i1}$ is discrete harmonic, vanishes on Γ_1 and, by the triangle inequality, $|w_i - w_{i1}|_{H^1(\Omega_i')} \leq 3|w_i|_{H^1(\Omega_i')}$. We can therefore construct w_{i2} with support in $\Omega_2 \bigcup \Omega_3 \bigcup \Gamma_2$ in the same way:

$$w_{i2} = \begin{cases} w_i - w_{i1} \text{ on } \Omega_2 \cup \Gamma_2 \\ \text{reflection across } \Gamma_2 \text{ from } \Omega_2 \text{ onto } \Omega_3. \end{cases}$$

w_{i2} is discrete harmonic, $|w_{i2}|_{H^1(\Omega_i')} \leq 6|w_i|_{H^1(\Omega_i')}$ and $|w_i - w_{i1} - w_{i2}|_{H^1(\Omega_i')} \leq 9|w_i|_{H^1(\Omega_i')}$. Finally, we construct w_{i3} with support in $\Omega_3 \bigcup \Omega_4 \bigcup \Gamma_3$:

$$w_{i3} = \begin{cases} w_i - w_{i1} - w_{i2} \text{ on } \Omega_3 \cup \Gamma_3 \\ \text{reflection across } \Gamma_3 \text{ from } \Omega_3 \text{ onto } \Omega_4. \end{cases}$$

Again, w_{i3} is discrete harmonic and $|w_{i3}|_{H^1(\Omega_i')} \leq 18|w_i|_{H^1(\Omega_i')}$.
Therefore $\sum_{j=1}^4 w_{ij} = w_i$ and $\sum_{j=1}^4 a(w_{ij}, w_{ij}) \leq C a(w_i, w_i)$.

b) We suppose now that w_i does not vanish identically on any Γ_j. Consider one of the four elements of Ω_i', for example Ω_1. w_i vanishes on the two edges of Ω_1 that are on the boundary of Ω_i' and differs from zero on the other two interior edges Γ_1 and Γ_4. We want to write w_i as a sum of two discrete harmonic functions each different from zero on only one of the interior edges. This can be done by using lemma 4.3; we can find two discrete harmonic polynomials w_{i1} and w_{i4} with nonzero values only on Γ_1 and Γ_4, respectively, such that

$$|w_{i1}|^2_{H^{1/2}(\partial\Omega_1)} + |w_{i4}|^2_{H^{1/2}(\partial\Omega_1)} \leq C(1+\log p)^2 |w_i|^2_{H^1(\Omega_1)} .$$

By the polynomial extension Theorem 4.3

$$|w_{i1}|^2_{H^1(\Omega_1)} \leq C|w_{i1}|^2_{H^{1/2}(\partial\Omega_1)}.$$

Therefore

$$|w_{i1}|^2_{H^1(\Omega_1)} \leq C(1+\log p)^2 |w_i|^2_{H^1(\Omega_1)}$$

and the same inequality holds for w_{i4}.
Repeating the process on the other elements in Ω_i', we obtain eight discrete harmonic polynomials. Combining the pairs corresponding to the same interior edge, where they have the same values by definition, we finally obtain the four discrete harmonic piecewise polynomials with the desired properties.

□

4.2.2 Proof of the main result

We can now proceed to prove Theorem 4.1. As in the case with refinement everywhere studied in [70], we base the proof on the Schwarz framework of Dryja and Widlund [36]. A constant upper bound for the spectrum of P is obtained by noting that for $i \geq 1$

$$a(P_i u_p, u_p) = a(P_i u_p, P_i u_p) = a_{\Omega_i'}(P_i u_p, P_i u_p) \leq a_{\Omega_i'}(u_p, u_p).$$

Each point is covered by no more than four subregions Ω_i' and the norm of P_0 is equal to one, therefore $\lambda_{max} \leq 5$.

A lower bound is obtained by using Theorem 2.2 and Lemma 4.5 and defining a partition of $u \in V_r^p$ with the required properties. We start by defining $u_0 \in V_0^p$ by

specifying its values at the interior nodes x_i:

$$u_0(x_i) = \begin{cases} u(x_i) & \text{if } x_i \notin \Omega_r \\ Q_H u(x_i) & \text{if } x_i \in \Omega_r \end{cases}$$

Here Q_H is the L_2-projection of V_r^p onto V_0^p. It has the properties

$$\|(I - Q_H)v\|^2_{L^2(\Omega)} \leq CH^2 a(v,v)$$

$$a(Q_H v, Q_H v) \leq C a(v,v)$$

for every function $v \in H_0^1(\Omega)$, see Bramble and Xu, [22]. Since $u - u_0 \equiv 0$ outside Ω_r, we have

$$a_\Omega(u - u_0, u - u_0) = a_{\Omega_r}(u - u_0, u - u_0).$$

Using the triangle inequality, we have

$$a(u - u_0, u - u_0)^{1/2} = a_{\Omega_r}(u - Q_H u + Q_H u - u_0, u - Q_H u + Q_H u - u_0)^{1/2} \leq$$

$$a_{\Omega_r}((I - Q_H)u, (I - Q_H)u)^{1/2} + a_{\Omega_r}(Q_H u - u_0, Q_H u - u_0)^{1/2}$$

$$\leq C a(u,u)^{1/2} + a_{\Omega_r}(Q_H u - u_0, Q_H u - u_0)^{1/2}.$$

By definition $Q_H u - u_0 \equiv 0$ in Ω_r except in those elements Ω_i with at least one vertex on $\partial\Omega_r$. Therefore, expanding $Q_H u - u_0 = \sum_i (Q_H u - u_0)(x_i)\phi_i$ in the nodal basis and using the properties of Q_H, we obtain:

$$a_{\Omega_r}(Q_H u - u_0, Q_H u - u_0) \leq C \sum_{x_i \in \partial\Omega_r} (Q_H u - u_0)^2(x_i) \leq$$

$$\leq CH^{-2} \|Q_H u - u_0\|^2_{L^2(\Omega)} \leq C a(u,u).$$

We have then proved

$$a(u - u_0, u - u_0) \leq C a(u,u). \tag{4.3}$$

Reasoning in the same way, we prove

$$\|u - u_0\|^2_{L^2(\Omega)} \leq CH^2 a(u,u). \tag{4.4}$$

We need now to decompose $w = u - u_0 \in H_0^1(\Omega_r)$. We use a result from the previous chapter for the case of refinement everywhere; see eqn. (3.24):

There exists a decomposition $w = \sum_{i=1}^{N} w_i$ *such that*

$$\sum_{i=1}^{N} a(w_i, w_i) \leq C(a(w,w) + \frac{1}{H^2}\|w\|^2_{L^2(\Omega)})$$

and $supp(w_i) \subset supp(w)$.

Here N is the number of all the interior nodes in Ω, therefore this is not the desired decomposition, since it contains terms w_i with $i \notin I_r$, which correspond to $x_i \notin \Omega_r$. Since $supp(w_i) \subset supp(w)$, these remaining terms w_i vanish unless $x_i \in \partial\Omega_r$. Therefore, we need to decompose only the terms w_i corresponding to $x_i \in \partial\Omega_r$ and distribute their contributions among the other terms. First, we write

$$w_i = w^0 + y$$

where $y = w_i$ on the mesh lines and is extended as a discrete harmonic polynomial inside each element. $w^0 = w_i - y$ vanishes on the mesh lines and is a-orthogonal to y:

$$a(w_i, w_i) = a(w^0, w^0) + a(y, y).$$

Since w^0 is nonzero only on elements in Ω_r, we can regard these elements as belonging to a unique subdomain Ω'_j with $j \in I_r$ and add the restrictions of w^0 on these elements to the corresponding term w_j. We still denote these modified terms by w_j. We now use lemma 4.5 to decompose and distribute y. If there are no isolated points on $\partial\Omega_r$, there is at least one edge of the mesh belonging to $\partial\Omega_r$ ending at x_i. Since $w_i = 0$ on this edge, y will vanish there too. Therefore, by lemma 4.5 $y = \sum_j y_j$ and

$$\sum_j a(y_j, y_j) \leq C\, a(y, y).$$

If x_i is an isolated point of $\partial\Omega_r$, then lemma 4.5 will give us a decomposition of y satisfying the weaker inequality:

$$\sum_j a(y_j, y_j) \leq C(1 + \log p)^2 a(y, y).$$

The functions y_j are nonzero only on one edge of the mesh and this edge can be assigned uniquely to a subdomain Ω'_k in Ω_r. We add the functions y_j to the corresponding w_k and still denote the modified terms by w_k. It follows then that we have found a decomposition

$$w = \sum_{i \in I_r} w_i$$

satisfying

$$\sum_{i \in I_r} a(w_i, w_i) \leq C(a(w,w) + \frac{1}{H^2}\|w\|^2_{L^2(\Omega)})$$

in the case without isolated points on $\partial\Omega_r$ and

$$\sum_{i \in I_r} a(w_i, w_i) \leq C(1+\log p)^2(a(w,w) + \frac{1}{H^2}\|w\|^2_{L^2(\Omega)})$$

in the general case. Using (4.3) and (4.4), we can conclude that the decomposition of u satisfies the inequality of Theorem 2.2 with a constant independent of p and N_r in the case without isolated points and with $C(1+\log p)^2$ otherwise. By Theorem 2.2, we then bound in the same way the smallest eigenvalue of the iteration operator P from below and conclude the proof of Theorem 4.1.

□

4.3 An overlapping ASM with local refinement in three dimensions

We now study local refinement in three dimensions. To date, the analysis is incomplete, since we have been able to generalize from two to three dimensions only some of the technical tools of Section 4.2. The most important missing tools are a polynomial extension theorem and a decomposition of discrete harmonic polynomials as in Lemma 4.3 and 4.5. Polynomial extension theorems are known in one and two dimensions, see Maday [54], Bernardi and Maday [11], Babuška, Craig, Mandel and Pitkäranta [8]. Very recently, Belgacem [10] proved a three dimensional polynomial extension theorem using results of Canuto and Funaro [27], but the paper is not yet available to us. More recent results of Canuto [26] about the uniform equivalence of finite element and spectral interpolation, could also lead to polynomial extension theorems. However, if we consider regular refinement regions, in the sense specified in the Theorem 4.4, these tools are not needed and an optimal bound can be established.

Our model problem is given on a region $\Omega \subset R^3$, which is the union of non-overlapping cubic elements Ω_i with mesh size H. Homogeneous Dirichlet boundary conditions are given on $\partial\Omega$. With the same notation used in the two dimensional case, we associated with each selected interior node x_i a subdomain Ω_i' defined as the open cube of side $2H$

centered at x_i. The refinement region is

$$\Omega_r = \bigcup \Omega_i' , \qquad i \in I^r$$

and the finite element space is decomposed as

$$V_r^p = V_0^p + V_1^p + \cdots + V_{N_r}^p ,$$

where

- $V_0^p = V_D^1$
- $V_i^p = V^p \bigcap H_0^1(\Omega_i')$ are the local spaces.

The definition of the algorithm in terms of projections is formally the same. The main result is:

Theorem 4.4 *Let $\Omega \subset R^3$. If there are no isolated points or edges on the boundary $\partial\Omega_r$ of the refinement region, then the operator P of the ASM defined by the spaces V_i^p satisfies the estimates*

$$\kappa(P) \leq const.$$

Proof. We repeat the two dimensional proof of the constant upper bound and the construction of the coarse component u_0 in the decomposition of $u \in V_r^p$. We then apply to $w = u - u_0$ the three dimensional decomposition result proved in the previous chapter for the case of refinement everywhere:

There exists a decomposition $w = \sum_{i=1}^N w_i$ such that

$$\sum_{i=1}^{N} a(w_i, w_i) \leq C(a(w,w) + \frac{1}{H^2}\|w\|^2_{L^2(\Omega)})$$

and $supp(w_i) \subset supp(w)$.

This is not the desired decomposition, since it contains terms with $i \notin I_r$. As before, we need to decompose only the terms w_i corresponding to $x_i \in \partial\Omega_r$. We write

$$w_i = w^0 + y$$

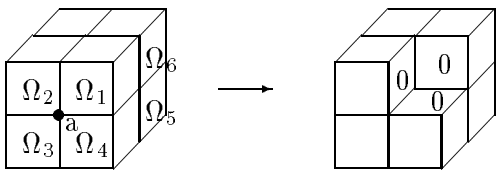


Figure 4.2: First step in the decomposition of w_i in 3-dim

where $y = w_i$ on the interface $\Gamma = \bigcup \partial\Omega_i$ and is extended as a discrete harmonic polynomial inside each element. $w^0 = w_i - y$ vanishes on the interface and is a-orthogonal to y:

$$a(w_i, w_i) = a(w^0, w^0) + a(y, y).$$

Since w^0 is nonzero only on elements in Ω_r, it can be distributed as in the two dimensional case. We now decompose and distribute y. If there are no isolated points or edges on $\partial\Omega_r$, then there is at least a face of the interface belonging to $\partial\Omega_r$ which contains x_i. Since $w_i = 0$ on this face, y will vanish there too. We then apply the following Lemma:

Lemma 4.6 *Consider an interior node x_i with its associated subdomain Ω_i' union of eight elements Ω_i (see the figure 4.2). Let $F_{ij} = \overline{\Omega}_i \bigcup \overline{\Omega_j}$ be the face between Ω_i and Ω_j and let $x_k, k = 1, \cdots, 6$ be the central nodes of the faces of Ω_i'. Let $w_i \in V_i^p = V^p \bigcap H_0^1(\Omega_i')$ be a discrete harmonic polynomial that vanishes at the interior node x_i . Also assume that w_i vanishes on at least one face F_{ij}. Then there exist discrete harmonic polynomials $w_{ik} \in V_k^p, k = 1, \cdots, 5$, such that $w_i = \sum_k w_{ik}$ and*

$$\sum_k a(w_{ik}, w_{ik}) \leq C a(w_i, w_i).$$

Proof. Let suppose that $w_i = 0$ on the face between Ω_1 and Ω_6 and denote by x_a the interior node shared by the elements $\Omega_1, \Omega_2, \Omega_3$, and Ω_4. We construct $w_a \in V_a^p$ by

reflection across faces:

$$w_a = \begin{cases} w_i \text{ in } \Omega_1 \bigcup F_{12} \bigcup F_{14} \\ \text{reflection across } F_{12} \text{ from } \overline{\Omega}_1 \text{ onto } \overline{\Omega}_2 \\ \text{reflection across } F_{14} \text{ from } \overline{\Omega}_1 \text{ onto } \overline{\Omega}_4 \\ \text{reflection across } F_{23} \text{ from } \overline{\Omega}_2 \text{ onto } \overline{\Omega}_3 \end{cases}$$

Clearly $w_a \in V_a^p$ and

$$|w_a|_{H^1(\Omega_a')} = 4|w_a|_{H^1(\Omega_1)} = 4|w_i|_{H^1(\Omega_1)} \leq 4|w_i|_{H^1(\Omega_i')} .$$

$w_i - w_a$ vanishes on F_{12} and F_{14} and it has support in $\Omega_i' \setminus \Omega_1$. This subregion is now an L-shaped subregion formed by seven cubic elements. This is the case treated in Lemma 3.4, which gives us the desired decomposition.

□

4.4 Numerical experiments

In this section, we describe some numerical experiments in two dimensions with the local refinement p-method. See Bramble, Ewing, Parashkevov, and Pasciak [18] for similar experiments with the h-version. More complete experiments with parallel machines and different methods can be found in Moe [65].

The programs, written in MATLAB, have been run on Sun Sparc workstations. In each set of experiments, we consider the Poisson equation in two dimensions with homogeneous Dirichlet boundary conditions. The equation $Pu_p = g_p$ is solved using the conjugate gradient method. The iteration process is stopped when the relative l^2-norm of the residual is less than 10^{-9}. In the following tables, $\kappa(P) = \frac{\lambda_{max}}{\lambda_{min}}$ is the condition number of the operator P.

1) We consider first a problem with no isolated points on $\partial\Omega_r$). We call this case *regular* refinement.

$$\begin{cases} -\Delta u &= -\frac{\pi^2}{8} sin(\frac{\pi}{4}x) sin(\frac{\pi}{4}y). & in\ \Omega, \\ u &= 0 & on\ \partial\Omega \end{cases}$$

The square region $\Omega = [0,8]^2$ has 16 elements and 5 refinement subregions, see figure 4.3. The condition number and the number of iterations are reported in table 4.1. In this case, λ_{max} is constant, while λ_{min} seems to converge as p increases.

2) Next, we consider an *irregular* refinement for problem 1. This selection of refinement interior points leaves an isolated point on $\partial\Omega_r$, see figure 4.4. From the results of table 4.2, we see that λ_{max} is still constant, but λ_{min} decreases considerably. A least square approximation shows that the condition number grows like $(\log p)^2$.

3) Finally, we consider a problem with an isolated edge on the boundary of Ω_r , see figure 4.5. Since in this case we do not have isolated points on $\partial\Omega_r$, this is a regular case and the theory predicts that the condition number is bounded by a constant. This is confirmed by the results of table 4.3.

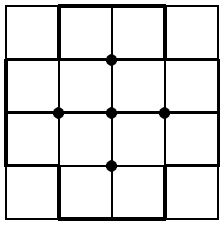


Figure 4.3: Regular choice of refinement points

degree p	unkn.	$\kappa(P)$	λ_{max}	λ_{min}	iter.
3	121	5.7256	3	0.5240	12
4	225	5.9885	3	0.5010	14
5	361	6.1371	3	0.4888	15
6	529	6.2328	3	0.4813	16
7	729	6.2986	3	0.4763	16
8	961	6.3466	3	0.4727	17

Table 4.1: Regular refinement for pb. 1

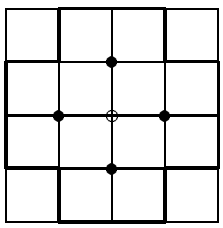


Figure 4.4: Irregular choice of refinement points

degree p	unkn.	$\kappa(P)$	λ_{max}	λ_{min}	iter.
3	121	8.1493	2	0.2454	12
4	225	10.4374	2	0.1916	13
5	361	12.4814	2	0.1602	13
6	529	14.3318	2	0.1395	14
7	729	16.0255	2	0.1248	14
8	961	17.5900	2	0.1137	15

Table 4.2: Irregular refinement for pb. 2

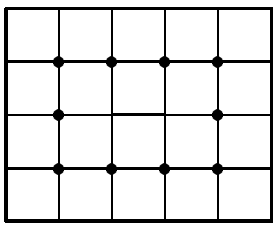


Figure 4.5: Refinement points with an isolated edge on $\partial\Omega_r$

degree p	$\kappa(P)$	λ_{max}	λ_{min}	iter.
3	9.0330	3	0.3321	22
4	10.0987	3	0.2971	23
5	10.8288	3	0.2770	23
6	11.2881	3	0.2658	24
7	11.6386	3	0.2578	24
8	11.8852	3	0.2524	25

Table 4.3: Regular refinement for pb. 3

Chapter 5

Iterative substructuring for the p-version finite element method in two dimensions

5.1 Introduction

Iterative substructuring methods are domain decomposition methods that use non-overlapping subdomains. They have been extensively studied in the current literature, see Bramble, Pasciak, and Schatz [19], [20] Dryja [34], Smith [77], and Widlund [85].

We adopt here the viewpoint of Dryja and Widlund [38], [39], where iterative substructuring methods are analyzed in the additive Schwarz framework. Their analysis is based on the h-version finite element method. The domain Ω is decomposed into non-overlapping subdomains Ω_i that are assumed shape regular and which form a triangulation τ^H with mesh size H. In each subdomain is then introduced a finer shape regular triangulation τ^h with mesh size h. Let Γ_{ij} be the common edge between two neighboring substructures Ω_i and Ω_j and define

$$\Omega_{ij} = \Omega_i \bigcup \Omega_j \bigcup \Gamma_{ij} .$$

Associated with the two triangulations τ^h and τ^H, we consider the linear finite element spaces V^h and V^H. An additive Schwarz iterative substructuring method is then defined by the decomposition

$$V^h = V^H + \sum_{ij} V_{ij},$$

where

- V^H is the coarse space,
- $V_{ij} = V^h \bigcup H_0^1(\Omega_{ij})$ are the local spaces associated with each edge Γ_{ij} .

Defining the additive Schwarz operator

$$P = P_H + \sum_{ij} P_{ij}$$

in terms of the projections onto the subspaces and denoting by u the exact solution of the discrete problem, the algorithm can be written:

Additive Schwarz iterative substructuring algorithm:

i) compute $g = Pu$;

ii) solve the operator equation

$$Pv = g$$

by the conjugate gradient method.

In [36], the following fundamental result is proved.

Theorem 5.1 (Dryja, Widlund) *The operator P of the additive algorithm defined by the spaces V^H and V_{ij} satisfies the estimate*

$$\kappa(P) \leq const.(1 + \log(H/h))^2 .$$

The same bound holds for other important iterative substructuring methods such as the BPS algorithm introduced in Bramble, Pasciak and Schatz [19]. We remark that the additive Schwarz framework developed in Dryja and Widlund [36] allows a quite elegant and direct proof of Theorem 5.1. In particular, a finite element extension theorem is no longer required. It is still an open problem to show that similar ideas can be applied to p-version iterative substructuring methods.

5.2 Iterative substructuring ASM for the p-version finite element method

We consider here the same model problem 1.1 on a region Ω which is the union of N_e non-overlapping square elements Ω_i of uniform size H. Homogeneous Dirichlet boundary

conditions are given on $\partial\Omega$. Our polynomial finite element space V_D^p is represented as:

$$V_D^p = V_0^p + \sum_{ij} V_{ij}^p,$$

where

- $V_0^p = V_D^1$ is the coarse space,
- $V_{ij}^p = V^p \bigcap H_0^1(\Omega_{ij})$ are the local spaces associated with each edge Γ_{ij} .

The iteration operator is

$$P = P_0 + \sum_{ij} P_{ij} \ ,$$

and the algorithm can then be described as in the previous section. A theorem analogous to Theorem 5.1 holds:

Theorem 5.2 *Let $\Omega \subset R^2$. The operator P of the additive algorithm defined by the spaces V_0^p and V_{ij}^p satisfies the estimate*

$$\kappa(P) \leq const.(1 + \log p)^2 \ .$$

Proof. As in Theorem 3.1, we use Theorem 2.2, i.e. we look for bounds for the constants $\rho(\mathcal{E})$ and C_0 . A constant upper bound for $\rho(\mathcal{E})$ is obtained as easily as before.

In order to obtain a good bound for C_0 , we use the technical tools of the previous chapter. First, we must define a decomposition of a function $u_p \in V_D^p$ into the subspaces V_0^p and V_{ij}^p . Since all the functions of V_{ij}^p vanish at the interior nodes, the coarse component $u_0 \in V_0^p$ must be the coarse interpolant of u_p at the nodes of Ω:

$$u_0 = I_H u_p = \begin{cases} u_p(x_i) \text{ for all nodes } x_i \in \Omega \\ \text{piecewise bilinear in each element } \Omega_i. \end{cases}$$

Then let $w = u_p - u_0$. On each element Ω_i, we can write

$$w_i = w|_{\Omega_i} = w_i^{ha} + w_i^0 \ ,$$

i.e w_i is the sum of a discrete harmonic polynomial and a polynomial that vanishes on the boundary of Ω_i . Note that w_i^{ha} vanishes at the interior nodes, in particular at the

vertices of Ω_i . We can then apply Lemma 4.3 to w_i^{ha} : there exist discrete harmonic polynomials $w_{ik} \in Q_p$, $k = 1,2,3,4$, each with nonzero boundary values only on one side Γ_{ik} of Ω_i, such that

$$w_i^{ha} = \sum_{k=1}^{4} w_{ik}$$

and

$$\sum_{k=1}^{4} |w_{ik}|^2_{H^{1/2}(\partial\Omega_i)} \leq C(1+\log p)^2 |w_i|^2_{H^1(\Omega_i)}$$

We can now define the component in V_{ij}^p of the decomposition of u_p :

$$u_{ij} = \begin{cases} w_{ij} + \frac{1}{4} w_i^0 & \text{in } \Omega_i \\ w_{ji} + \frac{1}{4} w_j^0 & \text{in } \Omega_j \, . \end{cases}$$

We clearly have

$$u_0 + \sum_{i,j=1}^{N_e} u_{ij} = u_p$$

and we claim that

$$|u_0|^2_{H^1(\Omega_i)} \leq C(1+\log p) |u_p|^2_{H^1(\Omega_i)} \, , \tag{5.1}$$

$$|u_{ij}|^2_{H^1(\Omega_i)} \leq C(1+\log p)^2 |u_p|^2_{H^1(\Omega_i)} \, . \tag{5.2}$$

Let us prove (5.1). Let $v_k = u_0(x_k) = u_p(x_k)$, where the x_k are the vertices of Ω_i , and let $v = \{v_k\}$. Then

$$|u_0|^2_{H^1(\Omega_i)} = a(u_0, u_0) = v^T K_i v,$$

where K_i is the local stiffness matrix associated with the element Ω_i . By a standard inverse inequality on an element Ω_i of diameter H, we have

$$a(u_0, u_0) \leq \frac{C}{H^2} \|u_0\|^2_{L^2(\Omega_i)} \, .$$

Since in two dimensions $\|u_0\|^2_{L^2}$ is equivalent to $H^2 \sum_k v_k^2$, we obtain

$$a(u_0, u_0) \leq C \sum_k v_k^2 \, ,$$

and for every constant α

$$a(u_0, u_0) = a(u_0 - \alpha, u_0 - \alpha) \leq C \sum_k (v_k - \alpha)^2.$$

Taking $\alpha = v_{average} = 1/4 \sum_k v_k$, we easily obtain the bound

$$(v_1 - \alpha)^2 \leq 2(\frac{v_1 - v_2}{4})^2 + 2(\frac{v_1 - v_3}{4})^2 + 2(\frac{v_1 - v_4}{4})^2 ,$$

and similarly for the other indexes. Therefore

$$|u_0|^2_{H^1(\Omega_i)} \leq C \sum_{k \neq j} (v_k - v_j)^2$$

and we conclude the proof of the inequality (5.1) by using Corollary 4.2.

Let us now prove (5.2). By the extension Theorem 4.3

$$|w_{ij}|^2_{H^1(\Omega_i)} \leq C |w_{ij}|^2_{H^{1/2}(\partial\Omega_i)} ,$$

Since

$$w_{ij} = \begin{cases} w \text{ on } \Gamma_{ij} \\ 0 \text{ on the other three sides of } \Omega_i , \end{cases}$$

we have

$$|w_{ij}|^2_{H^{1/2}(\partial\Omega_i)} = |w_{ij}|^2_{H^{1/2}_{00}(\Gamma_{ij})} = |w|^2_{H^{1/2}_{00}(\Gamma_{ij})},$$

and we claim that

$$|w|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C(1 + \log p)^2 |u_p|^2_{H^1(\Omega_i)} . \tag{5.3}$$

We can essentially repeat the proof of Lemma 4.3, that follows from the ideas presented in Widlund [84]. We bound the double integral defining the $H^{1/2}_{00}-$norm by a line integral. We divide this into two parts, the first bounded using Markov's inequality, the second using a discrete Sobolev inequality and obtain

$$|w|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C(1 + \log p) \|w\|^2_{L^\infty(\Omega_i)} .$$

Now, for every constant p_0

$$\|w\|^2_{L^\infty(\Omega_i)} = \|u_p - I_H u_p\|_{L^\infty(\Omega_i)} = \|u_p + p_0 - I_H(u_p + p_0)\|_{L^\infty(\Omega_i)} \leq 2\|u_p + p_0\|_{L^\infty(\Omega_i)}.$$

Therefore, by the discrete Sobolev inequality of Lemma 4.2

$$|w|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C(1 + \log p) \|u_p + p_0\|^2_{L^\infty(\Omega_i)} \leq C(1 + \log p)^2 \|u_p + p_0\|^2_{H^1(\Omega_i)} .$$

A quotient space argument (taking inf over p_0), concludes the proof of (5.3). Therefore

$$|w_{ij}|^2_{H^1(\Omega_i)} \leq C(1 + \log p)^2 |u_p|^2_{H^1(\Omega_i)} , \tag{5.4}$$

and, since $w_i^{ha} = \sum_{k=1}^{4} w_{ik}$, the same bound holds for w_i^{ha} . Combining this bound with eqn. (5.1), we have

$$|w_i^0|^2_{H^1(\Omega_i)} = |u_p - u_0 - w_i^{ha}|^2_{H^1(\Omega_i)} \leq C(1+\log p)^2 |u_p|^2_{H^1(\Omega_i)} \, . \tag{5.5}$$

Thus, by (5.4) and (5.5), we have proved (5.2). Since at most four components u_{ij} are different from zero in each element Ω_k , we can sum over i and j and obtain

$$|u_0|^2_{H^1(\Omega_k)} + \sum_{i,j=1}^{N_e} |u_{ij}|^2_{H^1(\Omega_k)} \leq C(1+\log p)^2 |u_p|^2_{H^1(\Omega_k)} \, .$$

Adding this inequalities over the elements Ω_k , we obtain the desired decomposition of u_p satisfying

$$|u_0|^2_{H^1(\Omega)} + \sum_{i,j=1}^{N_e} |u_{ij}|^2_{H^1(\Omega)} \leq C_0^2 |u_p|^2_{H^1(\Omega)} \, ,$$

with $C_0^2 \leq C(1+\log p)^2$.

□

5.3 Iterative substructuring ASM on the interface

As in Chapter 3, where we considered overlapping Schwarz methods, a faster iterative substructuring method is obtained by first eliminating the interior variables. The remaining Schur complement system is then solved with the algorithm described before. When the unknowns on the interface Γ have been computed, all the others are determined by solving separate subproblems on each element. It is easy to see that this corresponds to decomposing the space V_D^p into $N_e + 1$ subspaces

$$V_D^p = V_{ha}^p \oplus V_0(\Omega_1) \oplus \cdots \oplus V_0(\Omega_{N_e}).$$

Each subspace $V_0(\Omega_i)$ is associated with an element Ω_i and its interior functions that vanish on $\partial\Omega_i$. The elements of V_{ha}^p are discrete harmonic polynomials and they are a-orthogonal to all the other subspaces. We notice that an element in V_{ha}^p is uniquely determined by its values on the interface $\Gamma = \bigcup \partial\Omega_i$. The algorithm is defined by the decomposition of V_{ha}^p

$$V_{ha}^p = V_0^p + \sum_{ij} \tilde{V}_{ij}^p,$$

where

- $V_0^p = V_D^1$ is the coarse space,
- $\tilde{V}_{ij}^p = V_{ha}^p \bigcap H_0^1(\Omega_{ij})$ are the local spaces of discrete harmonic polynomials associated with each edge Γ_{ij} .

The following result follows from Theorem 5.2.

Theorem 5.3 *Let $\Omega \subset R^2$. The operator $\tilde{P}$ of the additive algorithm defined by the spaces V_0^p and $\tilde{V}_{ij}^p$ satisfies the estimate*

$$\kappa(\tilde{P}) \leq const.(1 + \log p)^2 .$$

Proof. The proof is the same as for Theorem 3.2. The upper bound for the spectrum of P is standard, while a lower bound is obtained by decomposing a function $\tilde{u}_p \in V_{ha}^p$ according to Theorem 5.2:

$$\tilde{u}_p = u_0 + \sum_{i,j=1}^{N_e} u_{ij}$$

with

$$|u_0|^2_{H^1(\Omega)} + \sum_{i,j=1}^{N_e} |u_{ij}|^2_{H^1(\Omega)} \leq C(1 + \log p)^2 |\tilde{u}_p|^2_{H^1(\Omega)} .$$

Then we restrict each component u_{ij} to Γ and extend it as a discrete harmonic polynomial $\tilde{u}_{ij} \in \tilde{V}_{ij}^p$. Since the discrete harmonic extension minimizes the energy, we obtain the desired decomposition.

□

A very successful idea of Smith to improve Schwarz methods in two and three dimansions, is based on introducing some additional spaces in the decomposition of the finite element space, see [76]. Besides the subspaces $\tilde{V}_{ij}$ associated with each edge, we consider vertex spaces associated with each interior node. This construction is well understood in the h-version finite element method: a function in the vertex space associated with the vertex x_i is obtained by setting to zero all values at the nodes on Γ which are at a distance greater than δ from x_i . This parameter measures the overlap introduced with the vertex spaces. For example, the maximal overlap $\delta = H$, gives us exactly the subspaces considered in the overlapping ASM on the interface of Chapter 3. The minimal overlap

in the h-version, $\delta = h$, seems to give the best overall performance in several numerical experiments on different architectures, see the references in Dryja and Widlund [40]. For the p-version finite element method, it is not clear yet what a vertex space might be. The basis functions in the p-version have support in the whole substructure and the local support property, fundamental in the h-version, is lost. Local concepts like vertex spaces and small overlap do not appear to have a p-version counterpart. Recent works by Mandel might be offering new insight in this field, see Mandel [60].

5.4 Numerical experiments

We now describe some numerical experiments with the iterative substructuring methods previously introduced. As before, MATLAB programs have been run on Sun Sparc workstations to solve the Poisson equation in two dimensions with homogeneous Dirichlet boundary conditions on square regions. The equation $Pu_p = g_p$ is solved iteratively using the conjugate gradient method. The iteration process is stopped when the relative l^2-norm of the residual ρ is less than 10^{-9}. In the following tables, $\kappa(P) = \frac{\lambda_{max}}{\lambda_{min}}$ is the condition number of the operator P, while $l^2 - err$ is the relative l^2- norm of the error $u - u_p$.

5.4.1 Iterative substructuring ASM

We consider our standard model problem with a trigonometric exact solution.

$$\begin{cases} -\Delta u &= -\frac{2\pi^2}{c^2} sin(\frac{\pi}{c}x) sin(\frac{\pi}{c}y). & in\ \Omega, \\ u &= 0 & on\ \partial\Omega \end{cases}$$

We consider two decompositions of a square region Ω with 9 and 16 elements, respectively with 12 and 24 subregions, each being the union of two elements. The numerical results are reported in table 5.1. λ_{max} is constant, while λ_{min} clearly decreases when p increases. The growth in the condition number is considerable. Since we have only six values for each experiment, it is not easy to find out how $\kappa(P)$ grows with p. A least square approximation appears to give the best residual for a $(\log p)^2$ approximation to $\kappa(P)$, but a good residual is also obtained with a linear function in p. As remarked before, the number of iterations is better than expected when N_e is small, but approaches the theoretical bound, as a function of the condition number, when N_e increases.

N	p	$\kappa(P)$	λ_{max}	λ_{min}	iter.	$l^2 - err$
12	3	16.0163	4	0.2497	9	0.0083
	4	23.1565	4	0.1727	11	8.4248e-4
	5	25.3644	4	0.1577	12	7.1026e-5
	6	31.0908	4	0.1287	14	5.1892e-6
	7	32.8787	4	0.1217	13	3.4483e-7
	8	37.7113	4	0.1061	15	3.0349e-8
24	3	16.8791	4	0.2370	17	0.0027
	4	23.9841	4	0.1668	19	2.0497e-4
	5	26.0059	4	0.1538	21	1.2917e-5
	6	31.8222	4	0.1257	22	7.0414e-7
	7	33.3884	4	0.1198	23	3.6437e-8
	8	38.3936	4	0.1042	23	3.6646e-8

Table 5.1: Iterative substructuring ASM

5.4.2 Iterative substructuring ASM on the interface

We consider now iterative substructuring on the interface Γ for the same model problem; see table 5.2. Now the condition number of P has improved, but essentially only because λ_{max} has improved. λ_{min} still decreases when p increases, essentially in the same way as before. Therefore, solving the problem on the interface just scales the growth of the condition number, but it does not change fundamentally as a function of p. The number of iterations is considerably smaller for this algorithm. For a better understanding, it is necessary to have more experimental data with larger N_e.

In conclusion, we report some experiments with "primitive" vertex spaces of dimension 5. For each interior node x_i, we introduce a subspace spanned by the nodal basis function associated with x_i and by the four quadratic basis functions associated with the four edges meeting at x_i. The new additive operator is obtained by simply adding the projections onto these vertex spaces to the previous operator. The results are reported in table 5.3. The introduction of these vertex spaces improves λ_{min}, but it seems just to scale it up. λ_{min} still decreases with p with the same law as without vertex spaces. λ_{max} is still constant, with respect to p and N_e, but greater than before. This is because now we have more subspaces and more overlap between subspaces than before. The resulting condition number and number of iterations are sligtly worse than without vertex spaces.

N	p	$\kappa(P)$	λ_{max}	λ_{min}	iter.	$l^2 - err$
	3	5.1895	1.4039	0.2705	3	0.0083
	4	5.9621	1.4043	0.2355	4	8.4248e-4
12	5	7.9567	1.4040	0.1765	5	7.1026e-5
	6	8.5796	1.4041	0.1637	6	5.1892e-6
	7	10.2524	1.4041	0.1370	7	3.4483e-7
	8	10.7776	1.4041	0.1303	7	3.0349e-8
	3	5.1912	1.3100	0.2524	3	0.0027
	4	6.7162	1.3096	0.1950	4	2.0497e-4
24	5	8.0724	1.3097	0.1622	5	1.2917e-5
	6	9.2957	1.3097	0.1409	6	7.0414e-7
	7	10.4129	1.3097	0.1258	6	3.6437e-8
	8	11.4434	1.3097	0.1145	6	3.6646e-8

Table 5.2: Iterative substructuring ASM on the interface

N	p	$\kappa(P)$	λ_{max}	λ_{min}	iter.	$l^2 - err$
	3	6.0351	2.6417	0.4377	3	0.0083
	4	7.0602	2.6435	0.3744	4	8.4248e-4
12	5	10.0665	2.6437	0.2626	5	7.1026e-5
	6	11.0034	2.6438	0.2403	6	5.1892e-6
	7	13.6006	2.6438	0.1944	7	3.4483e-7
	8	14.4173	2.6438	0.1834	7	3.0349e-8
	3	5.3730	2.9050	0.5407	4	0.0027
	4	7.5332	2.9050	0.3857	5	2.0497e-4
24	5	9.6539	2.9069	0.3011	6	1.2917e-5
	6	11.6170	2.9073	0.2503	7	7.0414e-7
	7	13.4569	2.9074	0.2161	8	3.6437e-8
	8	15.1813	2.9075	0.1915	8	3.6646e-8

Table 5.3: Iterative substructuring ASM on the interface with vertex spaces

Therefore these are not good candidates for vertex spaces. More research needs to be carried out.

Chapter 6

Future work and extensions

There are many possibilities for future research and extensions in various directions.

a) Completion of the theory developed in the thesis.

Polynomial extension theorem in three dimensions. This is the main missing technical tool needed for the three dimensional theory of Chapter 4 and 5. Such a result has recently been announced by Belgacem [10] as an extension of previous works of Bernardi and Maday [11] and Canuto and Funaro [27]. Recent work by Canuto [26] about the uniform equivalence of finite element and spectral interpolation, also opens new possibilities for proving polynomial extension theorems.

Local refinement in three dimensions. Such results are based on decomposition results for discrete harmonic polynomials and appear to require extension theorems. In case of *regular* refinement, due to the particular geometry defined by our brick-like elements, we have been able to prove such decomposition results using simple reflection arguments.

Iterative substructuring. The classic theory is based on results on decomposition of discrete harmonic polynomials. A more modern theory using the Schwarz framework has simplified the proofs for the h-version, eliminating the need of extension theorems; see Dryja and Widlund [39], Dryja, Smith, and Widlund [37]. It is still an open problem to establish if such an approach could work for the p-version, both in two and three dimensions. The main technical obstacles are the construction of a good partition of unity and a good interpolation operator as in Theorem 3.1.

The p-version of the vertex spaces idea is also an open problem, as well as the construction of a good coarse space in three dimension.

b) Numerical experiments with the algorithms proposed in the thesis.

Three dimensional numerical experiments. In this work, we have performed only two dimensional numerical computations. However, the theory has been extended in many cases to three dimensions. Therefore it will be of great interest to obtain numerical results to check our theoretical predictions.

Parallel implementation on different architectures. The algorithms proposed in the thesis are parallelizable and scalable. It would be of great practical importance to actually observe their performances on different parallel SIMD and MIMD architectures and on networks of workstations.

c) Ideas from the current research for h-version Schwarz methods.

Schwarz methods using the p-version finite elements can be applied to more general problems than the one considered in the thesis. Many ideas have already been developed for h-version methods: nonsymmetric and indefinite problems, small overlap theory, multilevel and multigrid methods, inexact solvers, more general problems (jump in the coefficients, $2m$-order, systems, mixed formulation, time dependent, nonlinear), more general elements (in particular triangles).

d) Connections with other numerical methods.

Spectral methods and spectral element methods. Even if p-methods and spectral methods have differences in applications and implementation, they are closely related since they use higher order polynomial approximation. Spectral methods in fact, expand the solution in high order Fourier or polynomial series and the coefficients of these expansions are determined by weighted residual projections, see Canuto, Hussaini, Quarteroni, and Zang [28], Patera [69], Canuto and Funaro [27]. Therefore, many of the technical tools developed in this field can be usefully applied to the analysis of p-version algorithms, and vice versa.

Non-conforming and mortar methods. These methods present very interesting ideas applicable to the p-version finite element method. In particular, the idea of weakening the continuity requirement on interelement boundaries and considering other

requirements such as integral matching conditions is quite promising; see Bernardi, Maday and Patera [12].

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